CHAPTER I

NEW MAPPINGS IN TOPOLOGICAL SPACES

The concept of continuous functions plays a central part in topology. It has been indicated in the previous chapter that several distinguished mappings have been introduced in the literature from time to time. The germs of most of these originations lie in the concept of continuous functions. Several turn up as a generalization of the concept of continuity whereas many are found to be independent. The aim in this chapter is to investigate and study some new mappings. It contains three sections.

1.1 SOME NEW MAPPINGS

In 1963, Levine [59] has given a generalization of continuity as follows:

**DEFINITION 1.1.A.** A mapping $f : X \rightarrow Y$ is termed semi continuous if the inverse image by $f$ of every open subset of $Y$ is semi open in $X$.

In 1961, Martin [77] had given a generalization of continuity as follows:
DEFINITION 1.1.B. A mapping \( f : X \rightarrow Y \) is said quasi-continuous if for every point \( x \in X \), and every open set \( V \) containing \( x \), and every open set \( U \) containing \( f(x) \), there exists a nonempty open set \( W \) such that \( W \subseteq V \) and \( f(W) \subseteq U \).

Another generalization that appeared in 1961, is due to Frolik [30]. It defines as follows:

DEFINITION 1.1.C. A mapping \( f : X \rightarrow Y \) is said almost continuous in the sense of Frolik if for every open subset \( V \) of \( Y \), \( f^{-1}(V) \subseteq \text{Cl} \ \text{Int} \ f^{-1}(V) \).

Dev [75] has proved that almost continuity in the sense of Frolik is equivalent to quasi-continuity and semi-continuity as well.

\( \varepsilon \)-continuous mappings were introduced by Fomin [27] in 1943, as a generalization of continuity. In 1968, Singal and Singal [125] have given another generalization termed almost continuity. They have also presented an example of a \( \varepsilon \)-continuous mapping which is not almost continuous in their sense. Papp [98a] and Kim [48a] showed that almost continuity in the sense of Singal and Singal infact implies \( \varepsilon \)-continuity. Thus a concept has been introduced which lies between continuity and \( \varepsilon \)-continuity.
The present section investigates a mapping which lies between continuity and semi continuity. It is introduced as follows:

**DEFINITION 1.1.1.** A mapping \( f : X \rightarrow Y \) is said to be feebly continuous if the inverse image by \( f \) of every open subset of \( Y \) is feebly open in \( X \).

**REMARK 1.1.1.** Since every open set is feebly open it follows from Definitions (0.3.A) and (1.1.1) that every continuous mapping is feebly continuous. However, the converse need not be true. For,

**EXAMPLE 1.1.1.** Let \( X = \{ a, b, c, d \} \), \( T = \{ \emptyset, \{ b \}, \{ c \}, \{ b, c \}, \{ b, c, d \}, X \} \) and \( X^* = \{ a, b \}, T^* = \{ \emptyset, \{ a \}, X^* \} \). Define \( f : X \rightarrow X^* \) by \( f(a) = f(b) = f(c) = a \) and \( f(d) = b \). Then \( f \) is feebly continuous but is is not continuous.

**REMARK 1.1.2.** Since every feebly open set is semi open, it results from Definitions(1.1.A) and (1.1.1) then every feebly continuous mapping is semi continuous. The example below asserts that the converse may be false.

**EXAMPLE 1.1.2.** Let \( X = \{ a, b, c \} \),...
$T = \{ \emptyset, \{a\}, \{b\}, \{a, b\}, x \}$ and $x^*$, $T^*$ ebe as in Example (1.1.1). Define $f : x \rightarrow x^*$ by $f(a) = f(c) = a$ and $f(b) = b$. Then $f$ is semi continuous but it is not feebly continuous.

The following theorem gives several characterizations of feebly continuous mappings:

**Theorem 1.1.1.** Let $f : x \rightarrow y$ be a mapping. Then the following statements are equivalent:

(a) $f$ is feebly continuous.

(b) For any point $x \in x$ and any open set $V$ of $y$ containing $f(x)$, there exists a feebly open set $U$ in $x$ such that $x \in U$ and $f(U) \subseteq V$.

(c) The inverse image of each closed subset of $y$ is feebly closed in $x$.

(d) For each subset $A$ of $x$, $f(\text{Fcl} A) \subseteq \text{Cl} f(A)$.

(e) For each subset $B$ of $y$, $\text{Fcl} f^{-1}(B) \subseteq f^{-1}(\text{Cl} B)$.

We shall use the following results:

**Lemma 1.1.2.** $A$ is feebly closed if and only
if $A = \text{Fcl } A$. [131]

**LEMMA 1.1.2.** $A \subseteq B$ implies that $\text{Fcl } A \subseteq \text{Fcl } B$. [131]

**PROOF OF THEOREM 1.1.1.**

(a) $\iff$ (b). Suppose that (a) holds. Let $V$ be open in $Y$ and contains $f(x)$. Clearly, $x \in f^{-1}(V)$. Now by hypothesis $f^{-1}(V)$ is feebly open in $X$. Put, $U = f^{-1}(V)$. Then $p \in U$ and $f(U) \subseteq V$. Thus (b) holds.

Conversely, suppose (b) holds. Let $A$ be feebly open in $Y$ and let $x \in f^{-1}(A)$. Then $f(x) \in A$. By hypothesis, there exists a feebly open set $U_x$ in $X$ such that $x \in U_x$ and $f(U_x) \subseteq A$. And so, $x \in U_x \subseteq f^{-1}(A)$. This implies that $f^{-1}(A)$ is a union of feebly open sets in $X$. Therefore $f^{-1}(A)$ is feebly open in $X$. Hence, (a) holds.

(a) $\iff$ (c). This follows due to the fact that for any subset $B$ of $Y$, $f^{-1}(X - B) = X - f^{-1}(B)$.

(c) $\iff$ (d). Suppose that (c) holds and
let $A$ be a subset of $X$. Since $A \subseteq f^{-1}(f(A))$ we have, $A \subseteq f^{-1}(\text{Cl} f(A))$. Now, $\text{Cl} f(A)$ is closed in $Y$ and so $f^{-1}(\text{Cl} f(A))$ is feebly closed in $X$. Consequently, $\text{Fcl} A \subseteq f^{-1}(\text{Cl} f(A))$. And so, $f(\text{Fcl} A) \subseteq \text{Cl} f(A)$.

Conversely, suppose that $(d)$ holds for any subset $A$ of $X$. Let $K$ be a closed set in $Y$. Then, $f(\text{Fcl} f^{-1}(K)) \subseteq \text{Cl} f(f^{-1}(K)) \subseteq \text{Cl} K = K$. Therefore, $\text{Fcl} f^{-1}(K) = f^{-1}(K)$. This implies that $f^{-1}(K) = \text{Fcl} f^{-1}(K)$. Hence $f^{-1}(K)$ is feebly closed in $X$ by Lemma (1.1.A).

\[ (d) \iff (e). \]

Suppose that $(d)$ holds and $B$ is a subset of $Y$. Now, $f(\text{Fcl} f^{-1}(B)) \subseteq \text{Cl} f(f^{-1}(B)) \subseteq \text{Cl} B$. That is, $\text{Fcl} f^{-1}(B) \subseteq f^{-1}(\text{Cl} B)$. Thus $(e)$ holds.

Conversely, suppose that $(e)$ holds and let $B = f(A)$ where $A$ is a subset of $X$. Using Lemma (1.1.B) and the hypothesis we get, $\text{Fcl} A \subseteq \text{Fcl} f^{-1}(B) \subseteq f^{-1}(\text{Cl} B) = f^{-1}(\text{Cl} f(A))$. Therefore, $f(\text{Fcl} A) \subseteq \text{Cl} f(A)$. Hence, $(d)$ holds.//

The concept of a pre semi open mapping is due to Crossley and Hildebrandt [18]. It is defined as follows:

**DEFINITION 1.1.D.** A mapping $f : X \longrightarrow Y$
is termed pre semi open if the image under $f$ of each semi open set is semi open.

**Remark 1.1.1.** In 1963, Levine [59] showed that every open continuous mapping is pre semi open. In 1969, Biswas [5] proved that every semi open continuous mapping is pre semi open (a mapping under which the image of every open set is semi open is called semi open [5]). In 1977, Neubrunn [85a] showed that every some what open continuous mapping is pre semi open (a mapping $f : X \rightarrow Y$ is said to be some what open [33a] if for each nonempty open subset $U$ of $X$, there exists an open set $V$ of $Y$ such that $\emptyset \neq V \subseteq f(U)$). In 1979, Noiri [97] showed that every weakly continuous some what open mapping is pre semi open (a mapping $f : X \rightarrow Y$ is said to be weakly continuous [58], if for each $x \in X$ and each open neighbourhood $V$ of $f(x)$, there exists an open neighbourhood $U$ of $x$ such that $f(U) \subseteq \text{Cl} V$). He also proved that every almost continuous (Singal's sense) semi open mapping is pre semi open.

**Remark 1.1.3.** Inverse image of a feebly open set under a feebly continuous mapping need not be feebly open. Consider the following example:
EXAMPLE 1.1.3. Let \( X = \{a, b, c\} \),
\[ T = \{\emptyset, \{a\}, \{b, c\}, X\} \] and \( Y = \{x, y, z\} \),
\[ T^* = \{\emptyset, \{x\}, Y\} \). Define \( f : X \rightarrow Y \) by \( f(a) = x \),
\( f(b) = y \) and \( f(c) = z \). Then \( f \) is feebly continuous.
It is clear that \( B = \{x, z\} \) is feebly open in \( Y \) but
\( f^{-1}(B) \) is not feebly open in \( X \).

We have,

THEOREM 1.1.2. If \( f : X \rightarrow Y \) is a pre
semi open and feebly continuous mapping, then \( f^{-1}(B) \) is
feebly open in \( X \) for every feebly open set \( B \) in \( Y \).

In the proof of this theorem we shall make use
of the following lemmas:

LEMMA 1.1.C. \( x \in \text{Scl} A \) if and only if each
semi open set containing \( x \) meets \( A \). [19]

LEMMA 1.1.D. If \( A \) is feebly open in a
space \( X \) and \( A \subset B \subset \text{Scl} A \), then \( B \) is feebly open in
\( X \). [131].

PROOF OF THEOREM 1.1.2. Let \( f : X \rightarrow Y \) be
a pre semi open feebly continuous mapping and let \( B \) is a
feebly open set in \( Y \). There exists an open set \( V \) in \( Y \) such that \( V \subset B \subset \text{Scl} \ V \). We obtain that,
\[
f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(\text{Scl} \ V).
\]
Since \( f \) is feebly continuous and \( V \) is open in \( Y \), by Definition (1.1.1),
it follows that \( f^{-1}(V) \) is feebly open in \( X \). Now let \( x \in f^{-1}(\text{Scl} \ V) \). Then, \( f(x) \in \text{Scl} \ V \). Let \( U \) be any semi open set in \( X \) which contains the point \( x \). Then \( f(x) \in f(U) \) which is semi open in \( Y \) because \( f \) is pre semi open. Since \( f(x) \in \text{Scl} \ V \), it results by
Lemma (1.1.C) that \( f(U) \cap V \neq \emptyset \). Let \( f(a) \in f(U) \cap V \).
This implies that \( U \cap f^{-1}(V) \neq \emptyset \). This implies again in virtue of Lemma (1.1.C), that \( x \in \text{Scl} \ f^{-1}(V) \). And so,
\[
f^{-1}(\text{Scl} \ V) \subset \text{Scl} \ f^{-1}(V).
\]
Thus, we obtain that
\[
f^{-1}(V) \subset f^{-1}(B) \subset \text{Scl} \ f^{-1}(V).
\]
Consequently, by Lemma (1.1.D),
\( f^{-1}(B) \) is feebly open in \( X \). //

**REMARK 1.1.4.** Composition of two feebly continuous mappings may fail to be feebly continuous. For,

**EXAMPLE 1.1.4.** Let \( f : X \rightarrow Y \) be a mapping of Example (1.1.3). Let \( Z = \{ u, v, w \} \),
\[
T^{**} = \{ \emptyset, \{ u \}, Z \} \quad \text{and} \quad g : Y \rightarrow Z \text{ defined by}
\]
\( f(x) = f(y) = u \) and \( f(z) = w \). Then \( f \) is feebly continuous (infact, continuous), \( g \) is feebly continuous
but \( \text{gof} \) is not feebly continuous.

However, we obtain the following result as an immediate consequence of Theorem (1.1.2):

**Theorem 1.1.3.** If \( f : X \to Y \) is a pre semi open feebly continuous mapping and \( g : Y \to Z \) is a feebly continuous mapping, then \( \text{gof} \) is feebly continuous.

**Remark 1.1.5.** Restriction of a feebly continuous mapping may not be feebly continuous. For, the mapping \( f : X \to Y \) defined in Example (1.1.1) is feebly continuous. Consider \( A = \{ a, \ d \} \) as a set in \( X \). Then the restriction \( f \mid A : A \to Y \) is not feebly continuous. However, we have the following theorem:

**Theorem 1.1.4.** If \( f : X \to Y \) is a feebly continuous mapping and \( A \) is an open subset of \( X \), then the restriction \( f \mid A : A \to Y \) is feebly continuous.

The following result will be useful:

**Lemma 1.1.5.** If \( U \) is open in \( X \) and \( B \) feebly open in \( X \), then \( U \cap B \) is feebly open in \( U \). [131].
PROOF OF THEOREM 1.1.4. Since \( f \) is feebly continuous, for any open set \( V \) in \( Y \), \( f^{-1}(V) \) is feebly open in \( X \) by Definition (1.1.1.). Hence, by Lemma (1.1.3.), \( f^{-1}(V) \cap A \) is feebly open in \( A \). Since, \( (f \mid A)^{-1}(V) = f^{-1}(V) \cap A \), it follows that \( f \mid A \) is feebly continuous. //

The following concept is due to Tapi [131].

DEFINITION 1.1.3. A topological space \( X \) is feebly \( T_2 \) if for any pair of distinct points \( x, y \) of \( X \) there exist disjoint feebly open sets \( U \) and \( V \) such that \( x \in U, y \in V \).

Tapi [131] has shown that if a mapping \( f : X \rightarrow Y \) is one-one, open and continuous then \( X \) is feebly \( T_2 \) if \( Y \) is feebly \( T_2 \). We prove that,

THEOREM 1.1.5. Let a mapping \( f : X \rightarrow Y \) be injective, pre semi open and feebly continuous. If \( Y \) is feebly \( T_2 \) then \( X \) is feebly \( T_2 \).

PROOF Let \( x, y \in X \) and \( x \neq y \). Then \( f \) being injective, \( f(x) \neq f(y) \). Since \( Y \) is feebly \( T_2 \), let \( U, V \) be disjoint feebly open sets in \( Y \) such that
f(x) \in U, f(y) \in V. By Theorem (1.1.2), it follows that \( f^{-1}(U), f^{-1}(V) \) are feebly open in \( X \) because by hypothesis \( f \) is pre-semi open and feebly continuous. It is clear that \( x \in f^{-1}(U), y \in f^{-1}(V) \), and that \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Hence, \( X \) is feebly \( T_2 \).

THEOREM 1.1.6. Let a mapping \( f : X \to Y \) be injective feebly continuous. If \( Y \) is a \( T_2 \) space then \( X \) is feebly \( T_2 \).

The proof is similar to that of Theorem (1.1.5).

THEOREM 1.1.7. Let a mapping \( f : X \to X \) is feebly continuous. If \( X \) is \( T_2 \) then the set \( A = \{ x \mid f(x) = x \} \) is feebly closed.

We shall need the following lemmas:

**Lemma 1.1.F.** In a topological space, the intersection of an open set and a feebly open set is feebly open. [131].

**Lemma 1.1.G.** Each feebly open set containing \( p \) meets \( A \) if and only if \( p \in \text{Fcl} A \). [131].
PROOF OF THEOREM 1.1.7. Let \( a \in \text{Fcl}(A) \).

Suppose \( a \notin A \). Then, \( f(a) \neq a \). Since \( X \) is \( T_2 \), there exist disjoint open sets \( U \) and \( V \) in \( X \) such that \( f(a) \in U \) and \( a \in V \). Now \( f \) being feebly continuous, \( f^{-1}(U) \) is feebly open and it contains \( a \). By Lemma (1.1.F), it results that \( f^{-1}(U) \cap V \) is feebly open and it contains \( a \). Since \( a \in \text{Fcl}(A) \), by Lemma (1.1.G) \( f^{-1}(U) \cap V \cap A \neq \emptyset \). This leads to a contradiction that \( U \cap V \neq \emptyset \). And so, it follows that \( a \notin A \). Therefore, \( \text{Fcl}(A) = A \). Consequently, by Lemma (1.1.A) \( A \) is feebly closed. //

The definition below now investigates a mapping which is strictly stronger than feebly continuity but independent of continuity.

DEFINITION 1.1.2. A mapping \( f : X \rightarrow Y \) is termed strongly feebly continuous if the inverse image of every feebly open subset of \( Y \) is feebly open in \( X \).

REMARK 1.1.6. Since every open set is feebly open it follow from Definition (1.1.1) and (1.1.2) that every strongly feebly continuous mapping is feebly continuous. The converse may be false. For,
EXAMPLE 1.1.5. Let $X, T$ be as in Example (1.1.2) and $X^*=\{x, y, z\}$, $T^*=\emptyset, \{x\}, X^*$. Define $f: X \rightarrow X^*$ by $f(a) = x, f(b) = y, f(c) = z$. Then $f$ is feebly continuous (in fact, continuous) but it is not strongly feebly continuous.

Consider, the following example:

EXAMPLE 1.1.6. Let $X = \{a, b, c\}, T = \emptyset, \{a\}, X$ and $Y, T^*$ be as in Example (1.1.5). Define $f: X \rightarrow X^*$ by $f(a) = f(b) = x$ and $f(c) = z$. Then $f$ is strongly feebly continuous but it is not continuous.

THEOREM 1.1.8. Every pre semi open feebly continuous mapping is strongly feebly continuous.

PROOF. Follows from Theorem (1.1.2) and Definition (1.1.2).//

REMARK 1.1.7. The converse of Theorem (1.1.8) may be false. For,

EXAMPLE 1.1.7. Let $X, T$ be as in Example...
(1.1.3) and \( Y = \{ x, y, z \} \), \( T^* = \{ \emptyset, \{ x \}, \{ y \}, \{ x, y \}, Y \} \). Define \( f : X \longrightarrow Y \) by \( f(a) = z \), \( f(b) = f(c) = x \). Then \( f \) is strongly feebly continuous but it is not pre semi open feebly continuous.

The following is a generalization of Theorem (1.1.5):

**Theorem 1.1.9.** Let a mapping \( f : X \longrightarrow Y \) be injective and strongly feebly continuous. If \( Y \) is feebly \( T_2 \) then \( X \) is feebly \( T_2 \).

**Proof.** Similar to Theorem (1.1.5) and utilizes Definition (1.1.2).

**Theorem 1.1.10.** Composition of two strongly feebly continuous mapping is strongly feebly continuous.

The proof is straightforward application of Definition (1.1.2).

Theorem (1.1.3) gets generalized as follows:

**Theorem 1.1.11.** If a mapping \( f : X \longrightarrow Y \)
is strongly feebly continuous and a mapping \( g : Y \to Z \) is feebly continuous, then \( g \circ f \) is feebly continuous.

**Proof**: Follows from Definition (1.1.1) and (1.1.2).

**Remark 1.1.8.** Restriction of a strongly feebly continuous may not be strongly feebly continuous. For, the mapping \( f \) defined in Example (1.1.1) is strongly feebly continuous. But if we let \( A = \{a, d\} \) then \( f \mid A : A \to Y \) is not strongly feebly continuous. However we have the following theorem:

**Theorem 1.1.12.** Let a mapping \( f : X \to Y \) be strongly feebly continuous. If \( A \) be open in \( X \) then \( f \mid A : A \to Y \) is strongly feebly continuous.

The proof is similar to Theorem (1.1.4).

The concept of an irresolute mapping is due to Crossley and Hildebrandt [18] is as follows:

**Definition 1.1.8.** A mapping \( f : X \to Y \) is termed irresolute if the inverse image under \( f \) of each semi open set is semi open.
REMARK 1.1.8. Every irresolute mapping is semi continuous where as the concepts of irresolute and continuous are independent.[18].

We now proceed to introduce a concept which strictly lies between the notion of irresolute and semi continuous, and at the same time independent to continuity.

DEFINITION 1.1.3. A mapping $f : X \rightarrow Y$ is strongly semi continuous if the inverse image of every feebly open subset of $Y$ is semi open in $X$.

REMARK 1.1.9. Since every feebly open set is semi open, it follows from Definition (1.1.2) and (1.1.3) that every strongly feebly continuous mapping is strongly semi continuous. The converse may be false. For, the mapping $f$ in Example (1.1.5) is strongly semi continuous but it is not strongly feebly continuous.

REMARK 1.1.10. Since every open set is feebly open it results from Definition (1.1.A) and (1.1.3) that every strongly semi continuous mapping is semi continuous. The converse need not be true. For,
EXAMPLE 1.1.8. Let \( X = \{a, b, c, d\} \), \( T = \{\emptyset, \{a\}, \{b, c, d\}, X\} \) and \( Y, T^* \) be as in Example (1.1.5). Define \( f : X \rightarrow Y \) by \( f(a) = x, f(b) = y \) and \( f(c) = f(d) = z \). Then \( f \) is semi continuous (in fact, continuous) but it is not strongly semi continuous.

However, we have the following theorem:

THEOREM 1.1.13. Every pre semi open semi continuous mapping is strongly semi continuous.

We shall need the following result:

LEMMA 1.1.14. If \( A \) is semi open in \( X \) and \( A \subseteq B \subseteq \text{Cl} A \), then \( B \) is semi open in \( X \). [59].

PROOF OF THEOREM 1.1.13. Let a mapping \( f : X \rightarrow Y \) be pre semi open and semi continuous. Let \( V \) be a feebly open set in \( Y \). Then there exists an open set \( U \) in \( Y \) such that \( U \subseteq V \subseteq \text{Scl} U \). Therefore, \( f^{-1}(U) \subseteq f^{-1}(V) \subseteq f^{-1}(\text{Scl} U) \). Since \( f \) is semi continuous, by Definition (1.1.1) \( f^{-1}(U) \) is semi open. As in the proof of Theorem (1.1.2) we obtain that
\[ f^{-1}(\text{Scl } U) \subseteq \text{Scl } f^{-1}(U) \quad \text{because } f \text{ is pre semi open.} \]

And so, \( f^{-1}(U) \subseteq f^{-1}(V) \subseteq \text{Scl } f^{-1}(U) \subseteq \text{Cl } f^{-1}(U) \).

Therefore by Lemma (1.1.4), \( f^{-1}(V) \) is semi open in \( X \).

Hence, by Definition (1.1.3), \( f \) is strongly semi continuous. //

**REMARK 1.1.11.** The mapping \( f : X \rightarrow Y \) considered in Example (1.1.6) is strongly semi continuous but it is not continuous.

The following example show that the concept of strongly semi continuous may fail to be feebly continuous:

**EXAMPLE 1.1.9.** Let \( X, T, Y \) and \( T^* \) be as in Example (1.1.5). Define \( f : X \rightarrow Y \) by \( f(a) = f(c) = x \) and \( f(b) = z \). Then \( f \) is strongly semi continuous but it is not feebly continuous.

**REMARK 1.1.12.** The mapping defined in Example (1.1.3.) is feebly continuous but it is not strongly semi continuous.

**DEFINITION 1.1.6.** A topological space \( X \) is semi \( T_2 \) if for any two distinct points \( x, y \) of \( X \) there exist disjoint semi open sets \( U \) and \( V \) such that
\( x \in U \text{ and } y \in V. \) [65].

**Remark 1.1.3.** Every feebly \( T_2 \) space is semi \( T_2 \) but not conversely. [131].

**Theorem 1.1.14.** Let a mapping
\[ f : X \rightarrow Y \]
be injective and strongly semi continuous. If \( Y \) is feebly \( T_2 \) then \( X \) is semi \( T_2 \).

The proof is analogous to that of Theorem (1.1.9).

**Remark 1.1.13.** Composition of two strongly semi continuous mappings need not be strongly semi continuous. For,

**Example 1.1.10.** Let \( X = \{ x, y, z \} \), \( T = \{ \emptyset, \{ y \}, X \} \), \( Y = \{ a, b, c \} \), \( T^* = \{ \emptyset, \{ a \}, \{ b \}, \{ a, b \}, Y \} \) and \( Z = \{ a, b \} \), \( T^{**} = \{ \emptyset, \{ a \}, Z \} \). Define \( f : X \rightarrow Y \) by \( f(x) = f(y) = b, \ f(z) = c \) and \( g : Y \rightarrow Z \) by \( f(a) = f(c) = a, \ f(b) = b \). Then \( f \) and \( g \) are strongly semi continuous mappings but \( g \circ f \) is not strongly semi continuous.

However, we have,
**THEOREM 1.1.15.** If a mapping
\[ f : X \to Y \] is irresolute and \( g : Y \to Z \) is strongly semi continuous then \( g \circ f \) is strongly semi continuous.

**PROOF**
Results from Definitions (1.1.3) and (1.1.F).

**THEOREM 1.1.16.** If a mapping
\[ f : X \to Y \] is strongly semi continuous and \( g : Y \to Z \) is feebly continuous then \( g \circ f \) is semi continuous.

**PROOF**
Follows from Definitions (1.1.1), (1.1.3) and (1.1.A).

**REMARK 1.1.14.** Since every feebly open set is semi open it follows that every irresolute mapping is strongly semi continuous. However the converse may be false. For,

**EXAMPLE 1.1.11.** Let \( X = \{ a, b \} \),
\[ T = \{ \emptyset, \{ a \}, X \} \] and \( Y = \{ x, y, z \} \), \( \mathcal{T}^* = \{ \emptyset, \{ x \}, y \}, \{ x, y \}, Y \} \). Define \( f : X \to Y \) by \( f(a) = x, f(b) = z \). Then the mapping \( f \) is strongly feebly continuous and consequently it is feebly continuous and strongly semi continuous but it is not irresolute.
REMARK 1.1.15. Let \( X, Y \) and \( T, T' \) be as in Example (1.1.1). Define \( f : X \to Y \) by \( f(a) = f(b) = f(d) = a \) and \( f(c) = b \). Then \( f \) is irresolute but it is not feebly continuous and hence it is not strongly feebly continuous.

REMARK 1.1.16. Restriction of a strongly semi continuous mapping need not be strongly semi continuous. For, the mapping defined in Example (1.1.1) is strongly semi continuous. If \( A = \{ a, d \} \), then \( f \restriction A \) is not strongly semi continuous. However,

THEOREM 1.1.17. Restriction of a strongly semi continuous mapping over an open set is strongly semi continuous.

The proof is similar to Theorem (1.1.12) and utilizes the following result.

LEMMA 1.1.1. If \( U \) is open in a space \( X \) and \( V \) is semi open in \( X \) then \( U \cap V \) is semi open in \( U \). [65].
1.2 ALMOST FEEBLY CONTINUOUS FUNCTIONS

In this section we generalize the concept of feebly continuous mappings as follows:

**Definition 1.2.1.** A mapping \( f : X \to Y \) is termed almost feebly continuous if the inverse image of every regular open subset of \( Y \) is feebly open in \( X \).

**Remark 1.2.1.** Since every regular open set is open, it follows from Definitions (1.1.1) and (1.2.1) that every feebly continuous mapping is almost feebly continuous. However, the converse may be false. For, the mapping \( f : X \to Y \) considered in Example (1.1.2) is almost feebly continuous but it is not feebly continuous.

The following theorem gives a characterization of almost feebly continuity:

**Theorem 1.2.1.** Let \( f : X \to Y \). Then the following conditions are equivalent:

(a) \( f \) is almost feebly continuous.

(b) For each \( p \in X \) and each regular open set \( O \) in \( Y \) such that \( f(p) \in O \), there exists...
a feebly open set $A$ in $X$ such that $p \in A$ and $f(A) \subseteq O$.

**Proof** \( (a) \iff (b) \). Let $O$ be regular open in $Y$ and $f(p) \in O$. Then $p \in f^{-1}(O)$ and $f^{-1}(O)$ is feebly open. Let $A = f^{-1}(O)$. Thus, $p \in A$ and $f(A) \subseteq O$.

\( (b) \iff (a) \). Let $O$ be regular open in $Y$ and let $p \in f^{-1}(O)$. Then $f(p) \in O$. By (b), there is a feebly open set $A_p$ in $X$ such that $p \in A_p$ and $f(A_p) \subseteq O$. And so, $p \in A_p \subseteq f^{-1}(O)$. Therefore $f^{-1}(O)$ is a union of feebly open sets in $X$. Hence, $f^{-1}(O)$ is feebly open in $X$ for any union of feebly open sets is feebly open. Thus, (a) holds. //

The following definition is due to Stone [127].

**Definition 1.2.1.** A space $X$ is semi regular if for each point $x$ of $X$ and each open set $U$ containing $x$, there is an open set $V$ such that $x \in V \subseteq \text{Int} \text{ Cl} \ V \subseteq U$.

We have,
THEOREM 1.2.2. An almost feebly continuous mapping $f : X \rightarrow Y$ is feebly continuous if $Y$ is semi regular.

PROOF Let $x \in X$ and let $A$ be an open set containing $f(x)$. Since $Y$ is semi regular the family of regular open sets in $Y$ forms a base for the topology of $Y$. So there is an open set $M$ in $Y$ such that $f(x) \in M \subseteq \text{Int} \ Cl M \subseteq A$. Since $\text{Int} \ Cl M$ is regular open in $Y$ and $f$ is almost feebly continuous, by Theorem (1.1.1) there is a feebly open set $U$ in $X$ containing $x$ such that $f(x) \in f(U) \subseteq \text{Int} \ Cl M$. Thus $U$ is a feebly open set containing $x$ such that $f(U) \subseteq A$. Hence by Theorem (1.1.1(b)), $f$ is feebly continuous.//

The following theorem generalizes Theorem (1.1.6):

THEOREM 1.2.3. Let $f : X \rightarrow Y$ be injective and almost feebly continuous. If $Y$ is a $T_2$ space then $X$ is feebly $T_2$.

PROOF Let $x$ and $y$ be any two distinct point of $X$. Since $f$ is injective $f(x) \neq f(y)$. Now,
Y being a $T_2$ space there exist two disjoint open sets $U$ and $V$ such that $f(x) \in U$, $f(y) \in V$. Since $U$ and $V$ are disjoint open, we have $U \cap \text{Cl } V = \emptyset$ and hence $U \cap \text{Int Cl } V = \emptyset$. Similarly, we obtain, $\text{Int Cl } U \cap \text{Int Cl } V = \emptyset$. Evidently, $f(x) \in \text{Int Cl } U$ and $f(y) \in \text{Int Cl } V$. Since $\text{Int Cl } U$ and $\text{Int Cl } V$ are regular open sets in $Y$ and $f$ is almost feebly continuous it follows that $f^{-1}(\text{Int Cl } U)$ and $f^{-1}(\text{Int Cl } V)$ are disjoint feebly open sets containing $x$ and $y$ respectively. Hence, $X$ is feebly $T_2$.//

**REMARK 1.2.2.** Restriction of an almost feebly continuous function may not be almost feebly continuous. For, let us consider the space $(X, T)$ of Example (1.1.6) and the space $(Y, T^*)$ of Example (1.1.7) and the mapping $f : X \rightarrow Y$ as in Example (1.1.6). Then $f$ is almost feebly continuous (infact, feebly continuous). Now if $A = \{ b, c \}$ then it is easy to observe that the mapping $f|A : A \rightarrow Y$ is not almost feebly continuous.

However, we have the following theorem:

**THEOREM 1.2.4.** If $f : X \rightarrow Y$ is almost
feeably continuous and $A$ is open in $X$ then the restriction $f | A : A \to Y$ is almost feeably continuous.

**Proof** Since $f$ is almost feeably continuous, for any regular open set $V$ in $Y$, $f^{-1}(V)$ is feeably open in $X$. Hence by Lemma (1.1.E), $f^{-1}(V) \cap A$ is feeably open in $A$, because $A$ is open.

Since $(f | A)^{-1}(V) = f^{-1}(V) \cap A$, therefore it follows that $f | A$ is almost feeably continuous.\/

**Remark 1.2.3.** Composition of two almost feeably continuous mappings may not be almost feeably continuous. For,

**Example 1.2.1.** Let $(X, T)$ be the space of Example (1.1.3), $Y = \{a, b, c, d\}$,

$T^* = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\} ; \ Z = \{u, v, w\}$,

$T^{**} = \{\emptyset, \{u\}, \{v\}, \{u, v\}, Z\}$.

Define $f : X \to Y$ by $f(a) = a, f(b) = c, f(c) = d$ and $g : Y \to Z$ by $g(a) = g(b) = g(c) = u, g(d) = w$. Then, $f$ and $g$ are almost feeably continuous mappings but $gof$ is not almost feeably continuous because $u$ is regular open in $Z$ but $(gof)^{-1}\{u\} = \{a, b\}$ is not feeably open in $X$. 

However, we have,

**THEOREM 1.2.5.** If a mapping

\[ f : X \to Y \]

is strongly feebly continuous and

\[ g : Y \to Z \]

is almost feebly continuous, then \( g \circ f \) is almost feebly continuous.

**PROOF**

Follows from Definitions (1.1.2) and (1.2.1).

Singal and Singal [125] introduced the following concept:

**DEFINITION 1.2.B.** A mapping

\[ f : X \to Y \]

is almost continuous if the inverse image of every regular open subset of \( Y \) is open in \( X \).

**REMARK 1.2.A.** Every continuous mapping is almost continuous but the converse may be false. [125].

Recently, Maheshwari and Thakur [74a] has introduced the following concept:

**DEFINITION 1.2.C.** A mapping \( f : X \to Y \)
is termed \( \beta \)-continuous if the inverse image of every regular open (resp. regular closed) set in \( Y \) is regular open (resp. regular closed) in \( X \).

**Remark 1.2.4.** The concepts of continuity and \( \beta \)-continuity are independent. [74a].

**Remark 1.2.4.** Since every regular open set is open it is clear that \( \beta \)-continuity implies almost continuity. That the implication is not reversible in general is shown by the following example. For,

**Example 1.2.2.** Let \((X, T)\) be the space of Example (1.1.2). Define \( f : X \to X \) by \( f(a) = f(b) = a, f(c) = c \). Then \( f \) is almost continuous but it is not \( \beta \)-continuous.

**Remark 1.2.5.** The mapping considered in Remark (1.1.15) is almost continuous but it is not feebly continuous and hence it is not strongly feebly continuous.

**Remark 1.2.6.** A strongly feebly continuous mapping may fail to be almost continuous. For, let \((X, T)\) be the space of Example (1.1.6), \((Y, T^*)\) be the space
of Example (1.1.7) and consider the mapping \( f \) defined in Example (1.1.6). Then \( f \) is strongly feebly continuous but it is not almost continuous.

**Theorem 1.2.6.** Let \( f : X \to Y \), \( g : Y \to Z \). If \( g \) is almost continuous and \( f \) is feebly continuous then \( g \circ f \) is almost feebly continuous.

**Proof.** Let \( U \) be regular open in \( Z \). Then \( g^{-1}(U) \) is open in \( Y \) because \( g \) is almost continuous. Therefore, \( f^{-1}(g^{-1}(U)) \) is feebly open in \( X \) because \( f \) is feebly continuous. Since \( (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \), it follows that \( g \circ f \) is almost feebly continuous. //

Similarly, we have,

**Theorem 1.2.7.** Let \( f : X \to Y \) and \( g : Y \to Z \). If \( g \) is \( \beta \)-continuous and \( f \) is almost feebly continuous then \( g \circ f \) is almost feebly continuous.

**Proof.** Follows from Definitions (1.2.1) and (1.2.1). //
1.3 ALMOST SEMI CONTINUOUS FUNCTIONS

This section aims to obtain a generalization of semi continuity and generalizes some results of Noiri [88] and Maheshwari and Prasad [65]. The new concept is introduced as follows:

**DEFINITION 1.3.1.** A mapping \( f : X \to Y \)
is termed almost semi continuous if the inverse image of every regular open set of \( Y \) is semi open in \( X \).

**REMARK 1.3.1.** Since every regular open set is open it follows from Definitions (1.3.1) and (1.1.A) that every semi continuous mapping is almost semi continuous. However, the converse may be false. For,

**EXAMPLE 1.3.1.** Let \( X = \{ a, b, c, d \} \), \( P = \{ \emptyset, \{ a \}, \{ b \}, \{ a, b \}, X \} \) and let \( (Y, T^*) \) be the space of Example (1.1.3). Define \( f : X \to Y \) by \( f(c) = f(d) = x \), \( f(b) = z \) and \( f(a) = y \). Then \( f \) is almost semi continuous but it is not semi continuous. We observe that \( f \) is almost continuous also.

**REMARK 1.3.2.** Every almost continuous
mapping is almost semi continuous. However, the converse is not necessarily true. For,

**EXAMPLE 1.3.2.** Let $(X, T)$ be the space of Example (1.1.2) and let $Y = \{a, b\}$, $T^* = \{\emptyset, \{a\}, \{b\}, Y\}$. Consider $f : X \rightarrow Y$ as defined in Example (1.1.2). Then $f$ is strongly semi continuous (hence semi continuous and a fortiori almost semi continuous) but it is not almost continuous.

**REMARK 1.3.3.** Since every feebly open set is semi open, it results from Definitions (1.2.1) and (1.3.1), that every almost feebly continuous mapping is almost semi continuous. However, the converse may be false. For, the mapping $g : Y \rightarrow Z$ defined in Example (1.3.3) onwards, is almost semi continuous but it is not almost feebly continuous.

The following theorem obtains a characterization of almost semi continuous mapping:

**THEOREM 1.3.1.** Let $f : X \rightarrow Y$. Then the following conditions are equivalent:

(a) $f$ is almost semi continuous.
(b) For each \( p \in X \) and each regular open set \( O \) in \( Y \) such that \( f(p) \in O \), there exists a semi open set \( A \) in \( X \) such that \( p \in A \) and \( f(A) \subseteq O \).

The proof is analogous to that of Theorem (1.2.1).

The following theorem investigates a situation for almost semi continuity to imply semi continuity.

**Theorem 1.3.2.** An almost semi continuous mapping \( f : X \rightarrow Y \) is semi continuous if \( Y \) is semi regular.

We require the following lemma:

**Lemma 1.3.A.** A mapping \( f : X \rightarrow Y \) is semi continuous if and only if for any point \( x \in X \) and any open set \( V \) of \( Y \) containing \( f(x) \), there exists a semi open set \( U \) in \( X \) such that \( x \in U \) and \( f(U) \subseteq V \). [59].
PROOF OF THEOREM 1.3.2. Let \( x \in X \) and let \( A \) be an open set containing \( f(x) \). Since \( Y \) is semi regular, there is an open set \( M \) in \( Y \) such that \( f(x) \in M \subseteq \text{Int Cl} M \subseteq A \). Since \( \text{Int Cl} M \) is regular open in \( Y \) and \( f \) is almost semi continuous, by Theorem (1.3.1), there is a semi open set \( U \) in \( X \) containing \( x \) such that \( f(x) \in f(U) \subseteq \text{Int Cl} M \). Thus \( U \) is a semi open set containing \( x \) such that \( f(U) \subseteq A \). Hence \( f \) is semi continuous by Lemma (1.3.A).

THEOREM 1.3.3. Let \( f : X \rightarrow Y \) be injective and almost semi continuous. If \( Y \) is a \( T_2 \) space then \( X \) is semi \( T_2 \).

PROOF. Let \( x \) and \( y \) be any two distinct points of \( X \). Since \( f \) is injective, \( f(x) \neq f(y) \).

Now, \( Y \) being a \( T_2 \) space, there exist two disjoint open sets \( U \) and \( V \) such that \( f(x) \in U \), \( f(y) \in V \).

Since \( U \) and \( V \) are disjoint open, we have,
\[ U \cap \text{Cl} V = \emptyset \] and hence \( U \cap \text{Int Cl} V = \emptyset \).

Similarly, we obtain, \( \text{Int Cl} U \cap \text{Int Cl} V = \emptyset \).

Evidently, \( f(x) \in \text{Int Cl} U \) and \( f(y) \in \text{Int Cl} V \).

Since \( \text{Int Cl} U \) and \( \text{Int Cl} V \) are regular open sets and \( f \) is almost semi continuous it follows that
\( f^{-1}(\text{Int Cl } U) \) and \( f^{-1}(\text{Int Cl } V) \) are disjoint semi open sets containing \( x \) and \( y \) respectively. Hence, \( X \) is semi \( T_2 \). //

**COROLLARY 1.3.A.** Let \( f: X \to Y \) be injective and semi continuous. If \( Y \) is a \( T_2 \) space then \( X \) is semi \( T_2 \) [65, Theorem (5.6)].

**THEOREM 1.3.4.** If \( f: X \to Y \) is almost semi continuous and \( Y \) is a \( T_2 \) space then the set \( G(f) = \{ (x, f(x)) \mid x \in X \} \) is semi closed in \( X \times Y \).

We shall need the following lemma:

**LEMMA 1.3.B.** If \( A \) is semi open in a space \( X \) and \( B \) is semi open in a space \( Y \) then \( A \times B \) is semi open in the space \( X \times Y \). [59].

**PROOF OF THEOREM 1.3.4.** Let \( (x, y) \in X \times Y = G(f) \). Then \( y \neq f(x) \). Since \( Y \) is a \( T_2 \) space, there exist disjoint open sets \( W \) and \( V \) in \( Y \) such that \( f(x) \in W \), \( y \in V \). As before, \( \text{Int Cl } W \cap \text{Int Cl } V = \emptyset \). Since \( \text{Int Cl } W \) is regular open and \( f \) is almost semi continuous, by Theorem (1.3.1),
there exists a semi open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq \text{Int Cl } W$. Therefore, we obtain that $(x, y) \in U \times V \subseteq X \times Y - G(f)$. Since every open set is semi open, $V$ is semi open in $Y$. Therefore, $U \times V$ is semi open in $X \times Y$ by Lemma (1.3.B). This implies that $X \times Y - G(f)$ is a union of semi open sets in $X \times Y$. Therefore, $X \times Y - G(f)$ is semi open in $X \times Y$. Hence, $G(f)$ is semi closed in $X \times Y$.//

**COROLLARY 1.3.B.** If $f : X \rightarrow Y$ is semi continuous and $Y$ is a $T_2$ space, then $G(f)$ is semi closed in $X \times Y$ [88, Theorem 3].

**THEOREM 1.3.5.** If $f : X \rightarrow Y$ is almost semi continuous and $Y$ is a $T_2$ space then the set $A = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is semi closed in $X \times X$.

**PROOF** If $(x_1, x_2) \notin A$ then $f(x_1) \neq f(x_2)$. Hence, there exist disjoint open neighbourhoods $V_1$ and $V_2$ of $f(x_1)$ and $f(x_2)$ respectively in $Y$. Since $V_1$, $V_2$ are disjoint and open we obtain

$\text{Int Cl } V_1 \cap \text{Int Cl } V_2 = \emptyset$. Since $f$ is almost semi continuous, $f^{-1}(\text{Int Cl } V_1)$ is a semi open set in $X$. 
containing $x_i$ for $i = 1, 2$. By Lemma (1.3.B), $f^{-1}(\text{Int Cl} \ V_1) \times f^{-1}(\text{Int Cl} \ V_2)$ is a semi open set in $X \times X$ containing $(x_1, x_2)$ and obviously it does not intersect $A$. Thus, $(X \times X) - A$ is a union of semi open sets in $X \times X$ and so it is semi open. Hence, $A$ is semi closed in $X \times X$. //

**COROLLARY 1.3.C.** If $f: X \rightarrow Y$ is semi continuous and $Y$ is a $T_2$ space, then the set \[(x_1, x_2) \mid f(x_1) = f(x_2)\] is semi closed in $X \times X$ [88, Theorem 4].

**REMARK 1.3.4.** Composition of two almost semi continuous mappings may fail to be almost semi continuous. For,

**EXAMPLE 1.3.3.** Let $X = \{ a, b, c, d \}$, $T = \{ \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c, d\}, X \};$ $Y = \{ a, b, c \}$, $T^* = \{ \emptyset, \{a\}, \{c\}, \{a, c\}, Y \};$ $Z = \{ a, b \}$, $T^{**} = \{ \emptyset, \{a\}, \{b\}, Z \}$. Define $f: X \rightarrow Y$ by $f(a) = a$, $f(b) = b$, $f(c) = f(d) = c$; $g: Y \rightarrow Z$ by $g(a) = g(b) = a$, $g(c) = b$. Then the mappings $f$ and $g$ are almost semi continuous but $gof$ is not almost semi continuous.
We have,

**THEOREM 1.3.6.** If \( f : X \rightarrow Y \) is irresolute and \( g : Y \rightarrow Z \) is almost semi continuous then \( g \circ f \) is almost semi continuous.

**PROOF** Follows from Definitions (1.1.1) and (1.3.1).

**THEOREM 1.3.7.** If \( f : X \rightarrow Y \) is semi continuous and \( g : Y \rightarrow Z \) is almost continuous then \( g \circ f \) is almost semi continuous.

**PROOF** Follows from Definitions (1.2.8) and (1.1.4).

**THEOREM 1.3.8.** If \( f : X \rightarrow Y \) is strongly semi continuous and \( g : Y \rightarrow Z \) is almost feebly continuous, then \( g \circ f \) is almost semi continuous.

**PROOF** Follows from Definitions (1.1.3) and (1.2.1).

**REMARK 1.3.5.** Restriction of an almost semi continuous mapping may not be almost semi continuous.
For,

**EXAMPLE 1.3.4.** Let \((X, T)\) be the space of Example (1.3.1); \(Y = \{x, y, z, u\}\), 
\(T' = \{\emptyset, \{x\}, \{z, u\}, \{x, z, u\}, Y\}\). Define \(f: X \to Y\) by \(f(a) = f(c) = x, f(b) = z\) and \(f(d) = u\). Then \(f\) is almost semi continuous but if \(A = \{b, c, d\}\) then \(f \mid A\) is not almost semi continuous.

**THEOREM 1.3.9.** If \(f: X \to Y\) is almost semi continuous and \(A\) is open in \(X\) then the restriction \(f \mid A: A \to Y\) is almost semi continuous.

**PROOF** Analogous to that of Theorem (1.2.4) and utilizes Lemma (1.1.I).

**LEMMA 1.3.C.** Let \(Y\) be a subspace of a space \(X\). Then \(A\) is semi open in \(Y\) is semi open in \(X\) iff \(Y\) is semi open in \(X\). [65].

**THEOREM 1.3.10.** Let \(f: X \to Y\) and \(\{A_\alpha \mid \alpha \in \Lambda\}\) be a semi open cover of \(X\) (i.e. \(A_\alpha\) is semi open in \(X\) for each \(\alpha \in \Lambda\) and \(\bigcup_{\alpha \in \Lambda} A_\alpha = X\)). If the restriction \(f \mid A_\alpha : A_\alpha \to Y\) is almost semi continuous for each \(\alpha \in \Lambda\) then \(f\) is almost semi continuous.
PROOF. For any subset $V$ of $Y$, we have, $(f \mid A_\alpha)^{-1}(V) = f^{-1}(V) \cap A_\alpha$ for each $\alpha \in \Lambda$. If $V$ is regular open in $Y$ then $f^{-1}(V) \cap A_\alpha$ is semi open in $A_\alpha$ for each $\alpha \in \Lambda$ because $f \mid A_\alpha$ is almost semi continuous by hypothesis. Since each $A_\alpha$ is semi open in $X$, $f^{-1}(V) \cap A_\alpha$ is semi open in $X$ for each $\alpha \in \Lambda$ by Lemma (1.3.C). We obtain that

$$\bigcup_{\alpha \in \Lambda} (f^{-1}(V) \cap A_\alpha) = f^{-1}(V) \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = f^{-1}(V) \cap X = f^{-1}(V)$$

which is semi open in $X$. Hence, $f$ is almost semi continuous. //
We conclude the chapter with the following diagram of implications:

\[
\begin{align*}
\beta-c & \rightarrow a.c \rightarrow a.f.c \rightarrow a.s.c \\
c & \rightarrow f.c \rightarrow s.c \\
s.f.c & \rightarrow s.s.c \\
irr & \\
\end{align*}
\]

where,

- **c** : continuous
- **a.c** : almost continuous
- **\(\beta-c\)** : \(\beta\)-continuous
- **f.c** : feebly continuous
- **s.c** : semi continuous
- **a.f.c** : almost feebly continuous
- **a.s.c** : almost semi continuous
- **s.f.c** : strongly feebly continuous
- **s.s.c** : strongly semi continuous
- **irr** : irresolute