Chapter 3

Extreme Points of

\( N(\alpha, 2\alpha + 1, \gamma) \)

3.1 Abstract

In the last chapter, we have defined and explored certain properties of the class \( N(\alpha, \beta, \gamma) \). In the present chapter, we further investigate this class with a particular condition \( \beta = 2\alpha + 1 \) and a more general condition \( \gamma < 1 \), instead of \( 0 \leq \gamma < 1 \). We obtain an integral representation for the functions of the class \( N(\alpha, 2\alpha + 1, \gamma) \) and hence find extreme points of the class. Sharp bounds for coefficients, \( |f(z)| \), \( \text{Re} \left\{ \frac{f(z)}{z} \right\} \) and \( \text{Re} \{ f'(z) \} \) will be obtained. A condition for univalence will also be obtained. It is significant to note that some well known results obtained earlier by Silverman [72], R.M. Ali [3], C Y Gao [23] can be obtained as special cases of our results, by taking \( \alpha = 0 \).

The results of this chapter are submitted for publication in Rocky Moun-
3.2 An Integral Representation

Theorem 3.2.1. Let $\alpha > 0$. A function $f \in A$ is in $N(\alpha, 2\alpha + 1, \gamma), \gamma < 1$ if and only if $f(z)$ can be represented as

$$f(z) = \int_{|x|=1} \left[ (2\gamma - 1)z + (2 - 2\gamma)\bar{x} \sum_{n=0}^{\infty} \frac{(xz)^{n+1}}{(n+1)^2(n\alpha + 1)} \right] d\mu(x) \quad (3.1)$$

where $\mu(x)$ is the probability measure defined on the unit circle.

Proof. By definition $f \in N(\alpha, 2\alpha + 1, \gamma)$ if and only if

$$\frac{\alpha z^2 f'''(z) + (2\alpha + 1)zf''(z) + f'(z) - \gamma}{1 - \gamma} \in P, \ z \in E.$$

Using Herglotz’s representation for functions in $P$, we have

$$\frac{\alpha z^2 f'''(z) + (2\alpha + 1)zf''(z) + f'(z) - \gamma}{1 - \gamma} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x)$$

where $\mu(x)$ is a probability measure on the unit circle $|x| = 1$, i.e.,

$$\int_{|x|=1} d\mu(x) = 1.$$

From the above equation, we obtain,

$$\frac{1}{\alpha} g'(z) + zg''(z) = \frac{1}{\alpha} \int_{|x|=1} \frac{1 + (1 - 2\gamma)xz}{1 - xz} d\mu(x)$$
where \( g(z) = zf'(z) \).

Thus we have,

\[
\begin{align*}
  z^{-\frac{1}{\alpha}} \int_0^z \left[ \frac{1}{\alpha} g'(t) + t g''(t) \right] t^{\frac{1}{\alpha} - 1} dt &= \frac{1}{\alpha} \int_{|x|=1} \left[ z^{-\frac{1}{\alpha}} \int_0^z \frac{1 + (1 - 2\gamma)xt}{1 - xt} t^{\frac{1}{\alpha} - 1} dt \right] d\mu(x) \\
  i.e.,
  g'(z) &= \int_{|x|=1} \left[ (2\gamma - 1) + (2 - 2\gamma)\bar{x} \sum_{n=0}^{\infty} \frac{(xz)^n}{n\alpha + 1} \right] d\mu(x)
\end{align*}
\]

where the power function takes the branch of principal value.

Integrating this equality we get,

\[
\begin{align*}
  f'(z) &= \int_{|x|=1} \left[ (2\gamma - 1) + (2 - 2\gamma)\bar{x} \sum_{n=0}^{\infty} \frac{(xz)^n}{n\alpha + 1} \right] d\mu(x).
\end{align*}
\]

Integrating again,

\[
\begin{align*}
  f(z) &= \int_{|x|=1} \left[ (2\gamma - 1)z + (2 - 2\gamma)\bar{x} \sum_{n=0}^{\infty} \frac{(xz)^{n+1}}{(n\alpha + 1)(n + 1)^2} \right] d\mu(x).
\end{align*}
\]

The process is deductive and hence the converse follows.

**Remark 3.2.1.** In fact, the expression (3.1) is also true if we let \( \alpha = 0 \), which says that, if \( f \in R(1, \gamma) \), then it can be expressed as

\[
\begin{align*}
  f(z) &= \int_{|x|=1} \left[ (2\gamma - 1)z + (2 - 2\gamma)\bar{x} \sum_{n=0}^{\infty} \frac{(xz)^{n+1}}{(n + 1)^2} \right] d\mu(x)
\end{align*}
\]
\[ \int_{|x|=1} \left[ \int_0^1 \frac{(2\gamma - 1)t + (2 - 2\gamma)x\log(1 - xt)}{t} \, dt \right] \, d\mu(x). \]

This result was obtained by Silverman [72].

**Corollary 3.2.1.** The extreme points of class \( N(\alpha, 2\alpha + 1, \gamma) \) are

\[ f_x(z) = (2\gamma - 1)z + (2 - 2\gamma)x \sum_{n=0}^{\infty} \frac{(xz)^{n+1}}{(n\alpha + 1)(n + 1)^2}, \quad |x| = 1. \quad (3.2) \]

**Proof.** Using the notation \( f_x(z) \), equation (3.1) can be expressed as,

\[ f_\mu(z) = \int_{|x|=1} f_x(z) \, d\mu(x). \]

The mapping \( \mu \to f_\mu \) is easily seen to be one-to-one since this property is known to hold for the Herglotz representation and the assertion follows (See [31]). \( \square \)

**Corollary 3.2.2.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N(\alpha, 2\alpha + 1, \gamma) \) then,

\[ |a_n| \leq \frac{2(1 - \gamma)}{\alpha(n - 1) + 1} \frac{1}{n^2}, \quad n \geq 2. \]

The results are sharp.

**Proof.** From equation (3.2), \( f_x(z) \) can be expressed as

\[ f_x(z) = z + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{x^{n-1}z^n}{\alpha(n - 1) + 1} \frac{1}{n^2}, \quad |x| = 1. \quad (3.3) \]

Since the coefficient bounds are maximized at an extreme point, the corollary follows from equation (3.3). \( \square \)
Corollary 3.2.3. If $f \in N(\alpha, 2\alpha + 1, \gamma)$ then, for $|z| = r < 1$,

$$|f(z)| \leq r + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{r^n}{|\alpha(n-1) + 1|n^2}.$$ 

Remark 3.2.2. If we let $r \to 1$, we get,

$$|f(z)| \leq 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{1}{|\alpha(n-1) + 1|n^2}.$$ 

Now, if we let $\alpha = 0$, we obtain,

$$|f(z)| \leq 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{1}{n^2}$$

which implies the following:

If $f \in R(1, \gamma)$, then

$$|f(z)| \leq 1 + 2(1 - \gamma) \left( \frac{\pi^2}{6} - 1 \right)$$

$$= (1 - \gamma) \left( \frac{\pi^2}{3} - 1 \right) + \gamma$$

This result was obtained by Silverman [72], which implies that the family $R(1, \gamma)$ is bounded in $E$ for all real $\gamma, \gamma < 1$.

3.2.1 Bounds for $\text{Re} \left\{ \frac{f(z)}{z} \right\}$ and $\text{Re} \left\{ f'(z) \right\}$

In this section, we obtain certain bounds for $\text{Re} \left\{ \frac{f(z)}{z} \right\}$ and $\text{Re} \left\{ f'(z) \right\}$. It is sufficient to investigate the extreme points of $N(\alpha, 2\alpha + 1, \gamma)$, since $L(f) = \text{Re} f'(z)$ and $L(f) = \text{Re} \left\{ \frac{f(z)}{z} \right\}$ are continuous linear operators. Also, the class $N(\alpha, 2\alpha + 1, \gamma)$ is rotationally invariant i.e. if $f \in N(\alpha, 2\alpha + 1, \gamma)$, then
\( e^{-i\theta} f(e^{i\theta} z) \in N(\alpha, 2\alpha + 1, \gamma), \ z \in E. \) Hence, we may restrict our attention to the extreme point
\[
g(z) = z + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{z^n}{\alpha(n-1) + 1 |n^2|}. \tag{3.4}
\]

**Theorem 3.2.2.** Let \( f \in N(\alpha, 2\alpha + 1, \gamma). \) Then, for \( |z| \leq r < 1, \)
\[
Re \left\{ \frac{f(z)}{z} \right\} \geq 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-r)^{n-1}}{\alpha(n-1) + 1 |n^2|} = 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\alpha(n-1) + 1 |n^2|}.
\]

The inequality is sharp.

**Proof.** We need only to consider the function \( g(z) \) defined by equation (3.4).

From (3.4),
\[
g(z) = 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{z^{n-1}}{\alpha(n-1) + 1 |n^2|} = 1 + \frac{2(1 - \gamma)}{\alpha} \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \frac{vz}{1 - tuvz} \, dv \right) \, du \right) \, dt.
\]

Thus,
\[
Re \left\{ \frac{g(z)}{z} \right\} = 1 + \frac{2(1 - \gamma)}{\alpha} \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \frac{z}{1 - tuvz} \, dv \right) \, du \right) \, dt \geq 1 - \frac{2(1 - \gamma)}{\alpha} \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \frac{vr}{1 + tuvr} \, dv \right) \, du \right) \, dt = 1 - \frac{2(1 - \gamma)}{\alpha} \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} \frac{vr}{1 + tuvr} \frac{(-1)^{n-1}(tuvr)^n \, dv} \right) \, du \right) \, dt.
\]
\[
= 1 - \frac{2(1 - \gamma)}{\alpha} \int_0^1 \frac{1}{t^\frac{1}{n}} \left[ \int_0^1 u \left[ \sum_{n=0}^{\infty} \frac{(-1)^n (tu)^n}{n+2} \right] \frac{1}{n+2} \right] \, du \, dt
\]

\[
= 1 - \frac{2(1 - \gamma)}{\alpha} \int_0^1 \frac{1}{t^\frac{1}{n}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(n+2)^2} \right] \, dt
\]

\[
= 1 - \frac{2(1 - \gamma)}{\alpha} \sum_{n=0}^{\infty} \frac{\alpha (-1)^n n^{n+1}}{(n+2)^2 [\alpha(n+1)+1]}
\]

\[
= 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\alpha(n-1)+1} n^2
\]

\[
> 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\alpha(n-1)+1} n^2.
\]

The sharpness can be seen from the extremal function

\[
g(z) = z + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{z^n}{\alpha(n-1)+1} n^2.
\]

**Remark 3.2.3.** If we let \( \alpha = 0 \), we get

\[
\text{Re} \left\{ \frac{g(z)}{z} \right\} > 1 + (1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2}.
\]

Thus, if \( f \in R(1, \gamma) \), then

\[
\text{Re} \left\{ \frac{f(z)}{z} \right\} > (1 - \gamma) \left( \frac{\pi^2}{6} - 1 \right) + \gamma.
\]

This result was obtained by the authors ([3], [23], [72]).

**Theorem 3.2.3.** If \( f \in N(\alpha, 2\alpha + 1, \gamma) \), then for \( |z| \leq r < 1 \),

\[
\text{Re} \{ f'(z) \} \geq 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-r)^{n-1}}{\alpha(n-1)+1} n
\]

\[
> 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\alpha(n-1)+1} n.
\]
The inequality is sharp.

Proof. Again we consider the function \( g(z) \) given by equation (3.4).

We have,

\[
g'(z) = 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{z^{n-1}}{\alpha(n - 1) + 1}n. \tag{3.6}
\]

It can be written as

\[
g'(z) = 1 + 2(1 - \gamma) \frac{1}{\alpha} \int_{0}^{1} t^{\frac{1}{\alpha}} \left[ \int_{0}^{1} v \Re \left[ \frac{z}{1 - tvz} \right] dv \right] dt.
\]

Thus,

\[
\Re \{g'(z)\} = 1 + 2(1 - \gamma) \frac{1}{\alpha} \int_{0}^{1} t^{\frac{1}{\alpha}} \left[ \int_{0}^{1} v \Re \left[ \frac{z}{1 - tvz} \right] dv \right] dt
\]

\[
\geq 1 - 2(1 - \gamma) \frac{1}{\alpha} \int_{0}^{1} t^{\frac{1}{\alpha}} \left[ \int_{0}^{1} v \Re \left[ \frac{vr}{1 + tvr} \right] dv \right] dt
\]

\[
= 1 - 2(1 - \gamma) \frac{1}{\alpha} \int_{0}^{1} t^{\frac{1}{\alpha}} \left[ \int_{0}^{1} vr \sum_{n=0}^{\infty} (-1)^{n}(tv)^{n} \right] dv \right] dt
\]

\[
= 1 - 2(1 - \gamma) \frac{1}{\alpha} \int_{0}^{1} t^{\frac{1}{\alpha}} \sum_{n=0}^{\infty} \frac{(-1)^{n}r^{n+1}}{n + 2} dt
\]

\[
= 1 - 2(1 - \gamma) \sum_{n=0}^{\infty} \frac{\alpha(-1)^{n}r^{n+1}}{(n + 2)[\alpha(n + 1) + 1]}
\]

\[
= 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-r)^{n-1}}{\alpha(n - 1) + 1}n
\]

\[
> 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{[\alpha(n - 1) + 1]n}.
\]
The sharpness can be seen from the extremal function
\[ g(z) = z + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{z^n}{\alpha(n-1)+1} n^2. \]

**Remark 3.2.4.** If we let \( \alpha = 0 \), in the above corollary, we find that
\[ f \in R(1, \gamma) \] and that
\[ \Re \{ f'(z) \} > 1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \]
\[ = (1 - \gamma)(2 \log 2 - 1) + \gamma. \]

This result was obtained by the authors ([3], [23], [72]).

**Corollary 3.2.4.**
\[ \sum_{k=1}^{\infty} \frac{\cos k \theta}{(\alpha k + 1)(k + 1)} \geq \sum_{k=1}^{\infty} \frac{(-1)^k}{(\alpha k + 1)(k + 1)}. \]

**Proof.** From equation (3.6),
\[ \Re \{ g'(z) \} = 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \frac{r^k \cos k \theta}{(\alpha k + 1)(k + 1)}. \]

This is minimized when \( \theta = \pi \). Thus,
\[ 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \frac{r^k \cos k \theta}{(\alpha k + 1)(k + 1)} \geq 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \frac{r^k (-1)^k}{(\alpha k + 1)(k + 1)}. \]

Letting \( r \to 1 \), we get,
\[ 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \frac{\cos k \theta}{(\alpha k + 1)(k + 1)} \geq 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \frac{(-1)^k}{(\alpha k + 1)(k + 1)} \]
from which the corollary follows.
Remark 3.2.5. If we let \( \alpha = 0 \), in the above corollary, we find that \( f \in R(1, \gamma) \) and that
\[
\sum_{k=1}^{\infty} \frac{\cos k \theta}{k+1} \geq \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} = \log 2 - 1
\]
This result was obtained by Silverman [72].

Corollary 3.2.5. \( N(\alpha, 2\alpha + 1, \gamma) \subset S \) for \( \gamma \geq \gamma_0 \), and this result cannot be extended to \( \gamma < \gamma_0 \), where
\[
\gamma_0 = 1 + \frac{1}{2} \left[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1) + 1} \right]^{-1}
\]

Proof. Let \( f \in N(\alpha, 2\alpha + 1, \gamma) \). From inequality (3.5), we note that \( f \in N(\alpha, 2\alpha + 1, \gamma) \), will be in \( S \), if
\[
1 + 2(1 - \gamma) \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1) + 1} \geq 0,
\]
i.e., if \( \gamma \geq \gamma_0 \), we have \( N(\alpha, 2\alpha + 1, \gamma) \subset S \).

The result cannot be extended to \( \gamma < \gamma_0 \), because \( f'(-1) = 0 \) at \( \gamma = \gamma_0 \) and thus \( f'(-r) = 0 \) for some \( r = r(\gamma) < 1 \) when \( \gamma < \gamma_0 \).

Remark 3.2.6. If we let \( \alpha = 0 \), we get \( R(1, \gamma) \subset S \), for
\[
\gamma = \gamma_0 \geq \frac{1 - \log 2}{2 - 2\log 2}
\]
and this result cannot be extended to \( \gamma < \gamma_0 \).