Chapter 2

On a New Subclass $N(\alpha, \beta, \gamma)$ of Univalent Functions

2.1 Abstract

We introduce and investigate some properties and characteristics of a certain three parameter class $N(\alpha, \beta, \gamma)$ of $\mathbb{A}$, where $\alpha \geq 0, \beta \geq 0$ and $0 \leq \gamma < 1$. We show that a function in this class is univalent and that for any function $f$ in this class, $\text{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2}$. As a consequence, we show that this class is closed under convolution although its superset $K$ is not.

If $f \in N(\alpha, \beta, \gamma)$ and $g \in \mathbb{A}$ is any function with $\text{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}$, then we show that $f \ast g \in N(\alpha, \beta, \gamma)$ and hence obtain some interesting convolution properties of this class.

Majority of the results of this chapter are obtained using convolution techniques. A pivotal role in the proofs of first four theorems is played by lemmas.
due to Duren P L [15] and Fejer L [17] pertaining to convex hull of the image of $E$ under a function $f$ and analytic functions with convex null sequences as their Taylor coefficients, respectively.

A major part of this chapter has appeared in the paper titled “On a New Subclass of Univalent Functions”, Far East Journal of Mathematical Sciences(FJMS) [32].

2.2 Preliminaries

In the investigation of functions of the newly defined subclass, we need the following definitions and results.

**Definition 2.2.1** (15). A sequence $\{C_n\}_{n=0}^{\infty}$ of non-negative real numbers is said to be a convex null sequence, if $C_n \to 0$ as $n \to \infty$ and

$$C_0 - C_1 \geq C_1 - C_2 \geq \ldots \geq C_n - C_{n+1} \geq \ldots \geq 0.$$ 

**Definition 2.2.2** (7). Let $c$ be a complex number. The Bernardi integral operator for a function $f \in \mathbb{A}$ is defined as

$$L_c(f) = c + 1 \int_0^{\frac{z}{c}} t^{c-1} f(t) dt.$$ (2.1)

Properties of this integral operator were studied by Bernardi for natural numbers.

If $c = 1$, we get the Libera’s operator [37] given by

$$L(f) = \frac{2}{z} \int_0^{\frac{z}{2}} f(t) dt.$$
Lemma 2.2.1 (17). Let \( \{C_n\}_{n=0}^{\infty} \) be a convex null sequence. Then the function
\[
q(z) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n z^n
\]
is analytic in \( E \) and \( \text{Re} \ q(z) > 0 \).

Lemma 2.2.2 (15). If \( p(z) \) is analytic in \( E \), \( p(0) = 1 \) and \( \text{Re} \ p(z) > \frac{1}{2} \), \( z \in E \), then for any function \( f \) analytic in \( E \), the function \( f * p \) takes values in the convex hull of the image of \( E \) under \( f \).

Lemma 2.2.3 (66). Let \( g(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \) be subordinate to
\[
G(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \text{ in } E. \text{ If } G(z) \text{ is univalent in } E \text{ and } G(E) \text{ is convex, then } |a_n| \leq |b_1| = |G'(0)|.
\]

Theorem 2.2.1 (68). Let \( c \) be a complex number with \( \text{Re} \ c > 0 \). Then
\[
g(z) = \frac{c + 1}{z^c} \int_{0}^{z} \frac{t^c}{1 - t} \, dt
\]
is convex univalent in \( E \).

2.3 The Class \( N(\alpha, \beta, \gamma) \)

In this section, we define a three parameter family \( N(\alpha, \beta, \gamma) \) of \( A \) and obtain univalence and other properties of this class.

Definition 2.3.1. Let
\[
N(\alpha, \beta, \gamma) = \{ f \in A : \text{Re}(\alpha z^2 f'''(z) + \beta z f''(z) + f'(z)) > \gamma, \ \alpha \geq 0, \beta \geq 0, \ 0 \leq \gamma < 1 \}. \tag{2.2}
\]
We note that, for $z \in E$,

1. $N(0, 0, 0) = P' = \{ f \in \mathbb{A} : \text{Re}(f'(z)) > 0 \}.$

2. $N(0, 0, \gamma) = P'_\gamma = \{ f \in \mathbb{A} : \text{Re}(f'(z)) > \gamma \}.$

3. $N(0, 1, 0) = R = \{ f \in \mathbb{A} : \text{Re}(zf''(z) + f'(z)) > 0 \}.$

4. $N(0, \beta, 0) = R(\beta) = \{ f \in \mathbb{A} : \text{Re}(\beta zf''(z) + f'(z)) > 0, \beta > 0 \}.$

5. $N(0, \beta, \gamma) = R(\beta, \gamma)
   = \{ f \in \mathbb{A} : \text{Re}(\beta zf''(z) + f'(z)) > \gamma, \beta > 0, 0 \leq \gamma < 1 \}.$

These classes have been studied extensively by several authors. (See [1], [11], [13], [60], [62], [61]). In fact, the classes $R(\beta)$ and $R(\beta, \gamma)$ were studied with the more general condition $\text{Re} \beta \geq 0$.

### 2.3.1 Univalence of the Class $N(\alpha, \beta, \gamma)$

Now, we show that any function in $N(\alpha, \beta, \gamma)$ is univalent.

**Theorem 2.3.1.** Let $f \in N(\alpha, \beta, \gamma)$. Then $f$ is univalent in $E$.

**Proof.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N(\alpha, \beta, \gamma)$.

Then,

\[
\text{Re} \left\{ \alpha z^2 f'''(z) + \beta z f''(z) + f'(z) \right\} = \text{Re} \left\{ 1 + \sum_{n=2}^{\infty} \left\{ \alpha (n-1)(n-2) + \beta (n-1) + 1 \right\} n a_n z^{n-1} \right\} > \gamma. \quad (2.3)
\]
Define a sequence \( \{C_n\}_{n=0}^\infty \) by
\[
C_0 = 1, 
C_n = \frac{1}{\alpha n(n-1) + \beta(n-1) + 1}, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}.
\]
Then, \( \{C_n\}_{n=0}^\infty \) is a convex null sequence. By lemma (2.2.1),
\[
q(z) = \frac{1}{2} + \sum_{n=2}^\infty \frac{1}{\alpha(n-1)(n-2) + \beta(n-1) + 1} z^{n-1}
\]
is analytic in \( E \) and \( \text{Re} \, q(z) > 0 \).
Define,
\[
p(z) = \frac{1}{2} + q(z) = 1 + \sum_{n=2}^\infty \frac{1}{\alpha(n-1)(n-2) + \beta(n-1) + 1} z^{n-1}.
\]
Then,
p(0) = 1, \( p(z) \) is analytic in \( E \) and \( \text{Re} \, p(z) > \frac{1}{2} \).
Now,
\[
f'(z) = 1 + \sum_{n=2}^\infty n a_n z^{n-1}
\]
\[= \left\{ 1 + \sum_{n=2}^\infty \{\alpha(n-1)(n-2) + \beta(n-1) + 1\} a_n z^{n-1} \right\}
\[\times \left\{ 1 + \sum_{n=2}^\infty \frac{1}{\alpha(n-1)(n-2) + \beta(n-1) + 1} z^{n-1} \right\}.
\]
Using (2.3) and lemma (2.2.2), it follows that \( \text{Re} \, f'(z) > \gamma \), i.e., \( f \) is close to convex of order \( \gamma \) and hence is univalent in \( E \).

Remark 2.3.1. \( N(\alpha, \beta, \gamma) \subset K \).

Theorem 2.3.2. If \( f \in N(\alpha, \beta, \gamma) \), then \( \text{Re} \left\{ \frac{f(z)}{z} \right\} > \gamma + \frac{1}{2} \).
Proof. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N(\alpha, \beta, \gamma) \).

Then

\[
\frac{f(z)}{z} = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}. \tag{2.4}
\]

Define a sequence \( \{C_n\}_{n=0}^{\infty} \) by

\[
C_0 = 1, \quad C_n = \frac{2}{(n+1)(\alpha n(n-1) + \beta n + 1)}, \quad n \in \mathbb{N}.
\]

Then, \( \{C_n\}_{n=0}^{\infty} \) is a convex null sequence. By lemma (2.2.1),

\[
q(z) = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{2}{n\{(\alpha n-1)(n-2) + (\beta n-1) + 1\}} z^{n-1}
\]

is analytic in \( E \) and Re \( q(z) > 0 \).

Define,

\[
p(z) = \frac{1}{2} + q(z) = 1 + \sum_{n=2}^{\infty} \frac{2}{n\{(\alpha n-1)(n-2) + (\beta n-1) + 1\}} z^{n-1}.
\]

Then, \( p(0) = 1 \), \( p(z) \) is analytic and Re \( p(z) > 1/2 \).

From (2.3),

\[
\text{Re} \left\{ 1 + \frac{1}{2} \sum_{n=2}^{\infty} \{\alpha(n-1)(n-2) + \beta(n-1) + 1\} na_n z^{n-1} \right\} > \frac{\gamma + 1}{2}. \tag{2.5}
\]

Also, from (2.4)

\[
\frac{f(z)}{z} = \left[ 1 + \frac{1}{2} \sum_{n=2}^{\infty} \{\alpha(n-1)(n-2) + \beta(n-1) + 1\} na_n z^{n-1} \right]
\]

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\[ * \left[ 1 + \sum_{n=2}^{\infty} \frac{2}{n\{\alpha(n-1)(n-2) + \beta(n-1) + 1\}} z^{n-1} \right] \]

The theorem follows from lemma (2.2.2) and equation (2.5).

\textbf{Remark 2.3.2.} If \( f \in N(\alpha, \beta, \gamma) \), then \( \text{Re}\left( \frac{f(z)}{z} \right) > \frac{1}{2} \).

\subsection*{2.3.2 Convolution Properties}

It is well known that if \( f, g \in K \), then \( f \ast g \) need not be in \( K \). In the following theorem we prove that if \( f, g \in N(\alpha, \beta, \gamma) \), a subclass of \( K \), then so does \( f \ast g \). i.e., we prove that \( N(\alpha, \beta, \gamma) \) is closed with respect to the Hadamard product.

\textbf{Theorem 2.3.3.} If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) are in \( N(\alpha, \beta, \gamma) \), then so does their convolution \( \phi(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \).

\textit{Proof.} Let \( \phi(z) = (f \ast g)(z) \).

Then,

\[ \alpha z^2 \phi'''(z) + \beta z \phi''(z) + \phi'(z) = \{\alpha z^2 f'''(z) + \beta z f''(z) + f'(z)\} * \frac{g(z)}{z}. \tag{2.6} \]

Now, \( f \in N(\alpha, \beta, \gamma) \) satisfies (2.3) and \( \text{Re}\left( \frac{g(z)}{z} \right) > \frac{1}{2} \) from theorem (2.3.2).

Hence, from equation (2.6) and lemma (2.2.2), we get,

\[ \text{Re}\{\alpha z^2 \phi'''(z) + \beta z \phi''(z) + \phi'(z)\} > \gamma. \]

Thus, \( \phi = f \ast g \in N(\alpha, \beta, \gamma) \). \qed
From the proof of the above theorem, it is clear that the following more general result holds:

**Theorem 2.3.4.** If \( f \in N(\alpha, \beta, \gamma) \) and \( g \in \mathbb{A} \) is such that
\[
\text{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}, \quad z \in E,
\]
then \( f \ast g \in N(\alpha, \beta, \gamma) \).

**Remark 2.3.3.** We observe that \( \text{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}, \quad z \in E \), need not even imply the univalence of \( g \) in \( E \).

Application of theorem (2.3.4), leads to the following convolution properties of the class \( N(\alpha, \beta, \gamma) \).

**Corollary 2.3.1.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is in \( N(\alpha, \beta, \gamma) \), then so does
\[
f_k(z) = z + \sum_{n=2}^{\infty} a_{nk+1} z^{nk+1}, \quad k = 1, 2, 3, ...
\]

*Proof.* Let
\[
g(z) = \frac{z}{1-z^k} = z + \sum_{n=2}^{\infty} z^{nk+1}.
\]
Then,
\[
\text{Re}\left(\frac{g(z)}{z}\right) = \text{Re}\left(\frac{1}{1-z^k}\right) > \frac{1}{2}, \quad z \in E.
\]
Now,
\[
f_k(z) = (f \ast g)(z)
\]
and the corollary follows from theorem (2.3.4). \( \square \)

**Corollary 2.3.2.** Let \( f \in N(\alpha, \beta, \gamma) \).

Then,
\[
F(z) = \int_{0}^{z} \frac{f(t)}{t} dt
\]
is also in \( N(\alpha, \beta, \gamma) \).

*Proof.* We have
\[
F(z) = (f \ast g)(z)
\]
where \( g(z) = \log \left( \frac{1}{1 - z} \right) \) is convex univalent in \( E \) and hence 
\[
\Re \left( \frac{g(z)}{z} \right) > \frac{1}{2} \quad \text{and the corollary follows.}
\]

**Remark 2.3.4.** It is well known that 
\( g \in S^*(1/2) \Rightarrow \Re \left( \frac{g(z)}{z} \right) > \frac{1}{2}, \quad z \in E. \)

Thus, it follows from theorem (2.3.4) that, if 
\( f \in N(\alpha, \beta, \gamma) \) and \( g \in S^*(1/2) \), then \( f \ast g \in N(\alpha, \beta, \gamma). \)

**Corollary 2.3.3.** Let \( f \in N(\alpha, \beta, \gamma). \)

Then, 
\[
F(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt, \quad |x| \leq 1, \quad x \neq 1,
\]
is also in \( N(\alpha, \beta, \gamma). \)

**Proof.** We have 
\[
F(z) = (f \ast g)(z)
\]
where 
\[
g(z) = \frac{1}{1 - x} \log \left( \frac{1 - xz}{1 - z} \right).
\]
The function \( g(z) \) is convex univalent in \( E \) and hence the corollary.

**Corollary 2.3.4.** Let \( f \in N(\alpha, \beta, \gamma). \) Then \( L_c(f) \) defined as in (2.1) is in \( N(\alpha, \beta, \gamma) \) for \( \Re c > 0. \)

**Proof.** We have 
\[
L_c(f) = (f \ast g)(z)
\]
where 
\[
g(z) = \frac{c + 1}{z^c} \int_0^z \frac{t^c}{1 - t} dt.
\]
From theorem (2.2.1), \( g(z) \) is convex univalent in \( E \), for \( \Re c > 0. \)
Hence \( \text{Re} \left( \frac{g(z)}{z} \right) > \frac{1}{2}, \ z \in E \) and the result follows.

Now, we explore some convolution properties of partial sums.

Let \( f \in A \) be given by (1.1).

Then,

\[
s_n(z) = z + \sum_{k=2}^{n} a_k z^k, \quad (n \in \mathbb{N} \setminus \{1\})
\]

is called the partial sum of \( f \).

**Corollary 2.3.5.** Let \( f \in N(\alpha, \beta, \gamma) \) be given by (1.1). Let

\[
s_n(z) = z + \sum_{k=2}^{n} a_k z^k, \quad (n \in \mathbb{N} \setminus \{1\}). \tag{2.7}
\]

Then

\[
\frac{1}{r_n} s_n(r_n z) \in N(\alpha, \beta, \gamma)
\]

where \( r_n \) is given by

\[
r_n = \sup \left\{ r : \text{Re} \left( \sum_{k=0}^{n-1} z^k \right) > \frac{1}{2}, (|z| = r < 1) \right\}, \quad (n \in \mathbb{N} \setminus \{1\}). \tag{2.8}
\]

**Proof.** Let

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N(\alpha, \beta, \gamma).
\]

Let

\[
g_n(z) = \sum_{k=1}^{n} z^k.
\]

Then,

\[
s_n(z) = (f \ast g_n)(z).
\]

From (2.8), it follows that

\[
\text{Re} \left( \frac{g_n(r_n z)}{r_n z} \right) > \frac{1}{2}, \quad z \in E \quad (n \in \mathbb{N} \setminus \{1\}).
\]

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An application of theorem (2.3.4), leads to

\[
\frac{1}{r_n}s_n(r_nz) = f(z) \ast \left( g_n(r_nz) \right) \in N(\alpha, \beta, \gamma), \quad (n \in \mathbb{N} \setminus \{1\}).
\]

\[\square\]

**Remark 2.3.5.** For \( r_n \) given by (2.8), we have \( r_2 = 1/2 \).

For \( n = 3 \) and \( z = re^{i\theta} \), we have

\[
\text{Re}\{z^2 + z + 1\} = \frac{7}{8} - r^2 + \frac{(4 \cos \theta + 1)^2}{8}
\]

which yields \( r_3 = \sqrt{6}/4 \).

For \( n \in \mathbb{N} \setminus \{1, 2, 3\} \) and \( |z| = 1/2 \), we get

\[
\text{Re}\left\{ \sum_{k=0}^{n-1} z^k \right\} = \text{Re}\left\{ \frac{1 - z^n}{1 - z} \right\}
\]

\[
= \text{Re}\left\{ \frac{1}{1 - z} \right\} - \text{Re}\left\{ \frac{z^n}{1 - z} \right\}
\]

\[
\geq \frac{1 + |z|}{1 - |z|} - \frac{|z|^n}{1 - |z|}
\]

\[
= \frac{2}{3} - \frac{1}{2^{n-1}}
\]

\[
\geq 13/24.
\]

Thus, we have, \( r_n > 1/2 \) \( (n \in \mathbb{N} \setminus \{1, 2\}) \).

**Remark 2.3.6.** MacGregor [45], proved that, if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathbb{A} \) satisfies \( \text{Re} f'(z) > 0 \), \( z \in E \), then the function \( s_n(z) \) is univalent in \( |z| < \frac{1}{2} \).

But this result is not sharp when \( n \in \mathbb{N} \setminus \{1, 2\} \).

**Corollary 2.3.6.** Let \( f \in N(\alpha, \beta, \gamma) \). Let \( s_n(z) \) be as defined in (2.7). Then
the function,

$$
\sigma_n(z) = \int_0^z \frac{s_n(t)}{t} dt, \quad (n \in \mathbb{N} \setminus \{1\})
$$

belongs to the class $N(\alpha, \beta, \gamma)$.

**Proof.** We have

$$
\sigma_n(z) = z + \sum_{k=2}^{n} a_k z^k = (f \ast g_n)(z), \quad (n \in \mathbb{N} \setminus \{1\}) \tag{2.9}
$$

where $f \in N(\alpha, \beta, \gamma)$ and $g_n(z) = z + \sum_{k=2}^{n} \frac{z^k}{k}$.

Since (see [68]),

$$
\text{Re} \left( \frac{g_n(z)}{z} \right) > \frac{1}{2}, \quad z \in E,
$$

it follows from (2.9) and theorem (2.3.4), that $\sigma_n(z) \in N(\alpha, \beta, \gamma)$. \hfill \qed

**Corollary 2.3.7.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N(\alpha, \beta, \gamma)$. Let $s_n(z)$ be as defined by (2.7). Then $\text{Re} \{s_n'(z)\} > \gamma$ for $|z| < r_n$, where $r_n$ is given by (2.8).

**Proof.** Let $s_n(z)$ be as defined in (2.7). Then,

$$
s_n'(z) = 1 + \sum_{k=2}^{n} k a_k z^{k-1}
= f'(z) \ast g_n(z)
$$

where

$$
g_n(z) = \sum_{k=0}^{n-1} z^k.
$$

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Since $\text{Re} \{ f'(z) \} > \gamma$ from theorem (2.3.1) and $\text{Re} \left\{ \frac{g_n(z)}{z} \right\} > \frac{1}{2}$, for $|z| < r_n$, application of lemma (2.2.2) leads to $\text{Re} \left\{ s'_n(z) \right\} > \gamma$ for $|z| < r_n$. 

**Remark 2.3.7.** From the above corollary, we see that $s_n(z)$ is close to convex of order $\gamma$ and hence is univalent in $|z| < r_n$. 

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