Chapter 6

On Srivastava-Attiya Integral Operator

6.1 Abstract

In this chapter, initially, we obtain certain subordination properties of Srivastava-Attiya operator. Then, we obtain some new results by considering effect of Srivastava-Attiya operator on some subclasses of the class $S$, such as $S^*_h$, $C_h$, $K_h$, etc. We introduce certain subclasses $S^*_{\mu,b}(h)$, $C_{\mu,b}(h)$ and $K_{\mu,b}(h)$ of analytic functions using the Srivastava-Attiya operator. Their inclusion properties will be obtained. Certain applications of these inclusion relationships will also be stated.

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6.2 Srivastava-Attiya operator

The generalised Hurwitz - Lerch Zeta function \( \phi(z, s, a) \) is defined as

\[
\phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(a + n)^s}
\]  

\( (a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \ Re s > 1, \text{ when } |z| = 1; \mathbb{Z}_0^- = \mathbb{Z} \setminus \mathbb{N}) \).

It contains, as its special cases, well known functions such as the Riemann and Hurwitz Zeta function, Lerch Zeta function, the polylogarithmic function and the Lipschitz Lerch Zeta function.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function \( \phi(z, s, a) \) can be found in the recent investigations by Choi and Srivastava [12], Ferreira and Lopez [18], Garg et al. [26], Lin and Srivastava [38], Lin et al [39].

Using this function, Srivastava and Attiya [73] introduced the following family of linear operator \( J_{\mu, b} : \mathbb{A} \to \mathbb{A} \), defined by

\[
J_{\mu, b}f(z) = G_{\mu, b} \ast f(z), \quad (z \in E, \ b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mu \in \mathbb{C})
\]

(6.2)

where the function \( G_{\mu, b} \) is given by

\[
G_{\mu, b}(z) = (1 + b)^\mu \left[ \phi(z, \mu, b) - b^{-\mu} \right] = z + \sum_{n=2}^{\infty} \left( \frac{b + 1}{b + n} \right)^\mu z^n, \quad z \in E.
\]

(6.3)

Using (6.3) in (6.2), we get

\[
J_{\mu, b}f(z) = z + \sum_{n=2}^{\infty} \left( \frac{b + 1}{b + n} \right)^\mu a_n z^n, \quad z \in E.
\]

(6.4)
For \( f \in \mathbb{A} \) and \( z \in E \),

\[
\begin{align*}
J_{0,b}f(z) &= f(z), \\
J_{1,0}f(z) &= \int_0^z \frac{f(t)}{t} \, dt = A(f), \\
J_{1,1}f(z) &= 2 \int_0^z f(t) \, dt = L(f), \\
J_{1,c}f(z) &= 2 \int_0^z t^{-1} f(t) \, dt = L_c(f), \\
J_{\sigma,1}f(z) &= z + \sum_{n=2}^{\infty} \left( \frac{2}{n+1} \right)^{\sigma} a_n z^n = I^\sigma f(z), \quad (\sigma > 0),
\end{align*}
\]

where \( A(f) \) denotes Alexander transform [2], \( L(f) \) denotes the Libera’s operator [37], \( L_c(f) \) denotes Bernardi integral operator [7], \( I^\sigma f(z) \) is the Jung - Kim - Srivastava integral operator [34], closely related to some multiplier transformations studied by Flett [20].

By using the equation (6.4), Srivastava and Attiya obtained the following relation:

\[
z(J_{\mu+1,b}f(z))^1 = (b + 1)J_{\mu,b}f(z) - bJ_{\mu+1,b}f(z).
\]

(6.5)

### 6.2.1 Certain Properties

In this section we obtain certain properties of Srivastava-Attiya integral operator.

**Theorem 6.2.1.** Let \( \lambda < 1 \) and \(-1 \leq B < A \leq 1\). If \( f \in \mathbb{A} \) satisfies
\[(1 - \lambda)\frac{J_{\mu,b}f(z)}{z} + \lambda \frac{J_{\mu+1,b}f(z)}{z} \prec \frac{1 + Az}{1 + Bz}, \quad z \in E, \quad (6.6)\]

then

\[\text{Re}\left\{\frac{J_{\mu+1,b}f(z)}{z}\right\} > \frac{b + 1}{1 - \lambda} \int_0^1 t^{\frac{\mu+1}{1-\lambda}-1} \left(\frac{1 - At}{1 - Bt}\right) dt, \quad (6.7)\]

for \(b > -1\). The result is sharp.

**Proof.** Let \(f \in \mathcal{A}\) satisfy equation (6.6). Put

\[p(z) = \frac{J_{\mu+1,b}f(z)}{z}. \quad (6.8)\]

Then, \(p(z)\) is analytic in \(E\) and \(p(0) = 1\). Using equation (6.5) and equation (6.8), we obtain,

\[\frac{J_{\mu,b}f(z)}{z} = p(z) + \frac{zp'(z)}{b + 1}. \quad (6.9)\]

From (6.6), (6.8) and (6.9), for \(b > -1\), we obtain,

\[p(z) + \frac{1 - \lambda}{b + 1}zp'(z) \prec \frac{1 + Az}{1 + Bz}.

Application of the lemma (??), leads to

\[p(z) \prec \frac{b + 1}{1 - \lambda} \int_0^1 z^{-\frac{\mu+1}{1-\lambda}} \int_0^z t^{\frac{\mu+1}{1-\lambda}-1} \left(\frac{1 + At}{1 + Bt}\right) dt, \]

i.e.,

\[\frac{J_{\mu+1,b}f(z)}{z} = \frac{b + 1}{1 - \lambda} \int_0^1 u^{\frac{\mu+1}{1-\lambda}-1} \left(\frac{1 + Auw(z)}{1 + Buw(z)}\right) du, \quad (6.10)\]
where \( w(z) \) is analytic in \( E \) with \( w(0) = 0 \) and \( |w(z)| < 1, \quad z \in E \).

If \( -1 \leq B < A \leq 1, \quad \lambda < 1 \) and \( b > -1 \), then from (6.10), it follows that

\[
\text{Re} \frac{J_{\mu+1,b}f(z)}{z} = \text{Re} \left\{ \frac{b + 1}{1 - \lambda} \int_0^1 \frac{u^{\frac{b+1}{1-\lambda}-1}}{u^{\frac{b+1}{1-\lambda}-1}} \left( \frac{1 + Auw(z)}{1 + Buw(z)} \right) du \right\} \\
> \frac{b + 1}{1 - \lambda} \int_0^1 \frac{u^{\frac{b+1}{1-\lambda}-1}}{u^{\frac{b+1}{1-\lambda}-1}} \left( \frac{1 - Au}{1 - Bu} \right) du, \quad z \in E,
\]

which is (6.7). To show the sharpness of (6.7), we take \( f \in A \) defined by

\[
\frac{J_{\mu+1,b}f(z)}{z} = \frac{b + 1}{1 - \lambda} \int_0^1 t^{\frac{b+1}{1-\lambda}-1} \left( \frac{1 + At}{1 + Bt} \right) dt.
\]

For this function, we find that

\[
(1 - \lambda) \frac{J_{\mu,b}f(z)}{z} + \lambda \frac{J_{\mu+1,b}f(z)}{z} = \frac{1 + Az}{1 + Bz},
\]

and that

\[
\frac{J_{\mu+1,b}f(z)}{z} \to \frac{b + 1}{1 - \lambda} \int_0^1 t^{\frac{b+1}{1-\lambda}-1} \left( \frac{1 - At}{1 - Bt} \right) dt, \quad \text{as} \quad z \to -1.
\]

Hence the proof is complete. \( \square \)

**Remark 6.2.1.** From the elementary inequality: \( \text{Re} \left\{ \frac{w^{\frac{1}{m}}}{} \right\} \geq \left( \text{Re} w \right)^{\frac{1}{m}} \) for \( \text{Re} w > 0 \) and \( m \geq 1 \), we obtain

\[
\text{Re} \left\{ \left( \frac{J_{\mu+1,b}f(z)}{z} \right)^{\frac{1}{m}} \right\} > \left( \frac{b + 1}{1 - \lambda} \int_0^1 t^{\frac{b+1}{1-\lambda}-1} \left( \frac{1 - At}{1 - Bt} \right) dt \right)^{\frac{1}{m}},
\]

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\[ \lambda < 1, \ b > -1 \text{ and } -1 < B < A < 1. \]

**Theorem 6.2.2.** Let \( 0 \leq \alpha < 1 \). Let \( \lambda \) be a complex number with \( \lambda \neq 0 \) and satisfy

\[
\text{either } |2\lambda(1-\alpha)(b+1)-1| \leq 1 \text{ or } |2\lambda(1-\alpha)(b+1)+1| \leq 1. \quad (6.11)
\]

Let \( f \in \mathbb{H} \) be such that

\[
\text{Re} \left\{ \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \right\} > \alpha, \quad z \in E. \quad (6.12)
\]

Then,

\[
\left( \frac{J_{\mu+1,b}f(z)}{z} \right)^{\lambda} \prec \frac{1}{(1-z)^{2\lambda(1-\alpha)(b+1)}} = q(z), \quad z \in E; \quad (6.13)
\]

and \( q(z) \) is the best dominant.

**Proof.** Let \( p(z) = \left( \frac{J_{\mu+1,b}f(z)}{z} \right)^{\lambda} \).

Then, by using equation (6.5), we obtain,

\[
z p'(z) = (b+1)\lambda p(z) \left[ \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} - 1 \right].
\]

from which we obtain,

\[
1 + \frac{zp'(z)}{\lambda(b+1)p(z)} = \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}.\]

Since, \( f \) satisfies (6.12), it follows that,

\[
1 + \frac{zp'(z)}{\lambda(b+1)p(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} = h(z), \quad z \in E. \quad (6.14)
\]
Let
\[ q(z) = \frac{1}{(1-z)^{2\lambda(1-\alpha)(b+1)}} , \quad \theta(w) = 1 \text{ and } \phi(w) = \frac{1}{\lambda(b+1)w}. \]

Then \( q(z) \) is univalent by the conditions (6.11) of the theorem and lemma (??). Also \( q(z), \theta(w) \) and \( \phi(w) \) satisfy conditions of lemma (??). In addition,
\[ Q(z) = zq'(z)\phi(q(z)) = \frac{2(1-\alpha)z}{1-z}, \]
is univalent starlike in \( E \) and
\[ \theta(q(z)) + Q(z) = \frac{1+(1-2\alpha)z}{1-z} = h(z). \]

Hence it readily follows that the conditions (1) and (2) of lemma (??) are satisfied. Hence the result follows from (6.14).

6.2.2 Srivastava-Attiya Operator and the Classes \( S_h^* \) and \( C_h \)

In this section, we show that the classes \( S_h^* \) and \( C_h \) are closed under Srivastava-Attiya operator, for \( \mu \in \mathbb{N}_0 \) and for \( \text{Re} \ b \geq 0 \).

Theorem 6.2.3. Let \( f \in S_h^* \). Then \( J_{n,b}(f) \in S_h^* \) for \( n \in \mathbb{N}_0, \text{ Re} \ b \geq 0 \).

Proof. Let \( f \in S_h^* \).

Then, \( J_{0,b}f(z) = f(z) \), and hence the theorem is true for \( n = 0 \).

For \( n = 1 \),
\[ J_{1,b}f(z) = \frac{1 + b}{z^b} \int_0^z t^{b-1} f(t) dt = L_b(f), \]

which is in \( S_b^* \) by theorem (??), for \( \text{Re } b \geq 0 \). Thus, the theorem is true for \( n = 1 \).

Assume that \( J_{n,b}f(z) \in S_h^* \), i.e.,

\[ \frac{z (J_{n,b}f(z))'}{J_{n,b}f(z)} \prec h(z). \]

Let

\[ p(z) = \frac{z (J_{n+1,b}f(z))'}{J_{n+1,b}f(z)}, \tag{6.15} \]

where \( p(z) \) is analytic and \( p(0) = 1 \).

Using (6.5), we get,

\[ b + p(z) = (b + 1) \frac{J_{n,b}f(z)}{J_{n+1,b}f(z)}. \]

On logarithmic differentiation, we get,

\[ \frac{p'(z)}{b + p(z)} = \frac{(J_{n,b}f(z))'}{J_{n,b}f(z)} - \frac{(J_{n+1,b}f(z))'}{J_{n+1,b}f(z)}. \]

Multiplication by \( z \) and using (6.15), we obtain

\[ \frac{zp'(z)}{b + p(z)} + p(z) = \frac{z (J_{n,b}f(z))'}{J_{n,b}f(z)} \prec h(z). \tag{6.16} \]

Since \( h \in H \) is convex univalent in \( E \) with \( h(0) = 1 \) and \( \text{Re } h(z) > 0 \), the subordination relation (6.16) and lemma (??) implies

\[ p(z) \prec h(z), \]

for \( \text{Re } b \geq 0 \). The theorem follows by induction. \( \square \)
Theorem 6.2.4. Let $f \in C_h$. Then $J_{n,b}f \in C_h$, $n \in \mathbb{N}_0$, $\Re b \geq 0$.

Proof. Let $f \in C_h$. Then

$$g(z) = zf'(z) \in S^*_h.$$ 

Hence $J_{n,b}g(z) \in S^*_h$ from theorem (6.2.3) for $\Re b \geq 0$. i.e.,

$$J_{n,b}(zf'(z)) \in S^*_h$$

$$\Rightarrow z (J_{n,b}f(z))' \in S^*_h$$

$$\Rightarrow J_{n,b}f(z) \in C_h.$$

Corollary 6.2.1. If $f \in S^*_h$, then $J_{n,b}(L_c(f)) \in S^*_h$,

for $\Re c \geq 0$, $\Re b \geq 0$, $n \in \mathbb{N}_0$.

Proof. The corollary follows from theorems (??) and (6.2.3).

Corollary 6.2.2. If $f \in C_h$, then $J_{n,b}(L_c(f)) \in C_h$,

for $\Re c \geq 0$, $\Re b \geq 0$, $n \in \mathbb{N}_0$.

Theorem 6.2.5. Let $f \in S^*_h$. Then,

$$\Re \frac{J_{n,b}f(z)}{J_{n+1,b}f(z)} > \frac{b + \alpha}{b + 1},$$ (6.17)

for all $n \in \mathbb{N}_0$, $b > 0$, where $\alpha = \min_{z \in E} \Re h(z)$.

Proof. From equation (6.5),

$$(b + 1) \frac{J_{n,b}f(z)}{J_{n+1,b}f(z)} = b + \frac{z(J_{n+1,b}f(z))'}{J_{n+1,b}f(z)}.$$ (6.18)
Since, $J_{n,b}f(z) \in S^*_{b,h}$ for all $n$ and for Re $b \geq 0$, we have

$$\frac{z(J_{n+1,b}f(z))'}{J_{n+1,b}f(z)} < h(z).$$

From (6.18), we get

$$(b + 1) \frac{J_{n,b}f(z)}{J_{n+1,b}f(z)} - b < h(z),$$

for all $b > 0$.

Let

$$\alpha = \min \text{Re } h(z).$$

Then, for each $z \in E$,

$$\text{Re} \left( (b + 1) \frac{J_{n,b}f(z)}{J_{n+1,b}f(z)} - b \right) > \alpha,$$

which is equivalent to (6.17), which holds for all $n \in \mathbb{N}_0$, if $b > 0$. 

\subsection*{6.2.3 New Subclasses}

In the following, we introduce classes of analytic functions defined by using the Srivastava-Attiya operator and investigate their inclusion relationships.

\textbf{Definition 6.2.1.}

\begin{align*}
S_{\mu,b,h}(h) &= \{ f \in \mathbb{A} : J_{\mu,b}f(z) \in S^*_{b,h} \} \\
&= \left\{ f \in \mathbb{A} : \frac{z(J_{\mu,b}f(z))'}{J_{\mu,b}f(z)} < h(z) \right\}.
\end{align*}

(6.19)
Definition 6.2.2.

\[ C_{\mu,b}(h) = \left\{ f \in \mathbb{A} : J_{\mu,b}f(z) \in C_h \right\} = \left\{ f \in \mathbb{A} : 1 + \frac{z}{(J_{\mu,b}f(z))'} < h(z) \right\}. \quad (6.20) \]

Remark 6.2.2. Clearly \( f(z) \in C_{\mu,b}(h) \) if and only if \( zf'(z) \in S^{*}_{\mu,b}(h) \).

Definition 6.2.3.

\[ K_{\mu,b}(h) = \left\{ f \in \mathbb{A} : \frac{z}{(J_{\mu,b}f(z))'} < h(z), \quad g \in S^{*}_{\mu,b}(h) \right\}. \quad (6.21) \]

We prove some inclusion relationships for above defined classes.

Theorem 6.2.6. Suppose that \( \text{Re } b \geq 0 \). Then \( S^{*}_{\mu,b}(h) \subset S^{*}_{\mu+1,b}(h) \).

Proof. Let \( f \in S^{*}_{\mu,b}(h) \). Then,

\[ \frac{z}{(J_{\mu,b}f(z))'} < h(z). \]

Let

\[ p(z) = \frac{z}{(J_{\mu+1,b}f(z))'} = \frac{(b+1)J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \quad (6.22) \]

where \( p(z) \) is analytic and \( p(0) = 1 \).

Using (6.5) in (6.22), we get,

\[ b + p(z) = \frac{(b+1)J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}. \]

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On logarithmic differentiation and multiplication by $z$, we arrive at
\[ p(z) + \frac{zp'(z)}{b + p(z)} = \frac{z (J_{\mu,b}f(z))'}{J_{\mu,b}f(z)} < h(z). \] *(6.23)*

Since $h \in H$ is convex univalent in $E$ with $h(0) = 1$ and $\Re h(z) > 0$, for $\Re b \geq 0$, lemma (??) and the subordination relation (6.23) gives
\[ p(z) < h(z), \text{ for } \Re b \geq 0, \]
proving the theorem. \qed

**Theorem 6.2.7.** Suppose that $\Re b \geq 0$. Then $C_{\mu,b}(h) \subset C_{\mu+1,b}(h)$.

**Proof.** Let $f \in C_{\mu,b}(h)$. i.e.
\[ J_{\mu,b}f(z) \in C_h \]
\[ \Rightarrow z (J_{\mu,b}f(z))' \in S_h^* \]
\[ \Rightarrow J_{\mu,b}(zf'(z)) \in S_h^* \]
\[ \Rightarrow zf'(z) \in S_{\mu,b}^*(h) \subset S_{\mu+1,b}^*(h) \]
\[ \Rightarrow f(z) \in C_{\mu+1,b}(h). \] \qed

**Theorem 6.2.8.** Suppose that $\Re b \geq 0$. Then $K_{\mu,b}(h) \subset K_{\mu+1,b}(h)$.

**Proof.** Let $f \in K_{\mu,b}(h)$. Then, by definition (6.2.3), there exists a function $g \in S_{\mu,b}^*(h)$ such that
\[ \frac{z (J_{\mu,b}f(z))'}{J_{\mu,b}g(z)} < h(z). \] *(6.24)*

Using theorem (6.2.6), we get $g(z) \in S_{\mu+1,b}^*(h)$. Thus,
\[ q(z) = \frac{z (J_{\mu+1,b}g(z))'}{J_{\mu+1,b}g(z)} < h(z), \] *(6.25)* 
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where \( q(z) \) is analytic. Let

\[
p(z) = \frac{z \left( J_{\mu+1,b} f(z) \right)'}{J_{\mu+1,b} g(z)},
\]

where \( p(z) \) is analytic and \( p(0) = 1 \). Now, making use of (6.5),

\[
\frac{z \left( J_{\mu,b} f(z) \right)'}{J_{\mu,b} g(z)} = \frac{J_{\mu,b} (zf'(z))}{J_{\mu,b} g(z)} \quad (6.25)
\]

\[
= \frac{z \left( J_{\mu+1,b} (zf'(z)) \right)'}{J_{\mu+1,b} g(z)} + b \frac{J_{\mu+1,b} (zf'(z))}{J_{\mu+1,b} g(z)} \quad (6.26)
\]

Simplifying the above using (6.25) and (6.26), and then using (6.24) we arrive at

\[
p(z) + \frac{zp'(z)}{q(z) + b} = \frac{z \left( J_{\mu,b} f(z) \right)'}{J_{\mu,b} g(z)} < h(z).
\]

Since, \( h \in H \) is convex univalent in \( E \) with \( h(0) = 1 \) and \( \text{Re} \, h(z) > 0 \), lemma (??) and the subordination relation (6.29) gives

\[
p(z) < h(z), \text{ for } \text{Re} \, b \geq 0,
\]

proving the assertion.

Above theorems can be applied with a view to obtain the following consequences.

In the following we give some results involving the Alexander Transform \( A(f) \).

**Corollary 6.2.3.** Suppose that \( f \in \mathbb{A} \). Then,
\[f \in S_{1,0}^*(h) \Rightarrow f \in S_{2,0}^*(h).\]

Equivalently, if
\[A(f) \in S_h^*\]
then,
\[f \in S_{n,0}^*(h), \ n \in \mathbb{N} \setminus \{1\}.\]

**Corollary 6.2.4.** Suppose that \(f \in A\). Then,
\[f \in C_{1,0}(h) \Rightarrow f \in C_{2,0}(h).\]

Equivalently, if
\[A(f) \in C_h\]
then,
\[f \in C_{n,0}(h), \ n \in \mathbb{N} \setminus \{1\}.\]

**Corollary 6.2.5.** Suppose that \(f \in A\). Then,
\[f \in K_{1,0}(h) \Rightarrow f \in K_{2,0}(h).\]

Equivalently, if
\[A(f) \in K_h\]
then,
\[f \in K_{n,0}(h), \ n \in \mathbb{N} \setminus \{1\}.\]

In the following we give some results involving the Bernardi’s integral operator \(L_b(f)\).

**Corollary 6.2.6.** Suppose that \(f \in A\) and \(\text{Re } b \geq 0\). Then,
\[f \in S_{1,b}^*(h) \Rightarrow f \in S_{2,b}^*(h).\]

Equivalently, if
\[L_b(f) \in S_h^*\]
then,
\[f \in S_{n,b}^*(h), \ n \in \mathbb{N} \setminus \{1\}.\]
Corollary 6.2.7. Suppose that \( f \in \mathbb{A} \) and \( \text{Re}\, b \geq 0 \). Then, 
\[
f \in \mathcal{C}_{1,b}(h) \Rightarrow f \in \mathcal{C}_{2,b}(h).
\]
Equivalently, if 
\[
L_b(f) \in \mathcal{C}_h
\]
then, 
\[
f \in \mathcal{C}_{n,b}(h), \ n \in \mathbb{N} \setminus \{1\}.
\]

Corollary 6.2.8. Suppose that \( f \in \mathbb{A} \) and \( \text{Re}\, b \geq 0 \). Then, 
\[
f \in \mathcal{K}_{1,b}(h) \Rightarrow f \in \mathcal{K}_{2,b}(h).
\]
Equivalently, if 
\[
L_b(f) \in \mathcal{K}_h
\]
then, 
\[
f \in \mathcal{K}_{n,b}(h), \ n \in \mathbb{N} \setminus \{1\}.
\]

In a similar manner, one can get many applications of the theorems (6.5.1), (6.5.2) and (6.5.3).