Chapter 5

On Noor Integral Operator

5.1 Abstract

In this chapter, we consider the Noor integral operator $T_n$ and study some of its properties. We show that the classes $S^*_h$ and $C_h$ are preserved under the Noor integral operator. We also show that for $f$ in respective classes, the Bernardi integral operator $L_c(f)$ also is in respective class, from which it follows that $T_n(L_c(f))$ is also in the respective class. Using Noor integral operator, new subclasses will be defined, analogous to the classes $S^*_h$, $C_h$, the class of strongly starlike functions of order $\alpha$ and the class of strongly convex functions of order $\alpha$ and certain inclusion relationships will be obtained.

A part of this work has appeared in the paper titled “On Integral Operators defined on some Subclasses of S”, International Journal of Mathematical Sciences and Applications [63]. A part of this work is submitted for publication in International Journal of Inequalities in Pure and Applied Mathematics.
5.2 Preliminaries

In our proofs, we need the following lemmas.

**Lemma 5.2.1.** Let $\beta$ and $\gamma$ be constants. Let $\psi$ be convex univalent in $E$ with $\psi(0) = 1$ and $\Re(\beta \psi(z) + \gamma) > 0$ for $z$ in $E$. Let $p(z)$ be analytic in $E$ with $p(0) = 1$. Then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \psi(z) \implies p(z) \prec \psi(z).$$

The above lemma is due to P Eenigenburg, S S Miller, P T Mocanu and M O Reade [16] and it is a strong basis in obtaining our results.

**Lemma 5.2.2** (57). Let $\beta$ and $\gamma$ be constants. Let $\psi$ be convex univalent in $E$ with $\psi(0) = 1$ and $\Re(\beta \psi(z) + \gamma) > 0$ for $z$ in $E$. Let $q(z)$ be analytic in $E$ with $q(z) \prec \psi(z)$. Let $p(z)$ be analytic in $E$ with $p(0) = 1$. Then

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \psi(z) \implies p(z) \prec \psi(z).$$

**Lemma 5.2.3** (65). The function $(1 - z)^\lambda \equiv e^{\lambda \log(1 - z)}, \lambda \neq 0$, is univalent in $E$ if and only if $\lambda$ is either in the closed disk $|\lambda - 1| \leq 1$ or in the closed disk $|\lambda + 1| \leq 1$.

**Lemma 5.2.4** (49). Let $q(z)$ be univalent in $E$ and $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $q(E)$ with $\phi(w) \neq 0$ when $w \in q(E)$. Set

$$Q(z) = zq'(z)\phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z)$$

and suppose that

1. $Q(z)$ is starlike(univalent) in $E$;
2. $\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0$, $z \in E$.

If $p(z)$ is analytic in $E$, with $p(0) = q(0)$, $p(E) \subset D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.
Lemma 5.2.5 (54). Let a function \( p(z) = 1 + p_1z + p_2z^2 + \ldots \) be analytic in \( E \) and \( p(z) \neq 0, \ z \in E \). If there exists a point \( z_0 \in E \) such that
\[
|\arg p(z)| < \frac{\pi}{2} \alpha \ (|z| < |z_0|) \text{ and } |\arg p(z_0)| = \frac{\pi}{2} \alpha \ (0 < \alpha \leq 1),
\]
then,
\[
\frac{z_0p'(z_0)}{p(z_0)} = ik\alpha,
\]
where
\[
k \geq \frac{1}{2} \left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = \frac{\pi}{2} \alpha\right),
\]
\[
k \leq -\frac{1}{2} \left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = -\frac{\pi}{2} \alpha\right),
\]
and \( (p(z_0))^{\frac{1}{a}} = \pm ia \ (a > 0) \).

Lemma 5.2.6. (see [69] and [41]) Given \( 0 \leq \alpha < 1 \). Let \( \phi(z) \in C \), \( g(z) \in S^{*}(\alpha) \), \( p(z) \) be analytic in \( E \) with \( \Re \{p(z)\} > \gamma \), \( (\gamma < 1) \). Then,
\[
\Re \left\{ \frac{\phi(z) \ast g(z)p(z)}{\phi(z) \ast g(z)} \right\} > \gamma.
\]

5.3 Noor Integral Operator

Let \( f \in A \). Denote \( D^{\alpha} : A \to A \) the operator defined by
\[
D^{\alpha} f(z) = \frac{z}{(1 - z)^{\alpha+1}} \ast f(z), \ (\alpha > -1).
\]

We note that
\[
D^{0} f(z) = f(z) \text{ and } D^{1} f(z) = zf'(z).
\]

For \( n \in N_{0} = N \cup \{0\} \)
\[
D^{n} f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}.
\]
The operator $D^n f(z)$ is called the Ruscheweyh derivative of $n^{th}$ order of $f$ [68]. Several classes of analytic functions, defined by using this operator, have been studied by many authors.

Inspired by this operator, in 1999, Noor [52] defined an integral operator $T_n : \mathbb{A} \rightarrow \mathbb{A}$ analogous to $D^n$ as follows.

**Definition 5.3.1.** Let

$$f_n(z) = \frac{z}{(1-z)^{n+1}}, \quad n \in \mathbb{N}_0.$$ 

Let $f_n^{(1)}(z)$ be defined such that

$$f_n(z) * f_n^{(1)}(z) = \frac{z}{(1-z)^2}.$$ 

Then

$$T_n f(z) = f_n^{(1)}(z) * f(z).$$

We note that

$$T_0 f(z) = z f'(z) \quad \text{and} \quad T_1 f(z) = f(z).$$

The operator $T_n$ is called the Noor integral operator of the $n^{th}$ order. The Noor integral operator satisfies the recurrence relation given by

$$z(T_{n+1}f(z))' = (n + 1)T_n f(z) - nT_{n+1} f(z). \quad (5.1)$$

This relation plays a significant role in obtaining our results. For the properties and applications of the Noor integral operator, see Noor [52] and Noor and Noor [53].

### 5.3.1 Certain Properties

In this section, we obtain certain properties of Noor integral operator.
Theorem 5.3.1. Let $\lambda < 1$ and $-1 \leq B < A \leq 1$. If $f \in \mathbb{A}$ satisfies

$$
(1 - \lambda) \frac{T_n f(z)}{z} + \lambda \frac{T_{n+1} f(z)}{z} \prec \frac{1 + A z}{1 + B z}, \quad z \in E,
$$

then,

$$
\text{Re} \left\{ \frac{T_{n+1} f(z)}{z} \right\} > \frac{n + 1}{1 - \lambda} \int_0^1 t^{n+1} - 1 \left( \frac{1 - A t}{1 - B t} \right) dt, \quad n \in \mathbb{N}_0.
$$

The result is sharp.

Proof. Suppose that $f \in \mathbb{A}$ satisfies equation (5.2).

Put

$$
p(z) = \frac{T_{n+1} f(z)}{z}.
$$

Then $p(z)$ is analytic in $E$ and $p(0) = 1$.

Using equation (5.1) and equation (5.4) we obtain,

$$
\frac{T_n f(z)}{z} = p(z) + \frac{zp'(z)}{n + 1}.
$$

From (5.2), (5.4) and (5.5), we obtain,

$$
p(z) + \frac{1 - \lambda}{n + 1} z p'(z) \prec \frac{1 + A z}{1 + B z}.
$$

Application of the lemma (??), leads to

$$
p(z) \prec \frac{n + 1}{1 - \lambda} z^{-\frac{n+1}{n+1}} \int_0^z t^{n+1} - 1 \left( \frac{1 + A t}{1 + B t} \right) dt,
$$
i.e.,

$$\frac{T_{n+1}f(z)}{z} = \frac{n + 1}{1 - \lambda} \int_0^1 u^{\frac{n+1}{1-\lambda}-1} \left( \frac{1 + Aw(z)}{1 + Bw(z)} \right) du,$$  \hspace{1cm} (5.6)$$

where $w(z)$ is analytic in $E$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in E$.

Since $-1 \leq B < A \leq 1$, $n \in \mathbb{N}_0$ and $\lambda < 1$, from (5.6) it follows that

$$\text{Re} \frac{T_{n+1}f(z)}{z} = \text{Re} \left\{ \frac{n + 1}{1 - \lambda} \int_0^1 u^{\frac{n+1}{1-\lambda}-1} \left( \frac{1 + Aw(z)}{1 + Bw(z)} \right) du \right\}$$

$$> \frac{n + 1}{1 - \lambda} \int_0^1 u^{\frac{n+1}{1-\lambda}-1} \left( \frac{1 - Au}{1 - Bu} \right) du, \quad z \in E,$$

which is (5.3). To show the sharpness of (5.3), we take $f \in \mathcal{A}$ defined by

$$\frac{T_{n+1}f(z)}{z} = \frac{n + 1}{1 - \lambda} \int_0^1 u^{\frac{n+1}{1-\lambda}-1} \left( \frac{1 + Az}{1 + Bz} \right) du.$$  

For this function, we find that

$$(1 - \lambda) \frac{T_n f(z)}{z} + \lambda \frac{T_{n+1}f(z)}{z} = \frac{1 + Az}{1 + Bz},$$

and that

$$\frac{T_{n+1}f(z)}{z} \to \frac{n + 1}{1 - \lambda} \int_0^1 u^{\frac{n+1}{1-\lambda}-1} \left( \frac{1 - Au}{1 - Bu} \right) du \quad \text{as} \quad z \to -1.$$  

Hence the proof is complete. \hspace{1cm} \Box

**Remark 5.3.1.** From the elementary inequality: $\text{Re} \left\{ w^\frac{1}{n} \right\} \geq (\text{Re} w)^\frac{1}{n}$ for
Re \( w > 0 \) and \( m \geq 1 \), we obtain

\[
\text{Re} \left\{ \left( \frac{T_{n+1}f(z)}{z} \right)^{\frac{1}{m}} \right\} > \left( \frac{n+1}{1-\lambda} \int_0^1 t^{n+1-1} \left( \frac{1-At}{1-Bt} \right) dt \right)^{\frac{1}{m}}.
\]

**Theorem 5.3.2.** Let \( n \in \mathbb{N}_0 \) and \( 0 \leq \alpha < 1 \). Let \( \lambda \) be a complex number with \( \lambda \neq 0 \) and satisfy

either \( |2\lambda(1-\alpha)(n+1) - 1| \leq 1 \) or \( |2\lambda(1-\alpha)(n+1) + 1| \leq 1 \). (5.7)

Let \( f \in \mathbb{A} \) be such that

\[
\text{Re} \left\{ \frac{T_n f(z)}{T_{n+1} f(z)} \right\} > \alpha, \quad z \in E.
\] (5.8)

Then,

\[
\left( \frac{T_{n+1}f(z)}{z} \right)^{\lambda} \prec \frac{1}{(1-z)^{2\lambda(1-\alpha)(n+1)}} = q(z), \quad z \in E
\] (5.9)

and \( q(z) \) is the best dominant.

**Proof.** Let \( p(z) = \left( \frac{T_{n+1}f(z)}{z} \right)^{\lambda} \).

Then, by using equation (5.1), we obtain

\[
z p'(z) = (n+1)\lambda p(z) \left[ \frac{T_n f(z)}{T_{n+1} f(z)} - 1 \right],
\]

from which we obtain,

\[
1 + \frac{zp'(z)}{\lambda(n+1)p(z)} = \frac{T_n f(z)}{T_{n+1} f(z)}.
\]
Since, \( f(z) \) satisfies (5.8), it follows that

\[
1 + \frac{zp'(z)}{\lambda(n+1)p(z)} < \frac{1 + (1-2\alpha)z}{1-z} = h(z), \quad z \in E. \tag{5.10}
\]

Let

\[
q(z) = \frac{1}{(1-z)^{2\lambda(1-\alpha)(n+1)}}, \quad \theta(w) = 1 \quad \text{and} \quad \phi(w) = \frac{1}{\lambda(n+1)w}.
\]

Then \( q(z) \) is univalent by the conditions (5.7) of the theorem and lemma (5.2.3). Also \( q(z), \theta(w) \) and \( \phi(w) \) satisfy conditions of lemma (5.2.4).

In addition,

\[
Q(z) = zq'(z)\phi(q(z)) = \frac{2(1-\alpha)z}{1-z},
\]

is univalent starlike in \( E \) and

\[
\theta(q(z)) + Q(z) = \frac{1 + (1-2\alpha)z}{1-z} = h(z).
\]

Hence it readily follows that the conditions (1) and (2) of lemma (5.2.4) are also satisfied. Hence the result follows from (5.10).

\[ \square \]

**Theorem 5.3.3.** Let \( 0 < \lambda < 1 \) and \( -1 \leq B < A \leq 1 \). If \( f \in \mathbb{A} \) satisfies

\[
(1-\lambda)\frac{T_nL_c(f(z))}{z} + \lambda\frac{T_nf(z)}{z} < \frac{1+Az}{1+Bz}, \quad z \in E, \tag{5.11}
\]

then,

\[
\text{Re} \left\{ \frac{T_nL_c(f(z))}{z} \right\} > \frac{c+1}{\lambda} \int_0^1 t^{\frac{1-\lambda+1}{\lambda}} \left( \frac{1-At}{1-Br} \right) dt, \tag{5.12}
\]

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\( n \in \mathbb{N}_0, \ c > 0. \text{ The result is sharp.} \)

**Proof.** Let \( f \in \mathbb{A} \) satisfy equation (5.11).

Put

\[
p(z) = \frac{T_n L_c(f(z))}{z}.
\]  \hspace{1cm} (5.13)

Then \( p(z) \) is analytic in \( E \) and \( p(0) = 1 \).

Using equation (5.1) and equation (5.13) we obtain,

\[
\frac{T_n f(z)}{z} = p(z) + \frac{zp'(z)}{c+1}.
\]  \hspace{1cm} (5.14)

From (5.11), (5.13) and (5.14), we obtain,

\[
p(z) + \frac{\lambda}{c+1}zp'(z) \prec \frac{1 + Az}{1 + Bz}.
\]

Application of the lemma (??), leads to

\[
p(z) < \frac{c+1}{\lambda} z^{-\frac{c+1}{\lambda}} \int_0^z t^{\frac{c+1}{\lambda}-1} \left( \frac{1 + At}{1 + Bt} \right) dt,
\]

i.e.,

\[
\frac{T_n L_c(f(z))}{z} = \frac{c+1}{\lambda} \int_0^1 u^{\frac{c+1}{\lambda}-1} \left( \frac{1 + Auw(z)}{1 + Buw(z)} \right) du,
\]  \hspace{1cm} (5.15)

where \( w(z) \) is analytic in \( E \), with \( w(0) = 0 \) and \( |w(z)| < 1, \ z \in E \).

Since \(-1 < B < A \leq 1, \ n \in \mathbb{N}_0, \ c > 0 \) and \( 0 < \lambda < 1, \) from (5.15) it follows that

\[
\text{Re} \left\{ \frac{T_n L_c(f(z))}{z} \right\} = \text{Re} \left\{ \frac{c+1}{\lambda} \int_0^1 u^{\frac{c+1}{\lambda}-1} \left( 1 + Auw(z) \right) du \right\}
\]
\[
> \frac{c + 1}{\lambda} \int_0^1 u^{\frac{c+1}{\lambda}} \left( \frac{1 - Au}{1 - Bu} \right) du, \quad z \in E,
\]

which is (5.12).

To show the sharpness of (5.12), we take \( f \in A \) defined by

\[
\frac{T_n L_c(f(z))}{z} = \frac{c + 1}{\lambda} \int_0^1 t^{\frac{c+1}{\lambda}} \left( \frac{1 + At \cdot z}{1 + Bt \cdot z} \right) dt.
\]

For this function, we find that

\[
(1 - \lambda) \frac{T_n L_c(f(z))}{z} + \lambda \frac{T_n f(z)}{z} = \frac{1 + Az}{1 + Bz}, \quad z \in E,
\]

and that

\[
\frac{T_n L_c(f(z))}{z} \to \frac{c + 1}{\lambda} \int_0^1 t^{\frac{c+1}{\lambda}} \left( \frac{1 - At}{1 - Bt} \right) dt, \quad \text{as} \quad z \to -1.
\]

Hence the proof is complete.

\[\square\]

5.3.2 Noor Integral Operator and the Classes \( S_h^* \) and \( C_h \)

In this section, first we show that the classes \( S_h^* \) and \( C_h \) are preserved under Noor integral Operator. Next, if \( L_c(f) \) denotes the Bernardi integral operator, then we show that \( (L_c(f)) \) is in the same class as that of \( f \), from which it follows that \( T_n L_c(f) \) is also in the respective class.

**Theorem 5.3.4.** Let \( f \in S_h^* \). Then \( T_n f \in S_h^* \), for \( n \in \mathbb{N} \).

**Proof.** Let \( f \in S_h^* \).

For \( n = 1 \), \( T_1 f(z) = f(z) \in S_h^* \).
Thus the theorem is true for \( n = 1 \).

Assume that \( T_n f(z) \in S_h^* \), i.e.,

\[
\frac{z (T_n f(z))'}{T_n f(z)} < h(z).
\]

Let

\[
p(z) = \frac{z (T_{n+1} f(z))'}{T_{n+1} f(z)},
\]

where \( p(z) \) is analytic and \( p(0) = 1 \).

Using (5.1), we get

\[
n + p(z) = (n + 1) \frac{T_n f(z)}{T_{n+1} f(z)}.
\]

On logarithmic differentiation, we get

\[
\frac{p'(z)}{n + p(z)} = \frac{(T_n f(z))'}{T_n f(z)} - \frac{(T_{n+1} f(z))'}{T_{n+1} f(z)}.
\]

Multiplying by \( z \) and using (5.16), we obtain

\[
\frac{zp'(z)}{n + p(z)} + p(z) = \frac{z (T_n f(z))'}{T_n f(z)} < h(z).
\]

Since \( h \in H \) is convex univalent in \( E \) with \( h(0) = 1 \) and \( Re \ h(z) > 0 \), the subordination relation (5.17) and lemma (5.2.1) implies

\[
p(z) < h(z),
\]

proving the theorem.

**Theorem 5.3.5.** Let \( f \in C_h \). Then \( T_n f \in C_h, \ n \in \mathbb{N} \).

**Proof.** Let \( f \in C_h \).
Then \( g(z) = zf'(z) \in S^*_h \).

Hence, \( T_n g(z) \in S^*_h \) from theorem (5.3.4). i.e.,
\[
T_n (zf'(z)) \in S^*_h \\
\Rightarrow z (T_n f(z))' \in S^*_h \\
\Rightarrow T_n f(z) \in C_h.
\]
\[\square\]

In the next two theorems, we show that the classes \( S^*_h \) and \( C_h \) are invariant under the Bernardi’s integral operator \( L_c(f) \).

**Theorem 5.3.6.** If \( f \in S^*_h \), then for \( \text{Re } c \geq 0 \), \( L_c(f) \) is in \( S^*_h \).

**Proof.** Let \( f \in S^*_h \). Then,
\[
\frac{zf'(z)}{f(z)} < h(z).
\]
Let
\[
p(z) = \frac{z(L_c(f)(z))'}{L_c(f)(z)}.
\]
Then \( p(z) \) is analytic and \( p(0) = 1 \).

Using (2.1), we get
\[
c + p(z) = (c + 1) \frac{f(z)}{L_c(f)(z)}.
\]
Logarithmic differentiation and multiplication by \( z \) leads to
\[
\frac{zp'(z)}{c + p(z)} + p(z) = \frac{zf'(z)}{f(z)} < h(z). \tag{5.18}
\]
Since \( h \in H \) is convex univalent in \( E \) with \( h(0) = 1 \) and \( \text{Re } h(z) > 0 \), equation (5.18) and lemma (5.2.1) implies
\[
p(z) < h(z), \text{ for } \text{Re } c \geq 0. \]
Theorem 5.3.7. If $f \in C_h$. Then $L_c(f)$ is in $C_h$, for $Re\,c \geq 0$.

Proof. Let $f(z) \in C_h$. Then,
\[
z f'(z) \in S^*_h, \text{ for } Re\,c \geq 0
\]
\[
\Rightarrow L_c(z f'(z)) \in S^*_h
\]
\[
\Rightarrow z(L_c f(z))' \in S^*_h
\]
\[
\Rightarrow L_c f(z) \in C_h.
\]

Corollary 5.3.1. If $f \in S^*_h$, then $T_n(L_c(f)) \in S^*_h$, for $Re\,c \geq 0$, $n \in \mathbb{N}$.

Corollary 5.3.2. If $f \in C_h$, then $T_n(L_c(f)) \in C_h$, for $Re\,c \geq 0$, $n \in \mathbb{N}$.

5.3.3 New Subclasses

In the following, we introduce some subclasses $S^*_n(h), C_n(h)$ and $K_n(h)$ of $\mathbb{A}$ defined by Noor integral operator, analogous to $S^*_h, C_h$ and $K_h$ as follows:

Let $h \in H$ and $n \in \mathbb{N}$.

Definition 5.3.2.

\[S^*_n(h) = \left\{ f \in \mathbb{A} : \frac{z (T_n f(z))'}{T_n f(z)} \prec h(z) \right\}.
\]

Definition 5.3.3.

\[C_n(h) = \left\{ f \in \mathbb{A} : 1 + \frac{z (T_n f(z))''}{(T_n f(z))'} \prec h(z) \right\}.
\]

Remark 5.3.2. Clearly $f \in C_n(h)$ if and only if $z f' \in S^*_n(h)$.

Definition 5.3.4.

\[K_n(h) = \left\{ f \in \mathbb{A} : \frac{z (T_n f(z))'}{T_n g(z)} \prec h(z), \, g(z) \in S^*_n(h) \right\}.
\]
Remark 5.3.3. We note that $S_h^*(h) = S^*_h$, $C_1(h) = C_h$ and $K_1(h) = K(h)$.

Now, we prove some inclusion relationships for the classes defined above.

Theorem 5.3.8. Let $f \in S_n^*(h)$. Then $f$ is also in $S_{n+1}^*(h)$, $n \in \mathbb{N}$.

Proof. Let $f \in S_n^*(h)$. Then

$$\frac{z(T_n f(z))'}{T_n f(z)} \prec h(z).$$

Let

$$p(z) = \frac{z(T_{n+1} f(z))'}{T_{n+1} f(z)},$$

where $p(z)$ is analytic in $E$ and $p(0) = 1$.

Using (5.1) and simplifying we get

$$n + p(z) = (n + 1) \frac{T_n f(z)}{T_{n+1} f(z)}.$$

On logarithmic differentiation and multiplication by $z$, we get

$$\frac{z p'(z)}{n + p(z)} + p(z) = \frac{z(T_n f(z))'}{T_n f(z)} \prec h(z). \quad (5.19)$$

Since $h \in H$ is convex univalent in $E$ with $h(0) = 1$ and $\text{Re } h(z) > 0$,

lemma (5.2.1) and the subordination relation (5.19) gives

$$p(z) \prec h(z),$$

completing the proof.

Remark 5.3.4. From the above theorem, the inclusion relationship

$S_n^*(h) \subset S_{n+1}^*(h)$ holds for all $n$, $n \in \mathbb{N}$.
**Theorem 5.3.9.** The inclusion relationship $C_n(h) \subset C_{n+1}(h)$, holds for each $n, n \in \mathbb{N}$.

**Proof.** Let $f \in C_n(h)$. Then,

$$zf'(z) \in S_n^*(h) \Rightarrow zf'(z) \in S_{n+1}^*(h) \Rightarrow f(z) \in C_{n+1}(h).$$

As consequences of the above two theorems and the theorems (5.3.6) and (5.3.7), we get the following interesting corollaries.

**Corollary 5.3.3.** Suppose that $f \in \mathbb{A}$ and $\text{Re } c \geq 0$. Then

$$f \in S_1^*(h) \Rightarrow f \in S_2^*(h).$$

Equivalently, if

$$f \in S_h^*$$

then

$$L_c(f) \in S_n^*(h), \ n \in \mathbb{N}.$$  

**Corollary 5.3.4.** Suppose that $f \in \mathbb{A}$ and $\text{Re } c \geq 0$. Then

$$f \in C_1(h) \Rightarrow f \in C_2(h).$$

Equivalently, if

$$f \in C_h$$

then

$$L_c(f) \in C_n(h), \ n \in \mathbb{N}.$$  

In the following, we obtain an interesting result pertaining to convolution using the above inclusion relations.

**Theorem 5.3.10.** If $f \in S_1^*\left(\frac{1-(2a-1)z}{1-z}\right)$, then $f \ast \phi \in S_n^*\left(\frac{1-(2a-1)z}{1-z}\right)$,

$\phi \in C, \ n \in \mathbb{N} \setminus \{1\}$.
Proof. Let \( f \in S^*_1\left(\frac{1-(2\alpha-1)z}{1-z}\right) = S^*(\alpha) \).

In theorem (5.3.4), we have proved that the class \( S^*_n \) is invariant under \( T_n \).

Hence,

\[
T_n f(z) \in S^*_1\left(\frac{1-(2\alpha-1)z}{1-z}\right) = S^*(\alpha).
\]

Thus, if

\[
p(z) = \frac{z(T_n f(z))'}{T_n f(z)},
\]

then,

\[
\Re \{p(z)\} > \alpha.
\]

Consider

\[
\frac{z(T_n(f \ast \phi))'}{T_n(f \ast \phi)} = \frac{z(T_n(f) \ast \phi)'}{T_n(f) \ast \phi} = \frac{z(T_n(f))' \ast \phi}{T_n(f) \ast \phi} = \frac{p(z)T_n(f) \ast \phi}{T_n(f) \ast \phi}.
\]

Now, \( T_n(f) \in S^*(\alpha), \ \phi(z) \in C \) and \( \Re p(z) > \alpha \). Applying lemma (5.2.6), we get,

\[
\Re \left\{ \frac{p(z)T_n(f) \ast \phi(z)}{T_n(f) \ast \phi(z)} \right\} > \alpha.
\]

i.e.,

\[
\Re \left\{ \frac{z(T_n(f \ast \phi))'}{T_n(f \ast \phi)} \right\} > \alpha.
\]

i.e.,

\[
\frac{z(T_n(f \ast \phi))'}{T_n(f \ast \phi)} < \frac{1-(2\alpha-1)z}{1-z}.
\]

\[
\Rightarrow f \ast \phi \in S^*_n\left(\frac{1-(2\alpha-1)z}{1-z}\right),
\]

as desired. \( \square \)

**Corollary 5.3.5.** If \( f \in C_1\left(\frac{1-(2\alpha-1)z}{1-z}\right) = C(\alpha) \), then

\[
f \ast \phi \in C_n\left(\frac{1-(2\alpha-1)z}{1-z}\right), \ \phi \in C, \ n \in \mathbb{N} \setminus \{1\}.
\]
Corollary 5.3.6. If $f \in S^*(\alpha)$, then

$$g_1(z) = \int_0^z \frac{f(t)}{t} dt, \quad g_2(z) = \frac{2}{z} \int_0^z f(t) dt,$$

$$g_3(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt, |x| \leq 1, x \neq 1, \text{ and } g_4(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) dt, \text{Re}\{c\} > 0,$$

are all in $S_n^*(\frac{1 - (2\alpha - 1)z}{1 - z})$, $n \in \mathbb{N} \setminus \{1\}$.

Proof. For each $i, i = 1, 2, 3, 4$, note that

$$g_i(z) = f(z) \ast \phi_i(z),$$

where

$$\phi_1(z) = -\log(1 - z), \quad \phi_2(z) = -2 \left( \frac{z + \log(1 + z)}{z} \right),$$

$$\phi_3(z) = \frac{1}{1 - x} \log \left( \frac{1 - xz}{1 - z} \right), \text{ and }$$

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1 + \zeta}{1 + \zeta} \frac{e^{zn}}{n!}, \text{ Re } c > 0,$$

and that $\phi_i(z), i = 1, 2, 3, 4$ are convex functions. \qed

Theorem 5.3.11. Suppose that $n \in \mathbb{N}$. Then $K_n(h) \subset K_{n+1}(h)$.

Proof. Let $f \in K_n(h)$. Then, by definition (5.3.4), there exists a function $g \in S_n^*(h)$ such that

$$z \left( T_n f(z) \right)' \prec h(z). \quad (5.20)$$

Using theorem (5.3.8), we get $g \in S_{n+1}^*(h)$. Thus,

$$q(z) = z \left( T_{n+1} g(z) \right)' \prec h(z), \quad (5.21)$$

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where \( q(z) \) is analytic. Let

\[
p(z) = \frac{z \left( T_{n+1}f(z) \right)'}{T_{n+1}g(z)},
\]

(5.22)

where \( p(z) \) is analytic and \( p(0) = 1 \).

Now, making use of (5.1),

\[
\frac{z \left( T_n f(z) \right)'}{T_n g(z)} = \frac{T_n \left( z f'(z) \right)}{T_n g(z)}
\]

\[
= \frac{z \left( T_{n+1} \left( z f'(z) \right) \right)'}{T_{n+1} g(z)} + \frac{n T_{n+1} \left( z f'(z) \right)}{T_{n+1} g(z)}
\]

\[
= \frac{z \left( T_{n+1} g(z) \right)'}{T_{n+1} g(z)} + n.
\]

Simplifying the above, using (5.21) and (5.22), and then using (5.20) we arrive at

\[
p(z) + \frac{z p'(z)}{q(z) + n} = \frac{z \left( T_n f(z) \right)'}{T_n g(z)} \prec h(z).
\]

(5.23)

Since, \( h \in H \) is convex univalent in \( E \) with \( h(0) = 1 \) and \( \text{Re} \ h(z) > 0 \), lemma (5.2.2) and the subordination relation (5.23) gives

\[
p(z) \prec h(z),
\]

proving the assertion.

Now, we introduce some classes of analytic functions, analogous to the class of strongly starlike functions of order \( \alpha \) and the class of strongly convex functions of order \( \alpha \) and study the properties of the newly defined classes.
Let \(0 < \alpha \leq 1, n \in \mathbb{N}\) and \(z \in E\).

**Definition 5.3.5.**
\[
\tilde{S}^*_n(\alpha) = \left\{ f \in \mathbb{A} : \left| \arg \left( \frac{z(T_nf(z))'}{T_nf(z)} \right) \right| < \frac{\pi}{2}\alpha, \quad \frac{z(T_nf(z))'}{T_nf(z)} \neq 0 \right\}
\]

**Definition 5.3.6.**
\[
\tilde{C}_n(\alpha) = \left\{ f \in \mathbb{A} : \left| \arg \left( 1 + \frac{z(T_nf(z))''}{(T_nf(z))'} \right) \right| < \frac{\pi}{2}\alpha, \quad 1 + \frac{z(T_nf(z))''}{(T_nf(z))'} \neq 0 \right\}
\]

**Remark 5.3.5.** Clearly \(f(z) \in \tilde{C}_n(\alpha)\) if and only if \(zf'(z) \in \tilde{S}^*_n(\alpha)\).

**Remark 5.3.6.** For \(n = 1\), \(\tilde{S}^*_1(\alpha)\) is the class of strongly starlike functions of order \(\alpha\) and \(\tilde{C}_1(\alpha)\) is the class of strongly convex functions of order \(\alpha\).

Now, we prove some inclusion relationships for the above defined classes.

**Theorem 5.3.12.** \(\tilde{S}^*_n(\alpha) \subset \tilde{S}^*_{n+1}(\alpha)\) for each \(n, n \in \mathbb{N}\).

**Proof.** Let \(f \in \tilde{S}^*_n(\alpha)\).

We set
\[
\frac{z(T_{n+1}f(z))'}{T_{n+1}f(z)} = p(z), \quad (5.24)
\]
where \(p(z)\) is analytic in \(E\), \(p(0) = 1\) and \(p(z) \neq 0\) for all \(z \in E\).

Using (5.1) and (5.24), we have
\[
(n + 1) \frac{T_nf(z)}{T_{n+1}f(z)} = n + p(z).
\]

On logarithmic differentiation and multiplication by \(z\), we get,
\[
\frac{z(T_nf(z))'}{T_nf(z)} = \frac{z(T_{n+1}f(z))'}{T_{n+1}f(z)} + \frac{zp'(z)}{n + p(z)}
\]

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\[ p(z) + \frac{zp'(z)}{n + p(z)}. \] (5.25)

Suppose that there exists a point \( z_0 \in E \), such that
\[
|\arg p(z)| < \frac{\pi}{2} \alpha \quad (|z| < |z_0|) \quad \text{and} \quad |\arg p(z_0)| = \frac{\pi}{2} \alpha.
\]

Then, by applying lemma (5.2.5), we can write
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i k \alpha \quad \text{and} \quad (p(z_0))^{\frac{1}{n}} = \pm i a (a > 0).
\]

Thus, if \( \arg p(z_0) = -\frac{\pi}{2} \alpha \), then from (5.25),
\[
\frac{z_0 (T_n f(z_0))'}{T_n f(z_0)} = p(z_0) \left[ 1 + \frac{z_0 p'(z_0)}{p(z_0) n + p(z_0)} \right] = a^\alpha e^{-i \alpha} \left[ 1 + \frac{i k \alpha}{n + a^\alpha e^{-i \alpha}} \right].
\]

This implies that,
\[
\arg \left\{ \frac{z_0 (T_n f(z_0))'}{T_n f(z_0)} \right\} = -\frac{\pi}{2} \alpha + \arg \left[ 1 + \frac{i k \alpha}{n + a^\alpha e^{-i \alpha}} \right] = -\frac{\pi}{2} \alpha + \tan^{-1} \left[ \frac{k \alpha \left( n + a^\alpha \cos \frac{\pi \alpha}{2} \right)}{a^{2 \alpha} + n^2 + a^\alpha \left( 2n \cos \frac{\pi \alpha}{2} - k \alpha \sin \frac{\pi \alpha}{2} \right)} \right] \leq -\frac{\pi}{2} \alpha,
\]

where \( k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \),

which contradicts the condition that \( f \in \tilde{S}_n^*(\alpha) \).
Similarly, if \( \arg p(z_0) = \frac{\pi}{2} \alpha \), then we obtain that

\[
\arg \left\{ \frac{z_0(T_n f(z_0))'}{T_n f(z_0)} \right\} \geq \frac{\pi}{2} \alpha,
\]

which also contradicts the condition that \( f \in \tilde{S}_n^*(\alpha) \). Thus we have,

\[
|\arg \ p(z)| < \frac{\pi}{2} \alpha, \ z \in E
\]
i.e.,

\[
\left| \arg \left\{ \frac{z(T_{n+1} f(z))'}{T_{n+1} f(z)} \right\} \right| < \frac{\pi}{2} \alpha, \ z \in E.
\]

\[
\Rightarrow f \in \tilde{S}_{n+1}^*(\alpha).
\]

**Theorem 5.3.13.** \( \tilde{C}_n(\alpha) \subset \tilde{C}_{n+1}(\alpha) \), for \( n \in \mathbb{N} \).

**Proof.** Let \( f \in \tilde{C}_n(\alpha) \). Then

\[
z f'(z) \in \tilde{S}_n^*(\alpha)
\]

\[
\Rightarrow z f'(z) \in \tilde{S}_n^{n+1}(\alpha)
\]

\[
\Rightarrow f(z) \in \tilde{C}_{n+1}(\alpha).
\]

**Theorem 5.3.14.** Let \( c > 0 \) and \( n \in \mathbb{N} \). If \( f \in \tilde{S}_n^*(\alpha) \) and

\[
\frac{z(T_n L_c(f)(z))'}{T_n L_c(f)(z)} \neq 0 \text{ for all } z \in E, \text{ then } L_c(f) \in \tilde{S}_n^*(\alpha).
\]

**Proof.** Let

\[
p(z) = \frac{z(T_n L_c(f)(z))'}{T_n L_c(f)(z)} \quad (5.26)
\]

where \( p(z) \) is analytic, \( p(0) = 1 \) and \( p(z) \neq 0, \ z \in E \).
Using (2.1), we obtain,

\[ z \left( T_n L_e(f)(z) \right)' = (c + 1) T_n f(z) - c T_n L_e(f)(z). \]  

(5.27)

The equations (5.26) and (5.27) lead to

\[ (c + 1) \frac{T_n f(z)}{T_n L_e(f)(z)} = c + p(z). \]

Logarithmic differentiation and multiplication by \( z \) yields

\[ \frac{z(T_n f(z))'}{T_n f(z)} = p(z) + \frac{zp'(z)}{c + p(z)}. \]  

(5.28)

Suppose that there exists a point \( z_0 \in E \), such that

\[ |\arg p(z)| < \frac{\pi}{2} \alpha, \quad (|z| < |z_0|) \]  

and \( |\arg p(z_0)| = \frac{\pi}{2} \alpha \).

Then, by applying lemma (5.2.5), we obtain

\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik \alpha \quad \text{and} \quad (p(z_0))^\frac{1}{2} = \pm ia(a > 0). \]

Thus, if \( \arg p(z_0) = \frac{\pi}{2} \alpha \), then from (5.28),

\[ \frac{z_0(T_n f(z_0))'}{T_n f(z_0)} \]

\[ = p(z_0) \left[ 1 + \frac{z_0 p'(z_0)}{p(z_0)} \right] \]

\[ = a^\alpha e^{\frac{i\pi}{2}} \left[ 1 + \frac{ik \alpha}{c + a^\alpha e^{\frac{i\pi}{2}}} \right]. \]

This shows that,

\[ \arg \left\{ \frac{z_0(T_n f(z_0))'}{T_n f(z_0)} \right\} \]
\[ \begin{align*}
= & \frac{\pi}{2} \alpha + \arctan \left[ \frac{ik\alpha}{c + a^\alpha e^{i\pi/2}} \right] \\
= & \frac{\pi}{2} \alpha + \tan^{-1} \left[ \frac{k\alpha \left( c + a^\alpha \cos \frac{\pi \alpha}{2} \right)}{a^{2\alpha} + c^2 + a^\alpha \left( 2c \cos \frac{\pi \alpha}{2} + k\alpha \sin \frac{\pi \alpha}{2} \right)} \right] \\
\geq & \frac{\pi}{2} \alpha, \quad \left( \text{where } k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \right),
\end{align*} \]

which is a contradiction to the hypothesis.

Similarly, we can prove the case: \( \arg p(z_0) = \frac{\pi}{2} \alpha. \)

Thus, we conclude that the function \( p(z) \) has to satisfy

\[ |\arg p(z)| < \frac{\pi}{2} \alpha, \quad z \in E. \]

i.e.,

\[ \left| \arctan \left\{ \frac{z(T_n Lc(f)(z))'}{T_n Lc(f)(z)} \right\} \right| < \frac{\pi}{2} \alpha, \quad z \in E. \]

\[ \Rightarrow Lc(f) \in \tilde{S}^*_{\alpha}(c) \]

completing the proof. \( \square \)

**Theorem 5.3.15.** Let \( c > 0 \) and \( n \in \mathbb{N} \). If \( f \in \tilde{C}_n(\alpha) \) and

\[ \frac{z(T_n Lc(f)(z))'}{T_n Lc(f)(z)} \neq 0 \text{ for all } z \in E, \text{ then } Lc(f) \in \tilde{C}_n(\alpha). \]

**Proof.** \( f \in \tilde{C}_n(\alpha) \)

\[ \Rightarrow zf'(z) \in \tilde{S}^*_{\alpha}(c) \]

\[ \Rightarrow Lc(zf'(z)) \in \tilde{S}^*_{\alpha}(c) \]

\[ \Rightarrow z(Lc f(z))' \in \tilde{S}^*_{\alpha}(c) \]

\[ \Rightarrow Lc f(z) \in \tilde{C}_n(\alpha). \] \( \square \)