CHAPTER II

SEMI-CLOSED GRAPH OF A MAPPING

0. INTRODUCTION:

When a mapping is considered with a set-theoretic view of point, it is well determined by its graph. Graph of a mapping \( f : X \rightarrow Y \) is denoted by \( G(f) \), where \( G(f) = \{(x, f(x)) : x \in X\} \) is a subset of the product space \( X \times Y \).

It is well-known that a space \( X \) is Hausdorff iff its diagonal \( \Delta = \{(x, x) : x \in X\} \) is closed in the product space \( X \times X \). The set \( \Delta \) is nothing but the graph of the identity mapping \( i : X \rightarrow X \). Utilizing these considerations, the result mentioned can be re-stated as follows:

**Theorem 0.1.** Space \( X \) is Hausdorff iff the graph of the identity mapping \( i : X \rightarrow X \) is closed.

Again, it is well-known that even a continuous mapping may not have closed graph, and that, on the converse part, a mapping having closed graph need not be continuous. These facts seem to recognize the interest of investigations motivated by the graph and its interaction appeared in Theorem 0.1. In the recent past, various conditions were obtained so that a 'nice' mapping has a closed graph and
also the conditions under which a mapping with closed graph
is of the 'nice' type whichever one may have in mind. These
two types of conditions, appeared in the above discussion,
are being illustrated below with the remark that, in the
choice of illustrations, we restrict ourselves only to
those conditions which have been imposed on the range.

**Theorem 0.2.** [5, Theorem 1.5(3), p. 140]: Let \( f : X \to Y \) be
continuous where \( X \) is arbitrary and \( Y \) is \( T_2 \). Then \( G(f) \) is
closed.

**Theorem 0.3.** [29, Exercise 16.10, page 130]: Let \( f : X \to Y \) be
any mapping where \( Y \) is compact and \( X \) is arbitrary. If \( G(f) \)
is closed, then \( f \) is continuous.

In fact, in the above results, taken together,
the continuity of a mapping of a space into a compact
Hausdorff space is characterised as its closed graph property.

Moreover, attempts have been made recently to
obtain some variants of closed graph itself and a study
of them has also been initiated. The notion of a mapping
with a strongly-closed graph was introduced by L.L. Herrington
and Paul E. Long and used to characterise \( H \)-closed spaces and
\( C \)-compact spaces in [11] and [12], respectively. Further,
Herrington and Long obtained a characterization of $H$-closed Urysohn space using the concept of mappings with $\ast$-closed graphs [13]. A.S. Mashhour et al. began a study of $S$-closed and strongly $S$-closed graphs in [25] wherein their own idea of supraclosed sets has been utilized. Mappings with $\alpha$-closed graphs and pre-closed graphs are due to I.A. Hasanein et al.[10]. James E. Joseph [15] introduced among others a mapping with $\Theta$-subclosed graph with which he succeeded in obtaining a generalization of the 'uniform-boundedness principle' from analysis and other results. Finally, a monograph entitled 'The Closed Graph and $P$-closed Graph properties in General Topology' by T.A. Hamlett and L.L. Herrington [9] is worth to mention.

These considerations inspired me to introduce two more variants of closed graph property of a mapping, namely, semi-closed graph and strongly semi-closed graph. A study devoted to the semi-closed graph concept is being given in the present chapter while the strongly semi-closed graph concept is reserved to be studied in the next Chapter III.

A mapping $f : X \rightarrow Y$ is defined to have a semi-closed graph $G(f) = \{(x, f(x)) : x \in X\}$ whenever $G(f)$ is semi-closed in the product space $X \times Y$. Obviously, if a
mapping has a closed graph, then its graph is semi-closed but not conversely (Example 2.1). If the graph of a mapping $f$ is semi-closed, then the mapping $f^{-1}$ (if exists) has also a semi-closed graph. A general investigation is made to show that, on one hand, various types of known mappings may fail to have semi-closed graphs and, on the other hand, a mapping having semi-closed graph may fail to be some 'nice' type of mappings (e.g., set-$s$-connected mapping introduced in the preceding Chapter 1 and in [4] may fail to have a semi-closed graph and, conversely, a mapping with semi-closed graph may fail to be set-$s$-connected, in general). Amongst others it has been found that a set-$s$-connected mapping $f$ of a space $X$ onto an extremally $s$-disconnected and semi-$T_2$ space $Y$ has a semi-closed graph. Finally, utilizing the concept of $s$-convergence of a net, a property of semi-closed graph is obtained.

1. TERMINOLOGY:

To make this chapter self-contained, the relevant preliminaries are being given with an excuse for their duplication naturally occurred somewhere else in the present study. In a space $X$, a set $A$ is semi-open [21] if there exists an open set $O$ in $X$ such that $O \subseteq A \subseteq \text{cl} O$, where $\text{cl} O$ denotes the closure of $O$ in $X$. Every open set is
semi-open[21]. So \( S_0(X) \) will denote the class of all semi-open sets in the space \( X \). A subset \( M \) of a space \( X \) is a semi-neighbourhood [1] of a point \( x \) of \( X \) if there exists a semi-open set \( A \) in \( X \) such that \( x \in A \subseteq M \). A set \( A \) in \( X \) is semi-open iff it is a semi-neighbourhood of each of its points [1].

A set \( A \) in \( X \) is called semi-closed [2] if \((X-A) \) is semi-open. The semi closure [2] of a subset \( A \) of \( X \), denoted by \( Scl \ A \), is the intersection of all the semi-closed sets containing \( A \).

Note that \( A \subseteq Scl \ A \subseteq Cl \ A \); \( Scl(Scl \ A) = Scl \ A \); \( A \subseteq B \) implies \( Scl \ A \subseteq Scl \ B \); and \( A \) is semi-closed iff \( A = Scl \ A \)[2].

By a semi-clopen set, we shall mean a set \( A \) which is both semi-open and semi-closed.

**Theorem 1.1:** \([A \in S_0(X) \text{ and } B \in S_0(Y)] \) implies \( A \times B \in S_0(X \times Y) \) [21].

2. **Mappings with Semi-Closed Graphs:**

If \( f:X \rightarrow Y \), then \( G(f) = \{(x,f(x)): x \in X\} \) is well known to be the graph of \( f \). Further, if the set \( G(f) \) is closed in the product space \( X \times Y \), then \( f \) is said to have a closed graph. Recall that a subset \( A \) in the product space \( X \times Y \) is closed iff, for each \( (x,y) \notin A \), there exist open \( U \) and \( V \) with \( x \in U \) and \( y \in V \) such that \((U \times V) \cap A = \emptyset \).

**Definition 2.1:** If the graph \( G(f) \) of a mapping \( f:X \rightarrow Y \) is semi-closed in the product space \( X \times Y \), then the mapping \( f \) is said to have a semi-closed graph.
Clearly, for \( f: X \to Y \), the graph \( G(f) \) is semi-closed if, for each \((x, y) \notin G(f)\), there exist semi-open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \((U \times V) \cap G(f) = \emptyset\). It is worth to note that the converse situation is not clear in the view that a semi-open set in the product space may not be expressed as a product of two semi-open sets [21].

A simple but useful condition for a graph of a mapping to be semi-closed is being recorded as given below.

**Theorem 2.1:** The mapping \( f: X \to Y \) has a semi-closed graph if, for each \( x \in X, y \in Y \) such that \( y \neq f(x) \), there exist semi-open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap V = \emptyset \).

**Proof:** Let \( x \in X, y \in Y \) such that \( y \neq f(x) \). Then \((x, y) \notin G(f)\). Since semi-open sets \( U \) and \( V \) are such that \( f(U) \cap V = \emptyset \), it follows that \( U \times V \) is a semi-open set, by Theorem 1.1, in \( X \times Y \) such that \((U \times V) \cap G(f) = \emptyset\). Hence, \( G(f) \) is semi-closed.

**Remark 2.1:** Each closed graph is, obviously, semi-closed. But the converse may fail to be true, this may be seen by the following example.
**Example 2.1:** Let \( X = \{a, b, c\} \) and \( T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) be the topology on \( X \). Let \( i : X \to X \) be the identity mapping. Then, obviously, \( G(i) \) is semi-closed but not closed.

The following characterization has already been obtained in Chapter I and also in [4].

\( f : X \to Y \) is set-s-connected iff, for any semi-clopen subset \( F \) of \( f(X) \), \( f^{-1}(F) \) is semi-clopen in \( X \).

**Remark 2.2:** Various types of known mappings, viz., a continuous mapping, a \( \sigma \)-continuous mapping [6], a weakly continuous mapping [20], a mapping almost continuous in the sense of Frolik [7], a semi-continuous mapping [21], a mapping almost continuous in the sense of Husain [14], a mapping almost continuous in the sense of M.K.Singal and A.R.Singal [28], a \( c \)-continuous mapping [8], a \( c^* \)-continuous mapping [26], an \( H \)-continuous mapping [22], an \( s \)-continuous mapping [17], a set-connected mapping [18], a set-\( s \)-connected mapping[4], an irresolute mapping[3], a connected mapping[27], a semi-connected mapping[19], and a weak semi-connected mapping[16] may fail to have a semi-closed graph. This may be seen by the following example.

**Example 2.2:** Let \( X \) be a space containing more than one point, with the indiscrete topology, and let \( i : X \to X \) be the identity mapping. Here, the graph \( G(i) \) is not semi-closed.
REMARK 2.3.: A mapping having a semi-closed graph may fail to be continuous, semi-continuous [21], irresolute [3], set-connected [18], set-s-connected [4], connected [27], semi-connected [19], weak semi-connected [16], s-continuous [17], H-continuous [22], c-continuous [8], c'-continuous [26], almost continuous in the sense of Frolik [7], almost continuous in the sense of Husain [14], almost continuous in the sense of M.K. Singal and A.R. Singal [28], weakly continuous [20], \( \Theta \)-continuous [6]. It may be seen by the following example.

**EXAMPLE 2.3.** Let \( X = \{a, b, c\} \), and consider on \( X \) the topologies

\[
\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \quad \text{and} \quad \mathcal{T}' = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.
\]

Then, obviously, the identity mapping \( i : (X, \mathcal{T}) \to (X, \mathcal{T}') \) has a semi-closed graph, but \( i \) is none of the mappings mentioned in Remark 2.3.

In the view of above Remark 2.2, a search for a condition under which the graph may be semi-closed arises in a natural way. Our next consideration with this objective requires some more concepts being given below.

**DEFINITION 2.2.** A space \( X \) is said to be extremally s-disconnected if the semi-closure of every semi-open set is semi-open [4].
**DEFINITION 2.3.1:** A space $X$ is semi-$T_2$ if to each pair of distinct points $x,y$ of $X$ there exist disjoint semi-open sets $A$ and $B$ in $X$ such that $x \in A$ and $y \in B$.

Equivalently, $X$ is semi-$T_2$ iff to each pair of distinct points $x,y$ of $X$ there exists a semi-open set $N$ in $X$ containing $x$ such that $y \not\in scl N$ [4].

**THEOREM 2.3.1:** Let $f: X \rightarrow Y$ be a set-$s$-connected surjection, and $Y$ be an extremally $s$-disconnected and semi-$T_2$ space. Then $G(f)$ is semi-closed in $X \times Y$.

**Proof:** Let $(x,y) \not\in G(f)$. Then $x \in X$, $y \in Y$ and $y \not\in f(x)$. Since $Y$ is semi-$T_2$, there exists a semi-open set $N \subset Y$ containing $y$ such that $f(x) \not\in scl N = V$. Then, $Y$ being extremally $s$-disconnected, $V$ is a semi-clopen subset of $Y$ and so, $f$ being set-$s$-connected surjection, $f^{-1}(V)$ is semi-clopen in $X$ and $x \not\in f^{-1}(V)$. Taking $U = X - f^{-1}(V)$, $U$ is a semi-open set containing $x$, and then $f(U) \cap V = \emptyset$. Consequently, $G(f)$ is semi-closed.

Recall that if a mapping $f: X \rightarrow Y$, $Y$ extremally $s$-disconnected and semi-$T_2$, is surjective with semi-closed graph, then $f$ may not be set-$s$-connected. This can be easily verified by the Example 23 considered earlier.

**THEOREM 2.3.2:** Let $f: X \rightarrow Y$ be a semi-continuous mapping, where $Y$ is $T_2$. Then $G(f)$ is semi-closed.
**Proof:** Let \((x, y) \notin G(f)\). Then \(x \in X, y \in Y,\) and \(y \neq f(x)\). Hence, \(Y\) being \(T_2\), there exist disjoint open sets \(U\) and \(V\) containing \(f(x)\) and \(y\), respectively. Since \(f\) is semi-continuous, \(f^{-1}(U)\) is a semi-open set containing \(x\). Then, clearly, \(f^{-1}(U) \times V\) is a semi-open subset containing \((x, y)\) in \(X \times Y\), by Theorem 1.1. Now, let \(x_1 \in f^{-1}(U)\) such that \(f(x_1) \notin V\). Then, obviously, we have \(U \cap V \neq \emptyset\), a contradiction. Hence, there is no \(x_1 \in f^{-1}(U)\) such that \(f(x_1) \in V\). Consequently, \((f^{-1}(U) \times V) \cap G(f) = \emptyset\) which implies \(G(f)\) is semi-closed.

**Theorem 2.4.** Let \(f : X \rightarrow Y\) be irresolute, where \(Y\) is semi-\(T_2\). Then \(G(f)\) is semi-closed.

The proof of the above Theorem 2.4 is, essentially, similar to that of Theorem 2.3.

The following corollary is an immediate consequence of Theorem 2.4.

**Corollary 2.1.** [24, Theorem 5.3]: If \(X\) is semi-\(T_2\), then the diagonal \(\Delta\) in \(X \times X\) is semi-closed.

**Definition 2.4.** [23]: A space \(X\) is locally \(s\)-connected if for every point \(x \in X\) and every open set \(O\) containing \(x\), there exists an open \(s\)-connected set \(G\) such that \(x \in G \subseteq O\).

**Definition 2.5.** [23]: Two subsets \(A\) and \(B\) of a space \(X\) are said to be semi-separated if \(A \cap \text{ scl } B = \emptyset = \text{ scl } A \cap B\).
**Theorem 2.5:** [23] If $A$ and $B$ are open, $e$-connected and non-semi-separated sets in the space $X$, then $A \cup B$ is $e$-connected.

**Theorem 2.6:** [23] A space $X$ is not $e$-connected iff it is the union of two non-empty, disjoint, semi-open sets.

**Definition 2.6:** A mapping $f: X \rightarrow Y$ is said to be almost irresolute on $X$ if for each $x \in X$ and for each semi-neighbourhood $V$ of $f(x)$, $scl f^{-1}(V)$ is a semi-neighbourhood of $x$.

An almost irresolute mapping need not have a semi-closed graph. For, the mapping $i$, defined in Example 2.2, is almost irresolute, but $G(i)$ is not semi-closed.

However, we have the following result.

**Theorem 2.7:** Let $f: X \rightarrow Y$ be an almost irresolute mapping and $Y$ a locally $e$-connected, $T_2$ space. If $f$ and $f^{-1}$ map open $e$-connected sets onto open $e$-connected sets, then $G(f)$ is semi-closed.

**Proof:** Let $(x, y) \notin G(f)$. Then $x \in X$, $y \in Y$, and $y \neq f(x)$. Since $Y$ is a $T_2$ locally $e$-connected space, there exist open $e$-connected subsets $U$ and $V$ containing $y$ and $f(x)$, respectively, such that $U \cap V = \emptyset$. Hence $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Then, we have $f^{-1}(U) \cap scl f^{-1}(V) = \emptyset$. For, if possible, let
$f^{-1}(U) \cap \text{scl } f^{-1}(V) \neq \emptyset$. Then, by Definition 2.5, by hypothesis and by Theorem 2.5, $f^{-1}(U) \cup f^{-1}(V)$ is open, s-connected. Again, by hypothesis, $f[f^{-1}(U) \cup f^{-1}(V)] = U \cup V$ is open s-connected, which is impossible in view of Theorem 2.6. Now, since $f$ is almost irresolute, $\text{scl } f^{-1}(V)$ is a semi-neighbourhood of $x$ and hence there exists a semi-open set $T \subset \text{scl } f^{-1}(V)$ containing $x$. Therefore, $f(T) \cap U = \emptyset$. Consequently, $G(f)$ is semi-closed.

**Remark 2.4:** A mapping having a semi-closed graph may fail to be almost irresolute. For, the mapping $i$, defined in Example 2.3, has a semi-closed graph but is not almost irresolute.

**Definition 2.7.** $f: X \rightarrow Y$ is said to be a semi-homeomorphism if $f$ is bijective, irresolute, and pre-semi-open. $X$ and $Y$ are said to be semi-homeomorphic.

**Definition 2.8.** A property which is preserved under semi-homeomorphisms is said to be a semi-topological property.

Hence, the property of a subset of a space being semi-closed is a semi-topological property.
THEOREM 2.8: If \( f: X \rightarrow Y \) is bijective and has a semi-closed graph, then \( f^{-1} \) has a semi-closed graph.

PROOF: Let \( G(f) \) and \( G(f^{-1}) \) denote the graphs of \( f \) and \( f^{-1} \), respectively. Then, clearly, \( G(f) = \{(x,y) : y = f(x), x \in X\} \), whereas \( G(f^{-1}) = \{(y,x) : x = f^{-1}(y), y \in Y\} \). Since
\[f^{-1}(y) = x \text{ iff } y = f(x), \]
it follows that \( G(f) \) and \( G(f^{-1}) \) are semi-homeomorphic under the semi-homeomorphism \((x,y) \rightarrow (y,x)\) of \( X \times Y \) onto \( Y \times X \). Since \( G(f) \) is semi-closed, so is \( G(f^{-1}) \).

DEFINITION 2.9a: A net \( \{x_\alpha : \alpha \in D\} \), in a space \( X \), s-converges to a point \( x_0 \in X \) if it is eventually in every semi-neighbourhood of \( x_0 \).

THEOREM 2.9b: If \( Y \) is a semi-open set in a space \( X \), then no net in \( X - Y \) can s-converge to a point of \( Y \).

PROOF: Let \( \{x_\alpha : \alpha \in D\} \) be a net in \( X \) such that \( \{x_\alpha : \alpha \in D\} \)
s-converges to a point \( x_0 \in Y \). Since \( Y \) is semi-open, and so,
a semi-neighbourhood of \( x_0 \), \( \{x_\alpha : \alpha \in D\} \) is eventually in \( Y \).
This means that \( \{x_\alpha : \alpha \in D\} \) is always in \( Y \), that is, \( \{x_\alpha : \alpha \in D\} \) is never in \( X - Y \).
The following corollary is an immediate consequence of Theorem 2.9.

**Corollary 2.2.** If $A$ is a semi-closed set in $X$, then no net in $A$ can s-converge to a point of $X - A$.

**Definition 2.10.** If a net $\{x_\alpha : \alpha \in D\}$ in space $X$, s-converges to a point $x_0 \in X$, then we say that $x_0$ is a semi-limit point of $\{x_\alpha : \alpha \in D\}$. Symbolically, we write $x_0 \in \text{semi-limit}_s(x_\alpha)$.

**Theorem 2.10.** Let $f: X \rightarrow Y$ be a mapping. If $G(f)$ is semi-closed and if, for each net $\{x_\alpha : \alpha \in D\}$ in $X$ s-converging to $x \in X$, the net $\{f(x_\alpha) : \alpha \in D\}$ s-converges to $y \in Y$, then $y = f(x)$.

**Proof:** Let $\{x_\alpha : \alpha \in D\}$ be a net in $X$ s-converging to $x \in X$ such that $\{f(x_\alpha) : \alpha \in D\}$ s-converges to $y \in Y$. Then, clearly, $(x_\alpha, f(x_\alpha)) \in G(f)$ for all $\alpha \in D$. Since $G(f)$ is semi-closed, by Corollary 2.2 and Definition 2.10, semi-limit$_s(x_\alpha, f(x_\alpha)) \in G(f)$. But semi-limit$_s(x_\alpha, f(x_\alpha)) = (x, y)$. Hence $(x, y) \in G(f)$ which implies $y = f(x)$. 
REFERENCES


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