CHAPTER 0

INTRODUCTION

1. TOPOLOGY:

Topology arose as a branch of geometry, but, in the course of its development, topology has outgrown its geometrical origins and today stands alongside analysis and algebra as one of the most fundamental branches of mathematics, and has interactions with many other branches of mathematics.

There are many domains in the broad field of topology, of which the following are only a few: the homology and cohomology theory of complexes, and of more general spaces as well; dimension theory; the theory of differentiable and Riemannian manifolds and of Lie groups; the theory of continuous curves; the theory of Banach and Hilbert spaces and their operators, and of Banach algebras; and abstract harmonic analysis on locally compact groups.

Historically speaking, topology has followed two principal lines of development. On one hand, in homology theory, dimension theory, and the study of manifolds, the
basic motivation appears to have come from geometry. In these fields, topological spaces are looked upon as geometrical configurations, and the emphasis is placed on the structure of the spaces themselves. On the other hand, in the theory of Banach and Hilbert spaces and Banach algebras, the modern theory of integration, and abstract harmonic analysis on locally compact groups, the main stimulus has been analysis. Continuous functions are the chief objects of interest here, and topological spaces are regarded primarily as carriers of such functions and as domains over which they can be integra-
ted.

Much of the beauty of mathematics is derived from the fact that it affords 'abstraction'. Consider the development of modern topology. Many different topological spaces were studied individually over long periods of time before the 'general and abstract' concept of 'topological space' was defined. Topology is qualitative mathematics. Roughly speaking, topology is concerned with those intrinsic qualitative properties of special configurations that are independent of size, location and shape. A topological space can be thought of as a set from which has been swept away all structure irrelevant to the continuity of functions defined on it.

Considerations of a topological nature, depending on the concepts of limit and continuity, originated together
with the oldest problems of geometry and mechanics such as the calculation of areas and the movement of figures.

Roots of the topological nature may be seen involved in the problems considered by Euler and Gauss. To Cauchy (1821) [18] and Abel (1823) [1] is due the credit for having defined the concepts of convergent sequences and series, and the concept of a continuous function with the rigor that is so familiar to us today.

The word 'topology' is derived from the Greek word 'τόπος', meaning 'a place'. The term 'Topology' apparently originated in Listing's 'Vorstudien zur Topologie', which was published in 1847 in Göttingen Studien, pp. 811–875. The synonymous terms 'Analysis situs', 'Geometria situs' were used even earlier; Gauss employed the latter term in 1833 [2, pp. 497–498]. The first mathematician, who attempted to isolate the idea of a topological space and who sensed its far reaching importance, was Niemann (1851) [94].

Then came Cantor's investigations of 1874. Simultaneously, the theory of real numbers was erected on a solid foundation by Dedekind and Cantor. The systematic study of the concepts of closed set, open set, perfect set, an accumulation point, situated in n-dimensional Euclidean space, is linked to the work of Cantor [17].
Despite the opposition of mathematicians who were hostile to new ideas, discoveries followed one another, particularly by Poincaré [90], Hadamard [51], Borel [14], Baire [9], Lebesgue, Klein, Mittag-Leffler, Ascoli [7], Pincherle. To Volterra [100], we owe a systematic study (1887) of line functions (or functionals according to the terminology adopted since Hadamard). An epoch-making step of progress was achieved, in [59], by the introduction of the so-called Hilbert spaces, later defined axiomatically by von Neumann (1927).

The existence of so many examples of spaces like the Euclidean spaces and their subspaces and various function spaces, in which topological considerations find natural applications, gave rise to the necessity of a synthesizing approach which would permit the study of the properties held simultaneously by all these spaces and would, consequently, bring about a better comprehension of the peculiar aspects of each one of them.

In 1906, M. Fréchet [49] introduced the idea of metric spaces and the still more general idea of topological spaces. However, whilst the idea of metric space was soon recognized as a very useful tool, the attempt of M. Fréchet to give a system of axioms defining topological spaces, as well as the efforts of F. Riesz [95] and [96], remained only attempts, and it
was F. Hausdorff [56] who succeeded in extracting a simple axiomatic system which is the corner-stone of the present-day general topology.

Series of systems of axioms were formulated, some equivalent to others, in order to define the notion of topological space. In these systems of axioms the basic terms used are open set by Bourbaki[15], closure operator by Kuratowski [61], neighbourhood by Hausdorff [50], accumulation point by Fréchet [49] and [50], convergence by Hilbert[59] and so on. The source of the axioms which define a topology on an abstract non-empty set \( X \) is some of the properties of the collection of all open subsets of real numbers.

Among the various systems of axioms which define a topological space, the following is established to be more convenient in due course of present-day development.

"A topology \( \mathcal{T} \) on a non-empty set of points \( X \) is a non-empty family of subsets of \( X \) such that the family is 'closed' under finite intersections and arbitrary unions. The members of \( \mathcal{T} \) are called open sets and the structure \((X, \mathcal{T})\), or, simply \( X \) when there is no confusion, is called a topological space."

To conclude we may add that the topology has grown up enormously in the recent years not only in depth but also in
breadth. As a result one may speak today of pure and applied
topology and it is surely a sign of the sound health of
mathematics. A summary of the recent past situation and main
developments in general topology may be found, to some extent,
in excellent but brief survey articles by P.S. Alexandroff[3],
[4] and [5].

2. SEMI-OPEN SETS :

To attain the generalizations of open sets, in general
topology, many mathematicians have shown their considerable
interest in the recent past. As a result, various sets, viz,
early open (pre-open) sets, \( \alpha \)-sets, semi-open sets (\( \beta \)-sets),
etc., have been introduced as the generalizations of open sets.
We shall confine ourselves here to one such generalization
which is called 'semi-open sets'. To Levine [62] is due the
credit for having introduced this concept of semi-open sets
in 1963. Since then the interest of the topologists continues
in the search and deeper studies of the implications of this
concept.

By employing the concept of semi-open sets some basic
tools, namely, semi-neighbourhood, semi-closed set, and semi-
limit point have been well developed in their own right. Semi-
open sets, evidently, give rise to new separation axioms,
covering axioms, mappings, connectedness property, etc. Involving semi-open sets some new separation axioms of the types of standard $T_i$-axioms, $R_1$-regularity axioms have been achieved. An axiomatic study of semi-closure operator was initiated and developed by Crossley and Hildebrand in [19]. Some basic concepts, like semi-topological property and semi-topological class, having fundamental nature, have been established by Crossley and Hildebrand [21]. With a categorical point of view, we feel that these fast growing concepts will play a fundamental role.

2.1. Basic Notions:

In 1963, Levine [62] introduced the concept of semi-open sets. There he also studied some of its basic properties.

**Definition 2.1.1 [62]:** A set $A$ in a topological space $X$ is termed semi-open if there exists an open set $O$ in $X$ such that $O \subseteq A \subseteq \text{cl} O$, where $\text{cl} O$ denotes the closure of $O$ in $X$.

It has been remarked there in [62] that in a topological space every open set is semi-open, but the converse need not be true. Consequently, in a topological space $X$, the family of all the open sets is, in general, a subfamily of the family of all the semi-open sets. Also, he [62] proved that any union of semi-open sets is semi-open, while the intersection of two semi-open sets is not necessarily semi-open.
Regarding the behaviour of semi-open sets in the product space, the following result is due to Levine [62]:

**Theorem 2.1.1 [62]:** Let \( X_1 \) and \( X_2 \) be topological spaces. If \( A_1 \in S \cup (X_1) \) and \( A_2 \in S \cup (X_2) \), then \( A_1 \times A_2 \in S \cup (X_1 \times X_2) \), where \( S \cup (x) \) denotes the class of all the semi-open subsets of the space \( x \).

Also, Levine [62] remarked that if \( X = X_1 \times X_2 \), \( X_i (i = 1,2) \) being topological spaces, and \( A \in S \cup (X) \), then it is not true, in general, that \( A \) is a union of the sets of the form \( A_1 \times A_2 \) where \( A_1 \in S \cup (X_1) \) and \( A_2 \in S \cup (X_2) \).

In [72] and [75], Noiri gave a generalization of Theorem 2.1.1 as follows:

**Theorem 2.1.2 [72],[75]:** Let \( \{ X_{\alpha} : \alpha \in \Delta \} \) be any family of topological spaces and \( A = \prod_{j=1}^{n} A_{\alpha j} \times \prod_{j=1}^{n} X_{\beta j} \) a non-empty subset of the product space \( \prod_{\alpha} X_{\alpha} \), where \( n \) is a positive integer. Then \( A \) is semi-open in \( \prod_{\alpha} X_{\alpha} \) iff \( A_{\alpha j} \) is semi-open in \( X_{\alpha j} \) for each \( j (j = 1,2,\ldots,n) \).

In 1965, Bohn and Lee [13] introduced the concept of semi-topological groups as a generalization of topological
groups. They achieved this generalization by introducing the notion of a semi-neighbourhood of a point as follows:

**Definition 2.1.2 [13]**: A set \( A \) of a topological space \( X \) is a semi-neighbourhood of a point \( x \in X \) if there exists a semi-open set \( A \in X \) such that \( x \in A \subseteq \overline{A} \).

In [62], Levine has remarked that, in general, the complement of a semi-open set is not semi-open. It seems that the above-cited remark led Biswas [11] in 1970, and others, namely, Crossley and Hildebrand [19] in 1971, and P. Das [26] in 1973, to introduce and to study, independently, the concept of semi-closed sets as follows:

**Definition 2.1.3**: A subset \( A \) of a topological space \( X \) is said to be semi-closed if there exists a closed set \( F \) in \( X \) such that \( \text{Int}(F) \subseteq A \subseteq F \) iff \( \text{Int(cl}\ A) \subseteq A \) iff \( A \subseteq \overline{A} \) is semi-open in \( X \), \( \text{Int} F \) denotes the interior of \( F \) in \( X \).

In fact, every closed set is necessarily semi-closed, but the converse is false, in general. Also, intersection of an arbitrary collection of semi-closed sets is semi-closed, but the union of two semi-closed sets is not necessarily semi-closed. The semi-closure of a set \( A \) is defined to be the intersection of all the semi-closed sets containing it, and hence \( A \) is semi-closed iff it equals to its semi-closure.
Bitwas [11] used $C_s(n)$, Crossley and Hildebrand [19] $A_s$, and P. Das [26] $A_s^a$, whereas we use $scl_\Lambda$ to denote the semi-closure of $\Lambda$.

Crossley and Hildebrand [19] introduced semi-interior of a set which also has been defined by P. Das [26] perhaps being unaware of the work of Crossley and Hildebrand [19].

**Definition 2.1.4:** In a topological space $\Lambda$, the union of all the semi-open sets contained in a set $\Lambda$ is called the semi-interior of $\Lambda$.

Crossley and Hildebrand [19] denoted the semi-interior of $\Lambda$ by $\Lambda_0$, and P. Das [26] by $A_s^O$. We denote it by $sint_{\Lambda}$.

Crossley and Hildebrand [19] and later P. Das [26], though, established independently several results concerning semi-closure and semi-interior, some of the results of P. Das [26] overlapped with those of Crossley and Hildebrand [19]. Some more results on semi-closure and semi-interior were obtained by Crossley and Hildebrand in [20].

Further, a pre-semi-closure operator was introduced in [19] as follows:

**Definition 2.1.5** [19]: Let $(\ )_c$ be an operator on subsets of $X$ such that:

(1) $\emptyset_c = \emptyset$;
(ii) \( A \subseteq A_c \)

(iii) \( (A_c)_c = A_c \)

(iv) \( (A \cup B)_c \supseteq (A_c \cup B_c) \), where \( A, B \) are the subsets of \( X \).

Then \( (\_)_c \) is called a pre-semi-closure operator.

It is demonstrated there in [19] that if a pre-semi-closure operator is related to a Kuratowski closure operator in a particular fashion, then the pre-semi-closure operator corresponds exactly to the semi-closure in the topological space generated by the Kuratowski closure operator. A method by which a Kuratowski closure operator may be constructed from a pre-semi-closure operator is also developed in [19]. Again, a criterion is found in [19] to guarantee that the pre-semi-closure operator is properly related to the Kuratowski closure operator constructed from it, in which case the pre-semi-closure operator is called a semi-closure operator. Also, it is shown in [19] that the topology generated by the constructed Kuratowski closure operator is the finest topology for which a semi-closure operator is the semi-closure in the topological space.

In 1973, Ochiai [36] defined a pre-semi-interior
operator on $\Lambda$ as a mapping $\mathcal{U}$ of the power set $\mathcal{P}(\Lambda)$ into $\mathcal{P}(\Lambda)$ satisfying the following conditions:

(i) $\mathcal{U}(\emptyset) = \emptyset$;

(ii) $\mathcal{U}(\Lambda) \subseteq \Lambda$;

(iii) $\mathcal{U}(\mathcal{U}(\Lambda)) = \mathcal{U}(\Lambda)$; and

(iv) $\mathcal{U}(\Lambda \cap B) \subseteq \mathcal{U}(\Lambda) \cap \mathcal{U}(B)$, where $\Lambda, B \in \mathcal{P}(\Lambda)$.

This leads him [86] to obtain the pre-semi-closure operator of Crossley and Hildebrand [19]. Giving several examples, Achiab studies in detail the inter-relations among various operators, viz, closure operator, interior operator, operator of taking the complements on $\Lambda$, pre-semi-interior operator, pre-semi-closure operator, semi-interior operator [86], semi-closure operator [86]. In the same year, i.e., in 1973, P. was [26] also introduced the notions of semi-limit point, semi-derived set, semi-border, semi-frontier, and semi-exterior of a set using the concept of semi-open sets and obtained some of their elementary properties.

2.2 Separation Axioms:

A number of new separation axioms have appeared recently wherein various degrees of separation are considered through the semi-open concept. Maheshwari and Prasad initiated
a study of separation axioms sought through the concept of semi-open sets. They introduced the separation axioms semi-$T_0$, semi-$T_1$ and semi-$T_2$ in [63], ($n_0$) in [65], s-regular in [64], s-normal in [66], and almost s-regular and almost s-normal in [91]. Vorsett introduced semi-$T_1$ axiom in [30] and established in [32] that this axiom is independent of the $K_1$-axiom. Recently, strongly s-regular, strongly s-normal, k-regular, and k-normal axioms have been introduced by Yadav [102]. For further work on this topic one may refer to [30], [32], [34], [35], [36], [38], [70], [78], [84], [92] and [93]. Bube and Sengar introduced a semi-$T_1$ space in [39] and studied some of its basic properties in [39] and [40].

2.3 Coverring Axioms:

Vorsett [37] introduced the concept of semi-compactness and characterised it in terms of semi-convergence of nets. He [37] also established relationship between semi-compactness and compactness. We may also refer to [33] and [36] for semi-compactness. Using semi-open sets, Yadav [102] introduces local s-compact spaces, s-Lindelöf and s-separable spaces, countable s-compact spaces. Therein, he [102] investigates some of the basic properties of these concepts and of s-compactness. Following the terminology of Yadav [102], by s-compact space means a space in which every semi-open cover
has a finite subcover. In [97], Thompson used semi-open sets to introduce the concept of \( \sigma \)-closed spaces given below:

**Definition 2.3.1 [97]**: A topological space \( X \) is \( \sigma \)-closed if, for every semi-open cover \( \{ U_\alpha : \alpha \in \Delta \} \) of \( X \), there is a finite subfamily \( \{ U_{\alpha_1} : i = 1, 2, \ldots, n \} \) so that \( \bigcup_{i=1}^{n} \text{cl}(U_{\alpha_i}) = X \).

These spaces have also been studied by Noiri in [77], [79], [82] and [83], Cameron in [16], Herrmann in [57] and [58], Asha Rathur in [70], Crossley in [25], Joseph and Kwack in [60], Hashour and Hasanein in [68], Sange in [101], Thompson in [98] and [99], Atia with Elleeb and Hasanein in [8], and others. \( \sigma \)-closed subspaces were investigated in [79] by Noiri. Also, Noiri defined locally \( \sigma \)-closed spaces in [79] and obtained some basic properties of such spaces in [79] and [83].

### 2.4. Connectedness

Semi-connectedness is defined by P. Das [27], Pipitone and Russo [69], and Maheshwari and Tapl [67] explicitly in terms of semi-open sets. I. Das [27] obtained some results on semi-connectedness analogous to those on connectedness. Vorsett [31] has further investigated this notion. An extensive investigation of this notion may be seen in [67] where this notion is renamed as \( e \)-connectedness. Two sets \( A \) and \( B \)
of a topological space $X$ are termed semi-separated if

\[ A \cap \text{scl} B = \emptyset = B \cap \text{scl} A, \]

A space $X$ is s-connected if it is not the union of two non-empty semi-separated sets iff no non-empty proper subset of $X$ is both semi-open and semi-closed (respectively semi-closed) sets. Every s-connected space is connected, but not conversely. Again, if $E$ be an open, s-connected set and $A \subset \subset \text{scl} E$, then $F$ is s-connected and, consequently, scl $F$ is s-connected. Image of an s-connected space under a semi-continuous surjection is connected while under an irresolute surjection its image is s-connected. Further, s-connectedness is found to be a topological property.

Recently, dubie and ranwar in [41] and [42] have introduced a symmetric relation between the pair of subsets of a space $X$ which is called 's-connectedness between sets'.

**Definition 2.4.1** [41]: A space $X$ is said to be 's-connected between subsets $A$ and $B$' if there exists no semi-clopen set $F$ of $X$ such that $A \subsetneq F$ and $F \cap B = \emptyset$ (semi-clopen set means a set which is both semi-closed and semi-open).

Clearly, the s-connectedness between sets is meaningful for a pair of non-empty sets. Further, if a space $X$ is s-connected between a pair of sets, then the space $X$ may fail to be s-connected. However, one may consider this as the notion
which localizes, to some extent, the requirement of the s-connectedness of the space. In fact, we have the following result.

**Theorem 2.4.1 [42]:** A space $\mathcal{X}$ is s-connected iff it is s-connected between every pair of its non-empty subsets.

**2.5. Mapping:**

Taking semi-open sets into consideration, Levine [62] introduced the concept of semi-continuous mappings as a generalization of the concept of continuous mappings as follows:

**Definition 2.5.1 [62]:** A mapping $f$ from a topological space $\mathcal{X}$ into a topological space $\mathcal{Y}$ is said to be semi-continuous if the inverse image under $f$ of every open subset of $\mathcal{Y}$ is a semi-open subset of $\mathcal{X}$.

Neither upper nor lower semi-continuous mappings in the usual sense (a mapping $f: \mathcal{X} \rightarrow \mathcal{Y}$ is lower [upper] semi-continuous iff for every $x$ in $\mathcal{X}$, $f^{-1}(x, \infty)$ [$f^{-1}(-\infty, x)$] is open in $\mathcal{X}$) are necessarily semi-continuous in the sense of Levine [62]. Some of the results of [62] are:
**Theorem 2.5.1** [62]: Let \( f : X \rightarrow Y \) be a continuous and open mapping. Then, if \( A \) is semi-open in \( X \), \( f(A) \) is semi-open in \( Y \).

**Theorem 2.5.2** [62]: Let \( f : X \rightarrow Y \) be semi-continuous and \( Y \) a second axiom space. Then the set of points of discontinuity of \( f \) is of the first category.

**Theorem 2.5.3** [62]: Let \( f_i : X_1 \rightarrow Y_1 \) be semi-continuous for \( i = 1, 2 \) and let \( f : X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) be defined by \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). Then \( f \) is semi-continuous.

**Theorem 2.5.4** [62]: Let \( f : X \rightarrow X_1 \times X_2 \) be semi-continuous where \( X, X_1, X_2 \) are topological spaces. Then \( f_i : X \rightarrow X_i \) is semi-continuous for \( i = 1, 2 \).

In 1967, Anderson and Jensen [6] proved the following theorem.

**Theorem 2.5.5** [6]: If \( f : X \rightarrow Y \) is a continuous and open mapping, then \( f^{-1}(B) \) is a semi-open subset of \( X \) for each semi-open subset \( B \) of \( Y \).

Results given in Theorems 2.5.3, 2.5.4 and 2.5.5 are sharpened by Noiri[72], respectively, as follows:
THEOREM 2.5.6 [72]: Let \( \{ X_\alpha : \alpha \in \Delta \} \) and \( \{ Y_\alpha : \alpha \in \Delta \} \) be any two families of topological spaces with the same index set \( \Delta \). For each \( \alpha \in \Delta \), let \( f_\alpha : X_\alpha \to Y_\alpha \) be a mapping. Then, a mapping \( f : \prod X_\alpha \to \prod Y_\alpha \) defined by \( f((x_\alpha)) = (f_\alpha(x_\alpha)) \) is semi-continuous iff \( f_\alpha \) is semi-continuous for each \( \alpha \in \Delta \).

THEOREM 2.5.7 [72]: Let \( \{ X_\alpha : \alpha \in \Delta \} \) be any family of topological spaces. If \( f : X \to \prod X_\alpha \) is a semi-continuous mapping, then \( p_\alpha \circ f : X \to X_\alpha \) is semi-continuous for each \( \alpha \in \Delta \), where \( p_\alpha \) is the projection of \( \prod X_\alpha \) onto \( X_\alpha \).

THEOREM 2.5.8 [72]: If \( f : X \to Y \) is an open and semi-continuous mapping, then \( f^{-1}(B) \) is semi-open in \( X \) for each semi-open \( B \) in \( Y \).

Also, in 1970, Biswas [11] succeeded in obtaining several characterizations of semi-continuous mappings. Some further work on semi-continuous mappings was carried out by Biswas in [10] and [12], Crossley and Hildebrand in [20], Noiri in [72], [74], [75], [80], [81] and [85], Neubrunnová in [71], Hamlett in [53], [54] and [55], Thompson in [98], Mashhour et al. in [69], Penot and Thera in [87] and others. In [54], T.N. Hamlett examines to what extent several
properties of continuous mappings hold for semi-continuous mappings. He [54] proved that the product of two semi-continuous mappings is semi-continuous if one of them is continuous, whereas it need not be semi-continuous in general. It is also shown that a discontinuity of a semi-continuous mapping is non-removable [54]. Dorsett [37] established relationship between semi-continuity and continuity.


**DEFINITION 2.5.2 [10]**: A subset \( A \) of a topological space \( X \) is simply open iff there exist two subsets \( B \) and \( C \) (either of which may be void), where \( B \) is open and \( C \) is nondense, such that \( B \cup C \subset A \subset \text{cl}(B \cup C) \).

He [10] remarked there that every semi-open set is simply open, but the converse may not hold. Utilising simply open sets, Biswas [10] introduced the concept of simply continuous mappings as follows:

**DEFINITION 2.5.3 [10]**: A mapping \( f : X \rightarrow Y \), where \( X \) and \( Y \) are topological spaces, is simply continuous if \( f^{-1}(B) \) is simply open in \( X \) for every open \( B \) in \( Y \).
It was observed there in [10] that the semi-continuity due to Levine [62] implies simple continuity but the converse may be false. He [10] also generalized Theorem 2.5.2 as follows:

**Theorem 2.5.9** [10]: If \( f : X \to Y \) is a simply continuous mapping and \( Y \) is a second axiom space, then the set of points of discontinuity of \( f \) is of the first category.

In addition to these, Biswas [10] introduced the concepts of semi-open mappings and semi-homeomorphism and established certain results related to them. According to him [10]

**Definition 2.5.4** [10]: A mapping \( f : X \to Y \) is semi-open if \( f(A) \) is semi-open in \( Y \) for each open \( A \) in \( X \).

Obviously, every open mapping is semi-open but the converse is not true, in general [10]. Also, by an example, he [10] showed that the image of a semi-open set under a semi-open mapping may fail to be semi-open. But he [10] succeeded in improving the Theorem 2.5.1 as follows:

**Theorem 2.5.10** [10]: If \( f : X \to Y \) be continuous and semi-open and if \( A \) is semi-open in \( X \), then \( f(A) \) is semi-open in \( Y \).
A mapping \( f : X \rightarrow Y \) is said to be a semi-homeomorphism in the sense of Biswas \([10]\) if it is injective, surjective, semi-open and continuous. Note that Crossley and Hildebrand \([21]\), also introduced the concept named semi-homeomorphism but in a different sense. To avoid the confusion, hereafter, the semi-homeomorphism in the sense of Biswas will be called semi-homeomorphism \((B)\) and by semi-homeomorphism we shall mean the semi-homeomorphism in the sense of Crossley and Hildebrand. Clearly, every homeomorphism is a semi-homeomorphism \((B)\) but the converse may not hold. We may also refer to \([71]\) for simple continuity. Later on, Noiri\([75]\), Dasgupta and Lahiri \([29]\) obtained some more properties of semi-open mappings. Noiri \([75]\) proved also the following theorem.

**Theorem 2.5.11** \([75]\): Let \( \{X_\alpha : \alpha \in \Delta\} \) and \( \{Y_\alpha : \alpha \in \Delta\} \) be arbitrary two families of topological spaces with the same set of indices, let \( f_\alpha : X_\alpha \rightarrow Y_\alpha \) be a mapping for each \( \alpha \in \Delta \) and surjective for all but at most a finite number of indices. Then, the product mapping \( f : \prod X_\alpha \rightarrow \prod Y_\alpha \) defined by \( f((x_\alpha)) = (f_\alpha(x_\alpha)) \) is semi-open iff \( f_\alpha \) is semi-open for each \( \alpha \in \Delta \).

The above Theorem 2.5.11 is a generalization of the following theorem.
**Theorem 2.5.12** [10]: Let \( f_i : x_i \to Y_i \) be a semi-open mapping for \( i = 1, 2 \), and let \( f : x_1 \times x_2 \to Y_1 \times Y_2 \) be a mapping defined as \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \), then \( f \) is semi-open.

In [73], Noiri introduced the notion of semi-closed mappings as a generalization of closed mappings as follows:

**Definition 2.5.5** [73]: A mapping \( f : x \to Y \) is semi-closed if \( f(F) \) is semi-closed in \( Y \) for each closed \( F \) in \( x \).

It is shown there in [73] that every closed mapping is semi-closed but the converse may not hold. He [73] also obtained some characterizations and basic properties of semi-closed mappings.

Again, Biswas [12] extended the class of semi-continuous mappings with the introduction of pseudo-continuous mappings as follows:

**Definition 2.5.6** [12]: A mapping \( f : x \to Y \) is said to be pseudo-continuous if, for every open set \( U \) in \( Y \), \( f^{-1}(U) = O \cup F \), where \( O \) is open and \( F \) is of first category in \( x \) (either of them may be empty).

He [12] observed that this class of pseudo-continuous mappings includes the class of upper and lower semi-continuous mappings.
mappings as well as the class of semi-continuous mappings defined by Levine [62]. Pseudo-continuity need not imply semi-continuity [12]. [71] may also be referred for pseudo-continuity.

Further, Crossley and Hildebrand [21] introduced the notions of irresolute and pre-semi-open mappings as follows:

**DEFINITION 2.5.7 [21]:** A mapping $f : X \to Y$ is irresolute [respectively pre-semi-open] if, for every semi-open subset $A$ of $Y$ [resp. of $X$], $f^{-1}(A)$ [resp. $f(A)$] is semi-open in $X$ [resp. in $Y$].

This diversion of terminology is perhaps due to the fact that the terms semi-continuous mappings and semi-open mappings were already reserved for the mappings for which inverses of open sets are semi-open [62] and for which direct images of open sets are semi-open [10], respectively. Every irresolute mapping is semi-continuous but not conversely, whereas the concepts of continuous and irresolute mappings are independent [21]. It is clear that every pre-semi-open mapping is semi-open but converse is not necessarily true. In [85], Noiri investigated some of the interrelations amongst the various concepts of weakly-continuous, semi-continuous,
irresolute, semi-open and pre-semi-open mappings. In [21], Crossley and Hildebrand also introduced the concept of semi-homeomorphism as follows:

DEFINITION 2.3.8 [21]: Topological spaces \( X \) and \( Y \) are said to be semi-homeomorphic if there exists a mapping \( f : X \rightarrow Y \) such that \( f \) is injective, surjective, irresolute and pre-semi-open. Such an \( f \) is called a semi-homeomorphism.

The notions of irresolute mappings, pre-semi-open mappings and semi-homeomorphisms, defined in [21], have also been introduced in [28] and are called demi-continuous mappings, demi-open mappings and demi-homeomorphisms, respectively. Semi-homeomorphisms have also been studied in [81] and [88]. If \( f : X \rightarrow Y \) is a homeomorphism, then \( f \) is a semi-homeomorphism [21]. A semi-homeomorphism need not be a homeomorphism [21]. It is also obvious that semi-homeomorphism \((\beta)\) and the semi-homeomorphism [21] are independent notions. Semi-homeomorphic is an equivalence relation between topological spaces [21].

Recently, Dube and Panwar have introduced some new mappings in [41], [42], [45], [46], [47] and [48]. Some of them imply while some of them are implied by the irresolute mappings.
2.6. SEMI-TOPOLOGICAL PROPERTIES AND CLASSES

Crossley and Hildebrand [21] defined a semi-topological property as:

**DEFINITION 2.6.1 [26]**: A property which is preserved under semi-homeomorphisms is said to be a semi-topological property.

A semi-topological property is a topological property [21]. Crossley and Hildebrand [21] established that $T_0$, $T_1$, regularity, complete normality, normality, $T_3$, $T_4$, $T_5$, compactness, paracompactness, local connectedness, first countability, Lindelöf, and metrizability are not semi-topological properties, whereas $T_2$, separability and connectedness are. The images of compact sets are not necessarily compact, and the images of connected sets are not necessarily connected under a semi-homeomorphism [21]. However, the image of an open, connected subset under a semi-homeomorphism is connected [21]. The property that a space be of the first (resp. second) category is a semi-topological property [21]. The strongly Hausdorff, Urysohn, semi-$T_0$, semi-$T_1$ and semi-$T_2$ properties of a topological space are shown to be the semi-topological properties by
Crossley [24]. A Hausdorff topological space $X$ is said to be strongly Hausdorff [52] if, for each infinite subset $A \subseteq X$, there is a sequence $\left\{ U_n : n \in \mathbb{N} \right\}$ ($\mathbb{N}$ is the set of positive integers) of pairwise disjoint open sets such that $A \bigcap U_n \neq \emptyset$ for each $n \in \mathbb{N}$. Also, Crossley [25] pointed out that semi-connectedness is a semi-topological property.

Hamlett [55] showed that the property of a topological space being a Baire space is semi-topological. In [25], Crossley also gives characterizations of several properties which are semi-topological, viz., first and second category, Hausdorff, separable, connected, Baire, strongly Hausdorff, and Urysohn, in terms of semi-open sets. Further, he [25] gives an axiomatic development of semi-open set structure.

In [98], Thompson proved that $s$-closed is a semi-topological property. Strongly $s$-regular, strongly $s$-normal, $s$-compact, $s$-Lindelöf, and $s$-separable all are shown to be semi-topological properties in [102].

It has been already pointed out that the concept of semi-open set generalizes the concept of an open set. That is, a topology $\mathcal{T}$ on $X$ is a subfamily of $SO(\mathcal{X}, \mathcal{T})$. Crossley and Hildebrand furnished two examples in [19] one of which shows that different topologies on $X$ may have the same family of semi-open sets and the other showed that there may exist topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_2 \subseteq \mathcal{T}_1$, on $X$ for which $SO(\mathcal{X}, \mathcal{T}_2) \subseteq SO(\mathcal{X}, \mathcal{T}_1)$. 


Crossley and Hildebrand [21] introduced the notion of semi-topological classes as follows:

'If \( \mathcal{X} \) is a set of points and if \((\mathcal{X}, \mathcal{T}_1)\) and \((\mathcal{X}, \mathcal{T}_2)\) are two elements of \( T(\mathcal{X}) \) \( T(\mathcal{X}) \) denotes the collection of all topological spaces which have \( \mathcal{X} \) as their set of points, then \((\mathcal{X}, \mathcal{T}_1)\) is semi-correspondent to \((\mathcal{X}, \mathcal{T}_2)\) if \( \mathcal{SO}(\mathcal{X}, \mathcal{T}_1) = \mathcal{SO}(\mathcal{X}, \mathcal{T}_2) \). Semi-correspondence is an equivalence relation on the collection \( T(\mathcal{X}) \). Thus \( T(\mathcal{X}) \) is partitioned into equivalence classes. The equivalence classes of \( T(\mathcal{X}) \) under the relation of semi-correspondence are called the semi-topological classes of \( \mathcal{X} \).'

Let \([\mathcal{T}]\) denote the equivalence class of topologies on \( \mathcal{X} \) with the same collection of semi-open sets as \( \mathcal{T} \).

It has been shown in [19] that \([\mathcal{T}]\) contains a maximal topology in the sense that the topology induced on \( \mathcal{X} \) by the semi-closure operator is finer than any other topology in \([\mathcal{T}]\), and, of course, the topology so induced gives a topology in \([\mathcal{T}]\). If \((\mathcal{X}, \mathcal{T})\) is a topological space, then the finest topology in \([\mathcal{T}]\) is denoted by \( F(\mathcal{T}) \). They [21] showed by an example that although a semi-topological class contains a finest topology, it need not contain a coarsest topology. Further, they [21] exhibited an example by which it is possible to have two semi-correspondent topologies on a
set \( X \) where one space is regular, completely normal, normal, \( T_3, T_4, T_5 \), paracompact, Lindelöf, and metrizable, and the other space satisfies none of these properties. A theorem in [21] states that image of a nowhere dense subset is nowhere dense under a semi-homeomorphism. Consequently, two semi-corrrespondent topologies on a set \( X \) determine precisely the same nowhere dense subsets and hence the same subsets of first category [53]. Some more results on semi-corrrespondent topologies are given in [53].

Again, Crossley [22] shows:

**THEOREM 2.6.1** [22] : (a) If \((X, \mathcal{T})\) is a topological space, and if \( \mathcal{T} \subset \mathcal{C}(X) \), then \( \sigma \in [\mathcal{T}] \).

(b) If \((X, \mathcal{T})\) and \((X, \mathcal{\sigma})\) are semi-corrrespondent, then, if \( \mathcal{T} \lor \mathcal{\sigma} \) is the usual join in the lattice of topologies \( \mathcal{T}(X) \), \( \mathcal{T} \lor \mathcal{\sigma} \in [\mathcal{T}] = [\mathcal{\sigma}] \).

(c) If \((X, \mathcal{T})\) and \((X, \mathcal{\sigma})\) are semi-corrrespondent, it is not necessarily the case that \( \mathcal{T} \cap \mathcal{\sigma} \in [\mathcal{T}] \).

(d) If \((X, \mathcal{T})\) is a topological space and \( V \) is the collection of all subsets which are nowhere dense in \((X, \mathcal{T})\), then \( F(\mathcal{T}) = \{ U \setminus N : U \in \mathcal{T} \text{ and } N \in V \} \).
Another characterization of $F(\mathcal{J})$ has also been given in [23].

2.7. SOME MORE CONCEPTS:

In [23], Crossley and Hildebrand introduced a semi-invertible space as follows:

**DEFINITION 2.7.1 [23]:** A space $X$ is said to be semi-invertible if, for each semi-open $\mathcal{S} \subseteq X$, there exists a semi-homeomorphism $h : X \rightarrow X$ such that $h(\mathcal{X} - \mathcal{S}) \subseteq \mathcal{S}$; equivalently, iff, for each non-void open set $O \subseteq X$, there exists a semi-homeomorphism $h : X \rightarrow X$ such that $h(\mathcal{X} - O) \subseteq O$.

Also, they [23] investigated several results regarding semi-invertible spaces some of which are as follows:

(a) Every invertible space is semi-invertible.

(b) If $(X, \mathcal{J})$ is semi-invertible, then $(X, F(\mathcal{J}))$ is invertible.

(c) If $(X, \mathcal{J})$ is semi-invertible, then each member of the semi-topological class $[\mathcal{J}]$ is semi-invertible.

(d) Every semi-invertible space is not invertible.
In [40], Bube and Sengar have recently obtained a condition under which two semi-$T_D$ spaces are semi-homeomorphic and, hence, in turn, are homeomorphic provided their families of semi-open sets are lattice isomorphic. It has been established there, in fact, that:

**Theorem 2.7.1** [40]: Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ be semi-$T_D$ spaces such that $SC(X, \mathcal{T})$ and $SC(Y, \mathcal{T}^*)$ both be closed under finite unions, $SC(X, \mathcal{T})$ and $SC(Y, \mathcal{T}^*)$ denote the classes of semi-closed subsets of $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$, respectively. If $SO(X, \mathcal{T})$ is lattice isomorphic to $SO(Y, \mathcal{T}^*)$, then

(a) $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ are semi-homeomorphic.
(b) $(X, F(\mathcal{T}))$ and $(Y, F(\mathcal{T}^*))$ are homeomorphic.
(c) As in particular if $F(\mathcal{T}) = \mathcal{T}$ and $F(\mathcal{T}^*) = \mathcal{T}^*$, then $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ are homeomorphic.

In [43] and [44], Bube and Hanwar have initiated the study of some variants of the idea of closed graph of a mapping. These variants, viz., semi-closed graph and strongly semi-closed graph, are achieved in such a way that semi-closed graph is a weaker form of the version of closed graph, and, strongly semi-closed graph is, on one hand, stronger than semi-closed graph and, on the other hand, independent of closed graph. Amongst other investigations,
various conditions are discussed under which the graph of some of 'nice' mappings is semi-closed or strongly semi-closed.

In this brief survey, we have attempted only to give an idea of a few concepts which are derived by the consideration of the concept of semi-open sets and the study of which has been initiated in the recent past. We have tried to make this survey exhaustive but it is, by no means, claimed to be complete.

The concept of semi-open sets still continues to be a widely utilized tool in the context of general topology. The important work of Crossley and Hildebrand [21] specially related to semi-topological classes is of fundamental nature. We conclude with the remark that this concept (of semi-topological classes) with a categorical point of view seems to play a fundamental role and the situation in a few years might be considerably more significant.
REFERENCES


[23] Crossley, S.Gene; Hildebrand, S.K.,

[24] Crossley, S.Gene,

[25] Crossley, S.Gene,

[26] Das, Phullendu,
Note on some applications of semi-open sets, Progr. Math. (Allahabad) 7 (1973), No. 1, 33-44. MR 48 # 9632.

[27] Das, Phullendu,

[28] Das, Phullendu,

[29] Dasgupta, H.; Lahiri, B.K.,

[30] Dorsett, C.,


[46] Dube, K.K. and O. S. Panwar,
A note on completely irresolute mappings (Under communication).

[47] Dube, K.K. and O. S. Panwar,
A note on weakly irresolute mappings (Under communication).

[48] Dube, K.K. and O. S. Panwar,
A generalization of irresolute mappings (Under communication).

[49] Fréchet, M.,

[50] Fréchet, M.,
Les Espaces abstraites,
Gauthier Villars, Paris, 1926.

[51] Hadamard, J.,

[52] Hajnal, A. and I. Juhasz,

[53] Hamlett, T. A.,


[85] Noiri, T.,
Semi-continuity and weak continuity,

[86] Ochiai, H.,

[87] Penot, J.P. and Thera, M.,

[88] Piotrowski, Z.,
MR81a: 54013.

[89] Pignone, V., and Russo, G.,

[90] Poincaré, H.,

[91] Prasad, R.,
The role of semi-open sets in topology, Ph.D. Dissertation (1976),
Univ. of Saugar, Sagar (M) India.

[92] Prasad, R. and Yadav, R.S.,


[100] Volterra, V., Theory of functionals, London-Glasgow (Blackie and Son), 1930.
     MR83a : 54023

[102] Yadav, R.S., Generalizations of certain topological spaces, Ph.D. dissertation (1982), Univ. of Saugar, Sagar( M), India.

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