CHAPTER VI

A GENERALIZATION OF IRRESOLUTE MAPPINGS

0. INTRODUCTION:

The concept of continuity in Topology is the central core for having its wide applications to analysis and geometry and also to any other flourishing branch of mathematics in its own way. Further, continuous mappings are found to be essential as the carrier of any of the topological properties.

The continuous mappings are characterized explicitly by means of open sets. More precisely, a continuous mapping is a mapping under which inverse of every open set is open. In the recent past, various generalizations of the continuous mappings have appeared as a pervading tool to attain various significant results fruitfully in the context of general topology. Here, we are to point out below a way which appears to be followed most commonly in the literature from time to time for getting a generalization of continuity.
"In this technique, simply a restriction is to be made to get ready only some of the open sets (of course, not all of the open sets) for having their inverse images as open ".

This way of approach is seen being pursued in obtaining the various generalizations of continuity, viz., \( c \)-continuity [6], \( c^* \)-continuity[11], \( \mathcal{M} \)-continuity [9], \( s \)-continuity [7], etc.

Further, in 1972, Crossley and Hildebrand [3] introduced the concept of irresolute mappings which are defined explicitly in terms of semi-open sets. The irresolute mappings are being seen to be established as a key tool for having their central role naturally in the context relevant to semi-open sets in general topology. The concept of irresolute mappings, being the carrier of semi-topological properties [3], is expected certainly to occupy its own importance centred around semi-open sets as the continuity has its importance regarding with open sets in topology.

With these considerations and employing the idea with which J.K. Kohli [7] introduced \( s \)-continuity as a
generalization of continuity, we initiate to obtain a
generalization of irresolute mappings. In fact, we introduce
c-irresolute mappings under which inverse image of any
semi-open set with connected complement is semi-open.

The concept of irresolute mappings is stronger
than that of semi-continuity introduced in [8] but it is
independent of the continuity [3]. In this chapter, the
introduced c-irresolute mapping is found to be independent
of semi-continuous mapping. Some of the characterizations
of a c-irresolute mapping have been noticed. Further, some
of the basic properties, like the graph, composition and
restriction of such mappings, are also investigated.

1. TERMINOLOGY

A set $A$ in a space $X$ is semi-open if there
exists an open set $O$ in $X$ such that $O \subseteq \text{Cl}_X(A)$, where
$\text{Cl}_X$ denotes the closure of $C$ in $X$. Every open set is
semi-open but not conversely. Complement of a semi-open
set in a space $X$ is a semi-closed set in $X$. Every closed
set is semi-closed but not conversely. The intersection
of all the semi-closed sets containing $A$ is the semi-
closure of $A$ in $X$ and is denoted by $\text{sc}l_A$. Since any
intersection of semi-closed sets is semi-closed, \( \text{sc} \{ A \} \) is always semi-closed if \( A \subseteq \text{cl} A \). A set \( A \subseteq X \) is semi-closed iff \( A = \text{sc} \{ A \} \). A set \( A \subseteq X \) is a semi-neighbourhood of a point \( x \in X \) if there exists a semi-open set \( O \) in \( X \) such that \( x \in O \cap X \). A set is semi-open in \( X \) iff it is a semi-neighbourhood of each of its points. A mapping \( f : X \rightarrow Y \) is irresolute if inverse of every semi-open set in \( Y \) is semi-open in \( X \), equivalently, iff \( f(\text{sc} \{ A \}) \subseteq \text{sc} f(A) \) for all \( A \subseteq X \) [3]. A space \( X \) is semi-\( T_2 \) if, to each pair of distinct points \( x, y \) of \( X \), there exist disjoint semi-open sets \( A \) and \( B \) in \( X \) such that \( x \in A, y \in B \). A space \( X \) is semi-\( T_1 \) if to each pair of distinct points \( x, y \) of \( X \) there exists a semi-open set \( A \) in \( X \) containing \( x \) but not \( y \) and a semi-open set \( B \) in \( X \) containing \( y \) but not \( x \). Every semi-\( T_2 \) space is semi-\( T_1 \). Any union of semi-open sets is semi-open.

2. c-IRRESOLUTE MAPPINGS:

**Definition 2.1:** A mapping \( f : X \rightarrow Y \) is said to be c-irresolute if, for each \( x \in X \) and each semi-open set \( V \) containing \( f(x) \) and having a connected complement, there exists a semi-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).
A characterization theorem for c-irresolute mappings is now given below.

**Theorem 2.1**: Let \( f : X \rightarrow Y \) be a mapping. Then the following statements are equivalent:

(a) \( f \) is c-irresolute.

(b) If \( V \) is a semi-open subset of \( Y \) with connected complement, then \( f^{-1}(V) \) is a semi-open subset of \( X \).

(c) If \( K \) is a semi-closed and connected subset of \( Y \), then \( f^{-1}(K) \) is semi-closed in \( X \).

**Proof**: (a) \( \Rightarrow \) (b). Let \( V \) be a semi-open subset of \( Y \) with connected complement. Let \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Therefore there is a semi-open subset \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \). Thus \( x \in U \subseteq f^{-1}(V) \). Therefore \( f^{-1}(V) \), being a semi-neighbourhood of each of its points, is semi-open.

(b) \( \Rightarrow \) (a): Let \( x \in X \) and let \( V \) be a semi-open set containing \( f(x) \) and having connected complement. Then \( f^{-1}(V) = U \) (say) is a semi-open set containing \( x \) such that \( f(U) \subseteq V \).
(b) $\implies$ (c). Let $K \subseteq Y$ be a semi-closed and connected set. Then, obviously, $Y - K$ is semi-open with connected complement and, therefore, $f^{-1}(Y - K) = x - f^{-1}(K)$ is semi-open. So $f^{-1}(K)$ is semi-closed.

(c) $\implies$ (b). Let $V \subseteq Y$ be a semi-open set with connected complement. Then, obviously, $Y - V$ is a semi-closed, connected set. Therefore $f^{-1}(Y - V) = x - f^{-1}(V)$ is semi-closed and so $f^{-1}(V)$ is semi-open.

**Corollary 2.1:** Every irresolute mapping is c-irresolute.

Following example shows that the converse of Corollary 2.1 is not true, in general.

**Example 2.1:** Let $X = \{a, b, c\}$ and consider a topology

$$\mathcal{T}_1 = \emptyset, \{a\}, \{b\}, \{a, b\}$$

and a discrete topology $\mathcal{T}_2$ on $X$. Then, evidently, the identity mapping $1 : (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$ is c-irresolute but is neither irresolute nor semi-continuous.

However, we have the following result.

**Theorem 2.2:** If $f : X \rightarrow Y$ is c-irresolute and $f(A)$ is connected for every $A \subseteq X$, then $f$ is irresolute.
PROOF: If $A \subseteq X$, then $\text{scl} f(A)$ is a semi-closed, connected subset of $Y$. Therefore $f^{-1}(\text{scl} f(A))$ is semi-closed in $X$. Furthermore, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\text{scl} f(A))$. Therefore $\text{scl} A \subseteq f^{-1}(\text{scl} f(A))$, and consequently, $f(\text{scl} A) \subseteq f(f^{-1}(\text{scl} f(A))) \subseteq \text{scl} f(A)$. Hence $f$ is irresolute.

Example 2.1 and the following Example 2.2 taken together show that the concepts of semi-continuity and $c$-irresolute mappings are independent.

**Example 2.2:** Let $X = \{a, b, c\}$ with topology 

$T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $Y = \{p, q, r\}$ with topology 

$U = \{\emptyset, Y, \{p\}, \{p, r\}\}$. Then the mapping $f : X \rightarrow Y$, defined by $f(a) = f(b) = r$ and $f(c) = q$, is, obviously, semi-continuous but is not $c$-irresolute.

**Theorem 2.3:** If $f : X \rightarrow Y$ is $c$-irresolute, then, for all connected $B \subseteq Y$, $\text{scl} f^{-1}(B) \subseteq f^{-1}(\text{scl} B)$.

**Proof:** Since $\text{scl} B$ is semi-closed and connected in $Y$, $f^{-1}(\text{scl} B)$ is semi-closed in $X$. Since $f^{-1}(B) \subseteq f^{-1}(\text{scl} B)$, it follows that $\text{scl} f^{-1}(B) \subseteq f^{-1}(\text{scl} B)$.
The composition of two $c$-irresolute mappings may fail to be $c$-irresolute. This is shown by the following example.

**Example 2.3:** Let $X = \{a, b, c\} = Y = Z$ and consider on $X$ the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ as in Example 2.1 and

$$\mathcal{T}_3 = \{\emptyset, Z, \{a\}, \{c\}, \{a, c\}\}.$$ Then, obviously, the identity mappings $i: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ and $i: (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)$ are $c$-irresolute but the composition mapping $i: (X, \mathcal{T}_1) \to (Z, \mathcal{T}_3)$ is not $c$-irresolute.

However, we have the result given below.

**Theorem 2.4:** If $f: X \to Y$ is irresolute and $g: Y \to Z$ is $c$-irresolute, then $g \circ f: X \to Z$ is $c$-irresolute.

**Proof:** Let $K$ be a semi-closed and connected subset of $Z$. Then $g^{-1}(K)$ is semi-closed and so $f^{-1}(g^{-1}(K)) = (g \circ f)^{-1}(K)$ is semi-closed. Hence $g \circ f$ is $c$-irresolute.

**Remark 2.1:** If $f$ is $c$-irresolute and $g$ is irresolute, then, in general, $g \circ f$ is not $c$-irresolute. For, in Example 2.3, $i: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is $c$-irresolute and $i: (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)$ is irresolute but their composition $i: (X, \mathcal{T}_1) \to (Z, \mathcal{T}_3)$ is not $c$-irresolute.
DEFINITION 2.2: For any mapping \( f : X \rightarrow Y \), the mapping \( g : X \rightarrow X \times Y \), defined by \( g(x) = (x, f(x)) \), is called the graph mapping with respect to the mapping \( f \).

LEMMA 2.1: [3]: If \( f : X \rightarrow Y \) is continuous and open, then \( f \) is irresolute.

THEOREM 2.5: If \( f : X \rightarrow Y \) is a mapping such that its graph mapping \( g \) is irresolute, then \( f \) is \( c \)-irresolute.

PROOF: Let \( p_Y \) be the projection of \( X \times Y \) onto \( Y \). Then \( p_Y \) is continuous and open and hence, in view of Lemma 2.1, is \( c \)-irresolute. Therefore, by Theorem 2.4, \( f = p_Y \circ g \) is \( c \)-irresolute.

THEOREM 2.6: If \( f : X \rightarrow Y \) is a mapping from a connected space \( X \) into a space \( Y \) such that the graph mapping \( g \) is \( c \)-irresolute, then \( f \) is \( c \)-irresolute.

PROOF: Let \( x \in X \) and \( V \) be a semi-open set containing \( f(x) \) such that \( Y - V \) is connected. By Lemma 2.1, each projection is irresolute. Therefore, \( p_Y^{-1}(V) \) is semi-open in \( X \times Y \). Since \( X \) and \( Y - V \) are connected, \( X \times (Y - V) = (X \times Y) - p_Y^{-1}(V) \) is connected. Thus \( p_Y^{-1}(V) \) is a semi-open set in \( X \times Y \) having a
connected complement. Since \( g \) is c-irresolute, there exists a semi-open set \( U \) in \( X \) containing \( x \) such that \( g(U) \subseteq p_Y^{-1}(V) \).

It follows that \( p_Y(g(U)) = f(U) \subseteq V \), so that \( f \) is c-irresolute.

**Theorem 2.7:** If \( f : X \to \prod_{\alpha \in \Delta} X_\alpha \) is irresolute, then, for each \( \alpha \in \Delta \), the mapping \( f_\alpha : x \mapsto x_\alpha \) defined as \( f(x) = (x_\alpha) \) where \( f(x) = (x_\alpha) \), is c-irresolute.

**Proof:** Obvious, since each projection \( p_\alpha \) from \( \prod_{\alpha \in \Delta} X_\alpha \) on to \( X_\alpha \) is c-irresolute.

**Remark 2.2:** Restriction of a c-irresolute mapping to a subset of its domain may fail to be c-irresolute. For, in Example 2.3, \( 1 | A : A \to (Y, \mathcal{T}_2) \), where \( A = \{b, c\} \) is a subset of \( (X, \mathcal{T}_1) \), is not c-irresolute.

**Lemma 2.2:** If \( A \) is open and \( U \) is semi-open in \( X \), then \( A \cap U \) is semi-open in \( A \).

**Theorem 2.8:** If \( f : X \to Y \) is c-irresolute and \( A \) is open in \( X \), then \( f|A : A \to Y \) is c-irresolute.
PROOF: Let \( U \) be a semi-open subset of \( Y \) with connected complement. Then \( f^{-1}(U) \) is semi-open in \( X \) and, hence, by Lemma 2.2, \( f^{-1}(U) \cap A = (f|A)^{-1}(U) \) is semi-open in \( A \). This proves the theorem.

In Theorem 2.8, if \( A \) is semi-open, then \( f|A \) is not always c-irresolute, as, in Example 2.3,
\[ 1|A : A \rightarrow (Y, \mathcal{T}_2), \text{where } A = \{b, c\} \text{ is semi-open in } (X, \mathcal{T}_1), \]
is not c-irresolute.

**Remark 2.3:** If \( f : X \rightarrow Y \) and \( A \subseteq X \) is such that \( f|A : A \rightarrow Y \) is c-irresolute, then it is not necessary that \( f \) is c-irresolute. For, in Example 2.3, \( 1|A : A \rightarrow (Z, \mathcal{T}_3) \), where \( A = \{c\} \) is a subset of \( (X, \mathcal{T}_1) \), is c-irresolute but \( 1 : (X, \mathcal{T}_1) \rightarrow (Z, \mathcal{T}_3) \) is not c-irresolute.

**Lemma 2.3** ([10]): Let \( A \) be a subset of a subspace \( Y \) of a space \( X \). Then \( A \), semi-open in \( Y \), is semi-open in \( X \) if \( Y \) is semi-open in \( X \).

**Theorem 2.9:** Let \( f : X \rightarrow Y \). If \( \{U_\alpha : \alpha \in \Delta\} \) be a semi-open cover of \( X \) and if, for each \( \alpha \in \Delta \), \( f|U_\alpha : U_\alpha \rightarrow Y \) is c-irresolute, then \( f \) is c-irresolute.
PROOF: Let \( U \) be a semi-open subset of \( Y \) with connected complement. Then, for each \( \alpha \in \triangle \), \((f|U_\alpha)^{-1}(U)\) is semi-open in \( U_\alpha \) and, since each \( U_\alpha \) is semi-open in \( X \), so \( f^{-1}(U) \cap U_\alpha \) for each \( \alpha \in \triangle \), and hence \( \bigcup_{\alpha \in \triangle} f^{-1}(U) \cap U_\alpha \) is semi-open in \( X \). Therefore, \( f^{-1}(U) \cap \bigcup_{\alpha \in \triangle} U_\alpha = f^{-1}(U) \cap X = f^{-1}(U) \) is semi-open in \( X \). Consequently, \( f \) is c- irresolute.

COROLLARY 2.2.1: Let \( f : X \to Y \). If \( \{ U_\alpha : \alpha \in \triangle \} \) be an open cover of \( X \) and if, for each \( \alpha \in \triangle \), \( f|U_\alpha : U_\alpha \to Y \) is c- irresolute, then \( f \) is c- irresolute.

PROOF: Obvious, since every open cover is semi-open.

LEMMA 2.4 [10]: A space \( X \) is semi-\( T_1 \) iff the singletons are semi-closed.

Also each singleton is connected.

THEOREM 2.10: Let \( f : X \to Y \) be c- irresolute and injective. If \( Y \) is semi-\( T_1 \), so is \( X \).

PROOF: Let \( Y \) be semi-\( T_1 \). Since \( f \) is injective, 
\( \{ x \} = f^{-1}(f(x)) \) for each \( x \in X \). Again, \( f(x) \) being singleton,
is semi-closed and connected. Hence under c-irresolute
\( f \), \{x\} is semi-closed. It follows then that \( X \) is semi-\( T_1 \).

**Definition 2.3:** A mapping \( f : X \rightarrow Y \) is said to be pre-
semi-closed if the image of every semi-closed set in \( X \) is
a semi-closed set in \( Y \) [5].

Recall that the following result has been
obtained earlier in the Chapter V and also in [5].

**Lemma 2.5:** Let \( f : X \rightarrow Y \) be a pre-semi-closed mapping.
Given any subset \( S \) in \( Y \) and any semi-open set \( U \) containing
\( f^{-1}(S) \), then there exists a semi-open set \( V = Y - f(X - U) \)
containing \( S \) such that \( f^{-1}(V) \subseteq U \).

**Definition 2.4**[12]: A space \( X \) is said to be strongly
s-normal if, to each pair of disjoint semi-closed subsets \( A \)
and \( B \) of \( X \), there exist disjoint semi-open subsets \( U \) and \( V \)
in \( X \) such that \( A \subseteq U \) and \( B \subseteq V \).

**Theorem 2.11:** Let \( f : X \rightarrow Y \) be a c-irresolute, pre-semi-
closed surjection from a strongly s-normal space \( X \) to a
space \( Y \). If either of the spaces \( X \) and \( Y \) is semi-\( T_1 \), then
\( Y \) is semi-\( T_2 \).
**Proof:** **Case I.** The space $Y$ is semi-$T_1$. Let $y_1, y_2$ be any two distinct points in $Y$. Then, by Lemma 2.4, $\{y_1\}$ and $\{y_2\}$ are semi-closed, connected subsets of $Y$. And so, $f$ being $c$- irresolute surjection, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint, semi-closed subsets of $X$. By strongly $s$-normality of $X$, there exist disjoint semi-open sets $U_1$ and $U_2$ containing $f^{-1}(y_1)$ and $f^{-1}(y_2)$, respectively. Since $f$ is pre-semi-closed, by Lemma 2.5, there exist semi-open sets $V_1 = Y - f(X - U_1)$ such that $y_1 \in V_1$ and $f^{-1}(V_1) \subseteq U_1$ and $V_2 = Y - f(X - U_2)$ such that $y_2 \in V_2$ and $f^{-1}(V_2) \subseteq U_2$.

Further, it can be easily verified that $V_1$ and $V_2$ are disjoint. Thus $Y$ is semi-$T_2$.

**Case II.** The space $X$ is semi-$T_1$. Then, by Lemma 2.4, singleton $\{x\}$ is semi-closed in $X$. And, since $f$ is pre-semi-closed, so $f(x)$ is semi-closed in $Y$. Since $f$ is surjective, each singleton in $Y$ is semi-closed. So $Y$ is semi-$T_1$ and the proof is complete in view of Case I.

**Theorem 2.12:** Strongly $s$-normality is invariant under irresolute pre-semi-closed surjections.

**Proof:** Let $X$ be strongly $s$-normal and $f : X \rightarrow Y$ be pre-semi-closed and irresolute surjection. Let $A, B$ be any
pair of disjoint semi-closed sets in $Y$. Then, since $f$ is surjective irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint semi-closed in $X$. Therefore, by strongly s-normality of $X$, there exist disjoint semi-open sets $U$ and $V$ in $X$ containing $f^{-1}(A)$ and $f^{-1}(B)$, respectively. Now, because $f$ is pre-semi-closed, Lemma 2.5 assures that there exist semi-open sets $G = Y - f(X - U)$ and $H = Y - f(X - V)$ such that $A \subset G$, $f^{-1}(G) \subset U$, $B \subset H$ and $f^{-1}(H) \subset V$. Evidently $G$ and $H$ are disjoint. Hence $Y$ is strongly s-normal.

**Definition 2.5:** A strongly s-normal and semi-$T_1$ space is said to be semi-$T_4$.

**Corollary 2.3:** Image of a strongly s-normal semi-$T_1$ space under an irresolute pre-semi-closed surjection is semi-$T_2$ and hence semi-$T_4$.

Every semi-$T_2$ space is semi-$T_1$. Therefore, by Theorem 2.11 and Theorem 2.12, the above result follows immediately.

The characterization of strongly semi-closed graph given below has already been obtained in the Chapter III and also in [4].
Lemma 2.6. [4]: A mapping \( f : X \rightarrow Y \) has a strongly semi-closed graph \( G(f) \) iff, for each \( x \in X \), \( y \in Y \) such that
\( y \neq f(x) \), there exist semi-open sets \( U \) in \( X \) and \( V \) in \( Y \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap \text{scl} \ V = \emptyset \).

Example 2.4. Let \( X = \{a, b, c\} \) with an indiscrete topology. Then, obviously, the identity mapping \( i : X \rightarrow X \) is c-irresolute but \( G(i) \) is not strongly semi-closed.

In view of the above Example 2.4, c-irresolute mapping need not have a strongly semi-closed graph. However, we have the following result.

Theorem 2.13: Let \( f : X \rightarrow Y \) be c-irresolute and let \( Y \) be a locally connected \( T_2 \) space. Then \( G(f) \) is strongly semi-closed.

Proof: Let \( x \in X \), \( y \in Y \) such that \( y \neq f(x) \). Then, since \( Y \) is \( T_2 \) and locally connected, there exists an open connected subset \( V \) of \( Y \) containing \( y \) such that \( f(x) \notin \text{cl} \ V \). Here \( \text{cl} \ V \) is closed and connected. Thus \( Y - \text{cl} \ V \) is a semi-open set containing \( f(x) \) and having connected complement. Hence, there exists a semi-open set \( U \) containing \( x \) such that \( f(U) \subseteq Y - \text{cl} \ V \). Consequently, \( f(U) \cap \text{scl} \ V = \emptyset \) and so, by Lemma 2.6, \( G(f) \) is strongly semi-closed.
REFERENCES


