CHAPTER VII

PROBABILISTIC ORDER LEVEL SYSTEM WITH LEAD TIME WHEN DELAY IN PAYMENTS IS PERMISSIBLE

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PROBABILISTIC ORDER LEVEL SYSTEM WITH LEAD TIME
WHEN DELAY IN PAYMENTS IS PERMISSIBLE

7.0 INTRODUCTION:

In this chapter, we consider the probabilistic order level system with lead time in which scheduling period T is fixed and the supplier offers a fixed credit period of T - time units to settle the account. An appropriate average expected total cost of the system and a formula for obtaining optimum value of order level Z = Z₀ have been derived. The only limitation of the model is that it can be applied to only those demand distributions for which f(x)/x is integrable w.r.t. x where f(x) denotes the probability density function (p.d.f.) of the demand variate x.

MODEL I

7.1 PROBABILISTIC ORDER LEVEL SYSTEM WITH LEAD TIME
WHEN DELAY IN PAYMENTS IS PERMISSIBLE

7.1.1 ASSUMPTIONS AND NOTATIONS:

The model under consideration is developed with following assumptions:

i) On hand inventory of the system is reviewed regularly at a prescribed time period of T - time units. At the time of review, units are ordered so as to bring the on hand inventory to an order level Z. Z is the decision variable.

ii) Lead time L is finite and known. The demand v during the lead
time \( L \) is a random variable with p.d.f. \( h(v) \), \( 0 \leq v < \alpha \).

iii) Shortages are allowed and are made up when next procurement is received by the system.

iv) The demand \( x \) during any scheduling period \( T \) is a random variable with p.d.f. \( f(x) \) and c.d.f. \( F(x) \), \( 0 \leq x < \alpha \) with

\[
\mu = \int_{0}^{\alpha} xf(x)dx \tag{1}
\]

as the mean demand during \( T \). It is assumed that the demand of \( x \) units occur in a uniform pattern over the cycle time.

v) The supplier gives a credit period of length \( T^* \) time units for settling the accounts. Thus, the system has not to pay interest charges on the purchase units for a fixed period of length \( T^* \) where

1) \( T^* \leq T \) or 2) \( T^* > T \).

vi) The unit cost \( C \) per unit, inventory holding cost excluding interest charges, \( C_1 \) per unit per time unit and shortage cost \( C_2 \) per unit per time unit are known and constant during the period under consideration.

For the period during which account of purchase quantity is not settled, the generated sales revenue is deposited in an interest bearing account which earns interest at the rate of \( I_e \) per rupee per time unit. After the account is settled, the system starts paying interest charges on the outstanding amount in inventory at the rate of \( I_c \) per rupee per time unit. It is assumed that only the unit cost from the generated revenue is deposited in an interest bearing account. The difference between the selling
price and unit cost is retained by the system for meeting regular expenses.

vii) As a period, we consider time interval between two successive realizations of orders into the inventory system.

7.1.2 DETERMINATION OF AVERAGE EXPECTED TOTAL COST AND OPTIMUM ORDER LEVEL:

The mathematical model for the system under consideration is derived in both the cases:

i) $T^i < T$

ii) $T^i > T$

($T^i$ and $T$ are known constants). After each review, units are ordered so that the amount on hand plus an order reaches the order level $Z$. The ordered quantity will be received by the system after a lapse of $L$ time units. If $v$ is the lead time demand and $v < Z$ then cycle starts with on hand inventory of $Z - v$ units and if $v > Z$ then a cycle starts with a shortage of $v - Z$ units. Let $x$ be the demand during the cycle time. When a demand of $v$ units occurs during the lead time $L$ preceding the cycle time, the demand of $x$ units occur during the cycle time and $Z$ is the order level then the instantaneous state of $Q(t|v,x,Z)$ of a system at time $t$ of a cycle is given by

$$\frac{dQ(t|v,x,Z)}{dt} = -\frac{x}{T}, 0 \leq t \leq T. \quad (2)$$

Under the boundary condition $Q(0|v,x,Z) = Z - v$, the solution of differential eq. (2) is

$$Q(t|v,x,Z) = Z - v - \frac{x}{T} t, 0 \leq t \leq T. \quad (3)$$
Hence,

\[ Q(T|v,x,Z) = Z - v - x. \]  

(4)

We may have one of the following three mutually exclusive cases:

i) \( Z - v \geq 0 \) and \( Z - v - x \geq 0 \)

that is, \( v \leq Z \) and \( Z - v \geq x \).

ii) \( Z - v \geq 0 \) and \( Z - v - x \leq 0 \)

implying, \( v \leq Z \) and \( Z - v \leq x \).

iii) \( Z - v \leq 0 \) and \( Z - v - x \leq 0 \)

giving, \( Z \leq v \) and \( Z - v \leq x \).

In the first case there will be no shortages and in the latter two cases a cycle will end with shortages. Hence, average expected shortages carried forward from the previous cycle is

\[ \frac{Z}{2} \int_0^\infty \int_{Z-v}^{\infty} (x+v-Z)f(x)h(v)dx dv + \int_0^\infty \int_0^{Z-v} (x+v-Z)f(x)h(v)dx dv \]  

(6)

Shortages are cleared first whenever a fresh stock arrives.

The number of units sold up to time \( t \), \( S(t|v,x,Z) \) are given by

\[ S(t|v,x,Z) = \frac{x}{T} t , \quad 0 \leq t \leq T \]  

(7)

a) On hand inventory per time unit is

\[ I_{11} = \frac{1}{T} \int_0^T Q(t|v,x,Z)dt \]

\[ = Z - v - x/2. \]

Thus, inventory holding cost per time unit is
\[ = C_1 \int_0^Z \int_0^{Z-v} \left[ Z - v - x/2 \right] f(x)h(v)dx dv. \]  

(8)

When shortages occur, suppose that the system carries inventory during \((0, t_1)\) and runs with shortages during \((t_1, T)\), then \(Q(t_1|v,x,Z) = 0\). Therefore,

\[ t_1 = \frac{(Z-v)T}{x}. \]  

(9)

b) Inventory on hand per time unit

\[ = \frac{1}{T} \int_0^{t_1} Q(t|v,x,Z)dt \]

\[ = \frac{(Z-v)^2}{2x}. \]

Hence, inventory holding cost per time unit is

\[ = C_1 \int_0^Z \int_{Z-v}^{Z} \frac{(Z-v)^2}{2x} f(x)h(v)dx dv. \]

(10)

Using eqs. (8) and (10), total inventory holding cost per time unit is

\[ C_1 \int_0^Z \left[ \int_0^{Z-v} \left( Z-v-x/2 \right) f(x)dx + \int_{Z-v}^{\alpha} \frac{(Z-v)^2}{2x} f(x)dx \right] h(v)dv. \]

(11)

Now for shortages:

c) \[ I_{21} = \frac{1}{T} \int_{t_1}^T -Q(t|v,x,Z) dt \]
Thus, total shortage cost per time unit is

\[
I_{22} = \frac{1}{T} \int_0^T Q(t \mid v, x, Z) dt = v - z + x/2.
\]

The value of \( t_1 \) defined in eq. (9) give rise to two subcases. Here, depending on the value of \( x \), we have

i) \( x < t_1 < T \) implying \( z - v < x < \frac{(z-v)T}{T} \)

and

ii) \( t_1 < x < T \) giving \( x > \frac{(z-v)T}{T} \).

c) Interest charged per time unit

\[
CI_c \left[ \frac{T}{T} \int_0^Z \int_{z-v}^{Z-v} Q(t \mid v, x, Z) dt \right] f(x)h(v)dx dv
\]

\[
= CI_c \left[ \frac{T}{T} \int_0^Z \int_{z-v}^{Z-v} f(x)h(v)dx dv \right]
\]

\[
= CI_c \left[ \frac{T}{T} \int_0^Z \int_{z-v}^{Z-v} f(x)h(v)dx dv \right]
\]
When shortages occur and $T^x \leq T$, interest is charged only in subcase (i) while it is zero in subcase (ii). Thus,

d) Interest charged per time unit

\[
\frac{CI_c}{T} \int_0^T \int_{z-v}^Z (z-v) T^x \int_0^T f(t|x,v,Z) dt f(x|v) dv dx
\]

\[
= \frac{CI_c}{T} \int_0^T \int_{z-v}^Z (z-v) T^x \left(\frac{xT^x}{T} - (z-v)\right)^2
\]

\[
\frac{2x}{2x} f(x|v) dv.
\] (16)

From eqs. (15) and (16), total interest charged per time unit is

\[
= \int_0^Z \left[ CI_c (1-T^x/T) \int_0^{z-v} (z-v - \frac{x}{2} (1+T^x/T)) f(x) dx +
\int_{z-v}^Z CI_c \int_0^{(z-v)T^x/T} \left(\frac{xT^x}{T} - (z-v)\right)^2
\frac{2x}{2x} f(x) dx \right] h(v) dv.
\] (17)

When $T^x < T$ and shortages do not occur

e) Interest earned per time unit

\[
= \frac{CI_e}{T} \int_0^T \int_0^{z-v} \int_0^{t^x} S(t|x,v,Z) dt f(x|v) dv dx
\]

\[
= \frac{CI_e}{T} \int_0^{z-v} \int_0^{z-v} xf(x|v) dv dx.
\] (18)

Also, when shortages occur,

f) Interest earned per time unit
\[
= \frac{C I}{T} \int_{0}^{Z} \int_{Z-v}^{T} (Z-v)T/T^* S(t|x,v,0) dt f(x,0) dv + \int_{0}^{Z} \frac{(Z-v)^2}{2T} f(x,0) dv, \quad T < t < T^*
\]

and

\[
= \frac{C I}{T} \int_{0}^{Z} \int_{Z-v}^{T} (Z-v)T/T^* S(t|x,v,0) dt f(x,0) dv + \int_{0}^{Z} \frac{(Z-v)^2}{2T} f(x,0) dv, \quad t_1 > T^*
\]

Using eqs. (18) - (20) and (6), total interest earned per time unit is

\[
= \frac{C I}{T} \int_{0}^{Z} \int_{Z-v}^{T} (x+v-Z) f(x,0) dv + \int_{0}^{Z} \int_{Z-v}^{T} (x+v-Z) f(x,0) dv
\]
\begin{align*}
&+ \int_0^Z \int_{(Z-v)T/T}^\infty (Z-v)f(x)h(v)dx dv \\
&+ \frac{C_1 T^2}{2T} \left[ \int_0^Z \int_{Z-v}^\infty xf(x)h(v)dx dv + \int_{Z-v}^\infty \int_x^\infty xf(x)h(v)dx dv \right] \\
&- \frac{C_1}{e} \int_0^Z \int_{(Z-v)T/T}^\infty \frac{(Z-v)^2}{2x} f(x)h(v)dx dv. \quad (21)
\end{align*}

Using eqs. (11), (14), (17) and (21), average expected total cost

\[ K_1(Z) \text{ per time unit when } T^* \leq T, \text{ is given by} \]

\[ K_1(Z) = \int_0^Z \int_{Z-v}^\infty C_1(Z-v-x/2)f(x)h(v)dx dv \]

\[ + \int_0^Z \int_{Z-v}^\infty \left[ C_1 \frac{(Z-v)^2}{2x} + C_2 \frac{(x-Z+v)^2}{2x} \right] f(x)h(v)dx dv \]

\[ + \int_0^\infty \int_0^x C_2(v-Z+x/2)f(x)h(v)dx dv \]

\[ + \int_0^Z \left[ C_1(1-T^*/T) \int_0^{Z-v} \left\{ Z-v - \frac{x}{Z} \left( 1 + T^*/T \right) \right\} f(x)dx \right] \\
+ \frac{C_1}{e} \int_0^Z \int_{(Z-v)T/T}^\infty \frac{(xT^*/(Z-v))^2}{2x} f(x) dx \right] h(v)dv \]

\[ - \frac{C_1 T^*}{e} \int_0^Z \int_{Z-v}^\infty (x+v-Z)f(x)h(v)dx dv \]
\[ -\int_{0}^{\alpha} \int_{0}^{\alpha} (x+v-Z)f(x)dx\ h(v)dv + \int_{0}^{\alpha} (Z-v)f(x)dx\ h(v)dv \]
\[ + \frac{1}{2T^2} \left[ \int_{0}^{Z} \int_{0}^{Z-v} x f(x) h(v) dx dv + \int_{0}^{Z} \int_{Z-v}^{Z-v} t f(x) h(v) dx dv \right] \]
\[ + \frac{Z-v}{2x} f(x) h(v) dx dv, \quad (22) \]

The optimum value of \( Z = Z_0 \) is the solution of

\[ (C_1 + C_2 + C_3) \int_{0}^{Z} f(x) h(v) dx dv + (C_1 + C_2) \int_{Z-v}^{Z} \frac{Z-v}{X} f(x) h(v) dx dv \]
\[ + \frac{1}{2T^2} \left[ \int_{0}^{Z} \int_{0}^{Z-v} x f(x) h(v) dx dv + \frac{1}{T} \int_{0}^{Z} \int_{0}^{Z-v} f(x) h(v) dx dv \right] \]
\[ - C(1 - I_e)^T \int_{0}^{Z} f(x) h(v) dx dv + C \int_{0}^{Z} \frac{Z-v}{X} f(x) h(v) dx dv \]
\[ - \frac{1}{T} \left[ \int_{0}^{Z} \int_{0}^{Z} f(x) h(v) dx dv + \int_{0}^{Z} \int_{Z-v}^{Z-v} f(x) h(v) dx dv \right] = C_2. \quad (23) \]

Here, the optimality condition is
\[
\frac{d^2 K_1(Z)}{dz^2} = (C_1 + C_2) \int_0^Z (Z-V) \int \frac{1}{x} f(x)h(v)dx dv
\]

\[
- C_1 \int_0^Z (Z-V)/(T/T)^x f(x)h(v)dx dv
\]

\[
+ C_1 e^{-(Z-V)/(Z-V)} \int_0^Z \int \frac{1}{x} f(x)h(v)dx dv > 0, \text{ for all } Z = Z_0
\]

(24)

CASE (II) \( T^* > T \), i.e. when account is to be settled after the end of scheduling period \( T \).

h) Interest charged = 0. \hspace{2cm} (25)

i) Interest earned per time unit is

\[
= C_1 e^{(T^*/T-1)} \int_0^Z \int x f(x)h(v)dx dv - C_1 e^{(Z-V)^2/(2x)} f(x)h(v)dx dv
\]

\[
+ C_1 e^{(Z-V)/(Z-V)} \{ \int_0^Z \int (x+v-Z)f(x)h(v)dx dv + \int_0^Z \int (x+v-Z)f(x)h(v)dx dv
\]

\[
= \int_0^Z \int (Z-V)f(x)dx h(v)dv.
\]

(26)

Using eqs. (11), (14), (25) and (26), average expected total cost \( K_2(Z) \) per time unit is

\[
K_2(Z) = \int_0^Z \int C_1 (Z-V+x/2)f(x)h(v)dx dv + \int_0^Z \int C_2 (v-Z+x/2)f(x)h(v)
\]
\[ \int_{0}^{Z} \int_{Z-v}^{\alpha} \left[ C_1 \frac{(Z-v)^2}{2x} + C_2 \frac{(Z-v+v)^2}{2x} \right] f(x)h(v)dxdv \]

\[ -C_1 \left( \frac{T^*}{T} - 1/2 \right) \int_{0}^{Z} \int_{0}^{Z-v} xf(x)h(v)dxdv \]

\[ + C_1 \int_{0}^{Z} \int_{Z-v}^{\alpha} \frac{(Z-v)^2}{2x} f(x)h(v)dxdv \]

\[ - \frac{C_1 T^*}{T} \left\{ \int_{0}^{Z} \int_{Z-v}^{\alpha} (x-v-Z)f(x)h(v)dxdv + \int_{0}^{\alpha} \int_{Z}^{\alpha} (x+v-Z)f(x)dx \right\} \]

\[ + \int_{0}^{Z} \int_{Z-v}^{\alpha} (Z-v)f(x)dx \int h(v)dv \right\} \]

\[ = C_2 \] (27)

The optimum value of \( Z = Z_0 \) is the solution of

\[ (C_1 + C_2) \int_{0}^{Z} \int_{Z-v}^{\alpha} f(x)h(v)dxdv + (C_1 + C_2) \int_{0}^{Z} \int_{Z-v}^{\alpha} \frac{Z-v}{x} f(x)h(v)dxdv \]

\[ + C_1 \int_{0}^{Z} \int_{Z-v}^{\alpha} \frac{Z-v}{x} f(x)h(v)dxdv + \frac{C_1 T^*}{T} \int_{0}^{\alpha} \int_{0}^{\alpha} f(x)h(v)dxdv \]

\[ = C_2 \] (28)

For cost to be minimum at order level \( Z = Z_0 \), we should have
\[
\frac{d^2 K_i(Z_0)}{dZ_0^2} = (C_1 + C_2 + C_1 C) \int_0^Z \int_{0}^{\alpha} \frac{1}{x} f(x)h(v)dx dv > 0,
\]

for all \( Z = Z_0 \). \hfill (29)

When delay in payments is not possible, \( t^s = 0 \), or \( I_e = 0 \), the average expected total costs (eqs. (22) and (27)) are

\[
K_1(Z) = K_2(Z) = (C_1 + C_1 C) \int_0^Z \int_{0}^{Z-v} (Z-v-x/2) f(x)h(v)dx dv
\]

\[
+ \int_0^Z \int_{Z-v}^\infty \left[ \frac{(C_1 + C_1 C)(Z-v)^2}{2x} + \frac{C_2(Z-Z+x)^2}{2x} \right] f(x)h(v)dx dv
\]

\[
+ C_2 \int_0^Z \int_{0}^{v-Z+x/2} f(x)h(v)dx dv. \hfill (30)
\]

The formula for optimum order level \( Z = Z_0 \) (eqs. (23) and (28)) is

\[
\frac{C_2}{C_1 + C_2 + C_1 C} = \int_0^Z \int_{0}^{Z-v} f(x)h(v)dx dv + \int_0^Z \int_{Z-v}^{Z-v} \frac{Z-v}{x} f(x)h(v)dx dv
\]

\hfill (31)

and the optimality condition (eqs. (24) and (29)) is

\[
\frac{d^2 K(Z_0)}{dZ_0^2} = (C_1 + C_2 + C_1 C) \int_0^{Z_0} \int_{Z_0-v}^{\alpha} \frac{1}{x} f(x)h(v)dx dv > 0,
\]

for all \( Z = Z_0 \) \hfill (32)

which are same as those given in Naddor [18].