CHAPTER 4

A RADIATING BLACK HOLE WITH INTERNAL MONOPOLE IN EXPANDING UNIVERSE

Verbitsky (1981) has obtained a generalization of McVittie's solution and described the gravitational field of a radiating black hole. In this solution, the space around the particle is empty and the geometry at large distances from the particle remains that of special relativity. McVittie's solution was the first attempt exploring the gravitational field of a radiating black hole in an expanding universe. The space around the particle is now occupied by a spherically symmetric distribution of matter with nonzero density and metric, as seen in general relativity.

Global monopoles may have been created during phase transition in the early universe [Linde (1976), Vilenkin (1981)]. Global monopoles are stable topological defects produced when
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4.1: INTRODUCTION

Vaidya (1951) has obtained a generalization of the Schwarzschild exterior solution which describes the gravitational field of a radiating star. In this solution, the space around the particle is empty and the geometry at large distances from the particle reduces to that of special relativity. McVittee (1933) gave the line element describing the gravitational field of a mass particle embedded in an expanding universe. The space around the particle is now occupied by a spherically symmetric distribution of matter with non-zero density and isotropic pressure which at large distances from the particle go over smoothly to the cosmic density and pressure in an expanding cosmological model. Vaidya and Shah (1957) have generalized McVittee's solution and have discussed the field of a radiating mass particle in an expanding universe.

Global monopoles may have been created during phase transition in the early universe [Kibble (1976), Vilenkin (1985)]. Global monopoles are stable topological defects produced when
global symmetry breaking occurs. The study of global monopoles and the space-times associated with them has become highly relevant and has received considerable attention. Barriola and Velinkin (1989) have derived the expression for the metric associated with the space-time of a static black hole with an internal monopole. Hong-Wei Yu (1993) has obtained an exact solution of Einstein equations with the energy-momentum tensor corresponding to a spherically symmetry global monopole plus a purely radial outgoing radiation. His solution describes the exterior field of a radiating black hole with an internal monopole. In the absence of the monopole, this solution reduces to the Vaidya radiating star solution.

Tikekar and Patel (1995) have discussed the gravitational fields of a radiating black hole with an internal monopole in the background of Einstein and deSitter universes. The purpose of the present investigation is to obtain a metric describing the field of a radiating black hole with internal monopole in the cosmological background of an expanding universe.

4.2: THE FIELD EQUATIONS

Hong-Wei Yu (1993) discussed the metric form

$$ds^2 = -2drdu + \left[1 - 8m^2 - \frac{2m(u)}{r}\right]du^2 - r^2 \left(\frac{du^2}{r^2 - \eta_o^2} + \sin^2 \theta \, d\phi^2\right)$$

(4.2.1)

where $m$ is an arbitrary function of $u$ and $\eta_o$ is a constant.

In order to consider the field of a radiating black hole with
an internal monopole in an expanding universe, we consider a line
element in the form [Patel and Desai, (1996)].

\[ ds^2 = e^{\omega} \left[ -2 \, dr \, du + (1+V) \, du^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right] \tag{4.2.2} \]

where \( V \) is an undetermined function of \( r \) and \( u \), and \( \omega \) is a
function of \( t=r-u \). Clearly (4.2.1) is a particular case of
(4.2.2).

The non-vanishing components of the Einstein mixed tensor \( G^i_k \)
\[ = R^i_k - (1/2) \delta^i_k \, R \] for the metric (4.2.2) are listed below for
readily reference:

\[ G^t_t \omega = (1+V) \left[ \frac{3}{4} \omega^2 + 2 \omega \frac{\omega'}{r} - \frac{\omega^2}{2(1+V)} + \frac{V'}{r(1+V)} + \frac{1}{r^2} \right] \tag{4.2.3} \]

\[ G^s_s \omega = G^r_r \omega = -2 \omega^{**} - \frac{1}{2} \omega^2 - \frac{\omega^2}{r} \]

\[ + (1+V) \left[ \omega^{**} + \frac{1}{4} \omega^2 + \frac{2}{r} + \frac{\omega^2}{2(1+V)} + \frac{V'}{r(1+V)} + \frac{V'}{r^2} \right] \tag{4.2.4} \]

\[ G^t_\phi \omega = (1+V) \left[ \omega^{**} + \frac{1}{4} \omega^2 + \frac{2}{r} + \frac{\omega^2}{2(1+V)} + \frac{\omega V'}{r(1+V)} + \frac{1}{r^2} \right] \]

\[ = \omega^{**} - \omega^2 = \frac{2}{r} = \frac{1}{r^2} \tag{4.2.5} \]
Here and in what follows an asterisk, a prime and a dot are used to denote differentiations with respect to $t$, $r$ and $u$ respectively. The coordinates are labelled as

$$x^4 = r, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = u$$

We assume that the physical content of the space-time (4.2.2) comprises of a spherically symmetric distribution of matter in the form of a perfect fluid accompanied by radially outgoing pure radiation with an internal global monopole. The energy momentum tensor for such a distribution can be taken in the form

$$T^i_k = (p + \rho) u^i u_k - p \xi^i_k + \sigma \xi^i_k \xi^i_k + T^i_k \text{(monopole)}$$

with

$$u^i u_i = 1, \quad \xi^i \xi_i = 0$$

Here $p$ and $\rho$ are respectively the fluid pressure and matter density.

Following Barriola and Vilenkin(1989), the $T^i_k \text{(monopole)}$ can be taken in the form
where \( \eta_0 \) is a constant associated with the internal monopole.

For the metric (4.2.2) we can take

\[
\nu^4 = \nu^2 = \nu^3 = 0 \quad \text{and} \quad \xi^2 = \xi^3 = \xi^4 = 0
\]

(4.2.11)

From equations (4.2.8), (4.2.9), (4.2.10) and (4.2.11) we have

\[
T_4 = -p + \frac{1}{2} \left( \eta_0^2 \right) e^{-\omega}, \quad T_5 = T_6 = -p,
\]

\[
T_4 = \rho + \frac{1}{2} \left( \eta_0^2 \right) e^{-\omega}, \quad T_4 = \omega \left( \xi^4 \right)^2 e^{-\omega}.
\]

(4.2.12)

The field equation \( G_k^i = -8\pi T_k^i \) then lead to

\[
-\frac{1}{2} \nabla^2 V + \frac{V}{r^2} = \frac{1}{2} \nabla^2 \omega - \frac{8\pi \eta_0^2}{r^2} + V \left( \omega^{**} - \frac{1}{2} \omega^2 - \frac{\omega^*}{r} \right)
\]

(4.2.13)

\[
8\pi p = G_4 + \frac{8\pi \eta_0^2}{r^2} e^{-\omega}
\]

(4.2.14)

\[
8\pi \rho = -G_4 - \frac{8\pi \eta_0^2}{r^2} e^{-\omega}
\]

(4.2.15)
\[ 8\pi\sigma = G^4 e^{-\omega} (\xi^4)^{-2} \quad (4.2.16) \]

It should be noted that when \( \eta_0 = 0 \), the above equations reduce to those discussed by Vaidya and Shah (1957). In the following section we obtain approximate solutions to the field equations (4.2.13)–(4.2.16) by the method developed by Vaidya and Shah (1957).

4.3: APPROXIMATE SOLUTIONS OF THE FIELD EQUATIONS

ZEROTH APPROXIMATION

In the zeroth approximation we put \( \omega = 0 \). Equation (4.2.13) then becomes

\[ \frac{1}{2} V'' + \frac{V}{r^2} = \frac{-8\pi \eta_0^2}{r^2} \quad (4.3.1) \]

The differential equation (4.3.1) can be easily solved. The solution is

\[ V = \frac{-2m}{r} + Ar^2 - 8\pi \eta_0^2 \quad (4.3.2) \]

where \( m \) and \( A \) are arbitrary functions of \( u \). The singularity at \( r = \infty \) can be avoided by choosing \( A = 0 \). The remaining field equations (4.2.14) to (4.2.16) yield

\[ p = \rho = 0, \quad 8\pi\sigma = \frac{2 \cdot \dot{m}}{r^8} \left( \xi^4 \right)^{-2} \quad (4.3.3) \]
which is the solution of Hong-Wei Yu (1993). Moreover if 
m = constant, we obtain the solution discussed by Barriola and
Vilenkin (1989). If \( \eta_0 = 0 \), we recover the shining star solution of
Vaidya (1951).

**FIRST APPROXIMATION**

In the first approximation, we neglect the terms in \( \omega^2 \), \( \omega' \),
\( \omega'' \), and \( \omega^{**} \) on the right hand side of the equation (4.2.13). Also,
we use \( V \) given by (4.3.2) in the right hand side of (4.2.13). We
note, of course, that the assumption, that the above quantities
are of the same order of smallness is a purely mathematical
device. Substituting from (4.3.2) on the right hand side of
(4.2.13) we get

\[
- \frac{1}{2} V'' + \frac{V}{r^2} = \frac{3}{r^2} \omega + \frac{8 \pi \eta^2 \omega}{r} \tag{4.3.4}
\]

which has the solution

\[
V = - \frac{2m}{r} + \left( 3 m + 8 \pi \eta^2 \right) r \omega - 8 \pi \eta^2 \tag{4.3.5}
\]

The remaining field equations (4.2.14)—(4.2.16) now give

\[
p = \rho = 0 \tag{4.3.6}
\]
(4.3.7)

\[ 8\pi \rho \xi^2 = e^{-2\omega} \left( \frac{1}{r^2} \left( 2 \frac{m}{r} \omega^* - \frac{2m}{r} \omega \right) \right) \]

When \( m = \text{constant and } \omega^* = 0 \), \( \omega \) vanishes and we get the Schwarzschild exterior solution with a global monopole. In this approximation the cosmic pressure and density are negligible.

**SECOND APPROXIMATION**

In the second approximation, we neglect the terms \( \omega^*, \omega^* \), \( \omega^{**}, \omega^*, \omega^{**} \) and \( \omega^{***} \) etc. The field equation (4.2.13) then becomes, on substitution the expression (4.3.5) in the coefficients of \( \omega, \omega^* \) and of \( \omega^{**} \) respectively,

\[ \frac{1}{2} v'' + \frac{v'}{r} = \frac{3m}{r^2} \omega^* - \frac{2m}{r} \left( \omega^{**} + \omega^* \right) \]

\[- \lambda \beta \eta_0^2 \left( \omega^{**} - \omega^* + \frac{1}{r^2} \right) \]

(4.3.8)

This equation has the solution

\[ v = \frac{-2m}{r} + \left( 3m + 8\pi \eta_0^2 r \right) \omega^* - 8\pi \eta_0^2 - 2m r \omega^{**} - 2m r \omega^* \]

(4.3.9)

The remaining field equations (4.2.14) — (4.2.16) then give
\[ \delta \eta e^\omega = \frac{-1}{4} \left( 1 - \frac{2m}{r} \right) \omega^2 - \left( 1 + \frac{m}{r} \right) \omega^{**} + 14\eta_o^2 \omega^2 + 4\eta_o^2 \omega^{**} \]  

(4.3.10)

\[ \delta \eta e^\omega = \frac{3}{4} \omega^2 \left( 1 - \frac{2m}{r} \right) + \frac{3m}{r} \omega^{**} - 18\eta_o^2 \omega^2 \]  

(4.3.11)

\[ \delta \eta o = (\xi^4)^{-2} e^{-2\omega} \left[ \frac{2m}{r} + \frac{\omega}{r} \left( \frac{2m}{r} + \frac{m}{r} \right) + \omega^{**} \left( \frac{m}{r} + 2 \frac{m}{r} \right) + 8\eta_o^2 \omega^{**} \right] \]  

(4.3.12)

However, if we let \( \eta_o = 0 \), we have

\[ \delta \eta o = (\xi^4)^{-2} e^{-2\omega} \left[ \frac{2m}{r} + \frac{\omega}{r} \left( \frac{2m}{r} + \frac{m}{r} \right) + \omega^{**} \left( \frac{m}{r} + 2 \frac{m}{r} \right) \right] \]  

(4.3.13)

Thus in the second approximation, we obtain the field of a radiating black hole with internal monopole embedded in an expanding universe. When \( \eta_o = 0 \), we get the field of a radiating particle in an expanding universe discussed by Vaidya and
Shah (1957) in which the density of flowing radiation surrounding the particle is given by equation (4.3.13).

The method of approximation used here is quite general and one can use it to obtain the solution of any desired degree of accuracy.

4.4: CONCLUDING REMARKS

In this chapter, we have attempted to find the space-time which describes the field of a radiating black hole with internal monopole embedded in an expanding universe. A scheme of obtaining approximate solutions has been developed. When the expansion of the universe is switched off, our solution gives the field of a radiating black hole with an internal monopole.

Also, in the absence of monopole our solution reduces to the solution discussed by Vaidya and Shah (1957).

The next chapter deals with two spherically symmetric non-static higher dimensional solutions of Einstein’s field equations. One of them represents the higher dimensional Vaidya metric in the background of Einstein universe and the other gives us higher dimensional Vaidya metric in the background of de Sitter universe.
REFERENCES