Chapter 7

Scalar hair for a static (3+1) black hole

7.1 Introduction

*I never thought that others would take them so much more seriously than I did,*

Albert Einstein.

No-hair conjecture [123] demands the non existence of any information other than mass, charge and angular momentum of a black hole. In order to prove the no-hair conjecture, no-hair theorems had been established by coupling the classical fields with the Einstein gravity [138]. It had been shown that the scalar field would be trivial if one demands a regular horizon at a finite distance from the centre of the black hole and also that stationary black hole solutions are hairless in a variety of cases, coupling different classical fields to gravity [122, 139, 140, 141]. In Chapter 6, we have noted
that the co-existence of non-trivial solution and a proper metric is difficult to evolve.

It is widely believed that black holes with scalar hair generally exist only when the scalar potential has negative region [70, 71, 72, 142]. There is a common belief that there are no static asymptotically flat and asymptotically AdS black holes with spherical scalar hair, if the scalar field theory coupled to gravity, satisfies the Positive Energy Theorem [143]. A charged de Sitter black hole in the Einstein-Maxwell-Scalar-Λ system possesses only unstable solutions [68]. But an unexpected development of scalar hair in AdS black hole with minimal [70] as well as nonminimal [144] coupling of scalar field, demanded a heuristic study of scalar hair [69].

As a strong interpretation, black hole has hair if there is a need to specify quantities other than the conserved charges defined at asymptotic infinity in order to characterize completely a stationary black hole solution [72, 128]. Efforts were done to reveal strong hair [124, 125] and they came up with a scalar solution conformally coupled to Einstein’s gravity through a metric for extremal case. Eventhough innocuous, the solution has a divergence at the horizon. It is given as

\[ \Phi = \frac{-r_0 \alpha^{-1/2}}{r - r_0}, \quad (7.1) \]

with \( \alpha = \frac{8\pi G}{\hbar} \). In Eq. (7.1), \( \Phi \) blows up at the horizon, which is against the principle that \( \Phi \) shall be finite everywhere [66]. In another attempt, a four dimensional solution of the Einstein equation with a positive cosmological constant coupled to a massless self interacting conformal scalar field was put forwarded [135]. The scalar
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solution in that case is

$$\Phi(r) = \sqrt{\frac{3}{4\pi}} \frac{\sqrt{G}M}{r - GM}. \quad (7.2)$$

But Eq. (7.2) does not give any information other than mass of black hole and the cosmological constant. Hence strong interpretation of scalar hair is not guaranteed. The motivation of the present work is to know whether a strong interpretation of the scalar hair can be possible in a static (3+1) black hole, which requires a nontrivial solution of scalar field in the vicinity of black hole with regular horizon.

In this chapter, we report a nontrivial black hole solution of a massive but self interacting scalar field showing no divergence at the horizon and asymptotically falling to the vacuum value. The proposed metric shows trace of scalar charge.

Whether a non-trivial scalar solution and a metric with a horizon are mutually compatible or not has been the objective of scalar hair investigations. Many contend that only when the solution is trivial that a metric with a horizon is established and for a non-trivial solution, the singularity will become naked. The criterion of scalar hair is the co-existence of non-trivial solution and a proper metric (having horizon and temperature) that holds the trace of scalar field.

The scheme of the chapter is as follows. In Sec. 7.2, non-trivial scalar hair solution is obtained for a (3+1) static black hole. In Sec. 7.3, the metric of the hairy black hole is obtained by solving the scalar stress-energy tensor. Entropy and mass of hairy black hole are also discussed in this section. Sec. 7.4, discusses thermodynamics of black hole. The conclusion is given in section 7.5.
7.2 Solution with a conformal coupling

A non-trivial radial solution of a scalar field, whose source is a scalar double well potential, in the vicinity of a static (3+1) black hole will be discussed in this section. We will restrict our consideration to the conformally coupled case. Consider the action

\[
I = \int d^4 x \sqrt{-g} \left[ \frac{R}{2k} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi \right. \\
\left. - \frac{1}{2} \xi R(\Phi - \Phi_0)^2 - V(\Phi - \Phi_0) \right],
\]

(7.3)

where \( \Phi \) is a massive, self interacting and conformally coupled scalar field. In the static case, \( \Phi \) represents a radial field. For a (3+1) case, \( \xi = \frac{1}{6} \). A double well potential of the type in Eq. (6.11) and Fig. (6.1) has been considered in Eq. (7.3). In Fig. (6.1), \( V \) has global minima at \( \Phi = \pm \frac{\mu}{2} \) and a local maximum at \( \Phi = \Phi_0 \). The scalar field equation is given by

\[
\Box(\Phi - \Phi_0) - \xi R(\Phi - \Phi_0) + \mu^2(\Phi - \Phi_0) - \delta^2(\Phi - \Phi_0)^3 = 0,
\]

(7.4)

where \( \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \) is the Laplace-Beltrami operator and \( R \) represents the Ricci scalar. For the present situation, we do not consider cosmological constant. Hence \( R = 0 \). The stress-energy tensor of scalar field under gravity can be given by the relation

\[
T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi + \\
\frac{1}{6} [g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + G_{\mu\nu}](\Phi - \Phi_0)^2 + \\
+ \frac{1}{2} g_{\mu\nu} \mu^2(\Phi - \Phi_0)^2 - \frac{1}{4} g_{\mu\nu} \delta^2(\Phi - \Phi_0)^4 - \frac{1}{4} g_{\mu\nu} \frac{\mu^4}{2},
\]

(7.5)
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with

$$\Box (\Phi - \Phi_0)^2 = 2(\Phi - \Phi_0)\Box \Phi + 2\nabla_\mu \Phi \nabla_\nu \Phi,$$

(7.6)

$$\nabla_\mu \nabla_\nu (\Phi - \Phi_0)^2 = 2(\Phi - \Phi_0)\nabla_\mu \nabla_\nu \Phi + 2\nabla_\mu \Phi \nabla_\nu \Phi.$$

Here $\nabla_\mu$ represents a covariant derivative in the metric $g_{\mu\nu}$. For a static and spherically symmetric space time, the $t - t$ component of scalar stress energy tensor is given as

$$T_0^0 = -\frac{1}{2} g^{11} (\nabla_1 \Phi)^2 + \frac{1}{6} G_0^0 (\Phi - \Phi_0)^2$$

(7.7)

$$+ \frac{1}{6} \mu^2 (\Phi - \Phi_0)^2 - \frac{1}{12} \delta^2 (\Phi - \Phi_0)^4 - \frac{1}{12} \frac{\mu^4}{3^2},$$

and the $r - r$ component of stress energy tensor is given as

$$T_1^1 = \frac{1}{6} g^{11} (\nabla_1 \Phi)^2 - \frac{1}{3} g^{11} (\Phi - \Phi_0) \nabla_1^2 \Phi + \frac{1}{3} (\nabla_1 \Phi)^2 + \frac{1}{6} G_1^1 (\Phi - \Phi_0)^2$$

$$+ \frac{1}{6} \mu^2 (\Phi - \Phi_0)^2 - \frac{1}{12} \delta^2 (\Phi - \Phi_0)^4 - \frac{1}{12} \frac{\mu^4}{3^2}. $$

(7.8)

The metric of a static (3+1) black hole may be given as

$$ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2.$$

(7.9)

In the above metric, $\lambda$ is a function of $r$ only and let $\nu = \lambda(r) + f(t)$. Here, $f(t)$ is an arbitrary function of $t$. There is no loss of generality in setting $f(t) = 0$, since it can be absorbed in the definition of $t$, i.e., by replacing $e^{f(t)} dt$ by $dt$. With this redefinition of the time
coordinate, \( \nu = -\lambda \). Then

\[
G_0^0 = G_1^1 = \frac{1}{r^2} \frac{d}{dr}[r(1-e^{2\nu})]. \tag{7.10}
\]

Applying Eq. (7.10) in Einstein’s equation \( G^\mu_\nu - \kappa T^\mu_\nu = 0 \), we get

\[
T_0^0 - T_1^1 = 0. \tag{7.11}
\]

The concept of scalar hair is applicable only to a static black hole, since only in that case that we will be able to solve the field equation. From Eq. (7.11), we get

\[
-2g^{11}(\nabla_1 \Phi)^2 - (\nabla_1 \Phi)^2 + g^{11}(\Phi - \Phi_0)\nabla_1^2 \Phi = 0. \tag{7.12}
\]

Eq. (7.12) is a covariant differential equation. In Eq. (7.12), properties such as mass and self interaction terms of scalar field do not come explicitly. In Eq. (7.12), \( \nabla_1^2 \Phi \) can be written in the ordinary derivative as

\[
\nabla_1^2 \Phi = \partial_i^2 \Phi - \Gamma^i_{11} \partial_i \Phi, \tag{7.13}
\]

where \( \Gamma \) is the usual Christoffel symbol and \( i \) runs from 0 to 3. In the above case, all the Christoffel symbols except \( \Gamma^1_{11} \) are zeroes and \( \Gamma^1_{11} = \lambda' = -\nu' \). Now, Eq. (7.12) gets modified as,

\[
-2g^{11}(\partial_1 \Phi)^2 - (\partial_1 \Phi)^2 + g^{11}(\Phi - \Phi_0)[\partial_1^2 \Phi + \partial_1 \nu \partial_1 \Phi] = 0. \tag{7.14}
\]

In quest of scalar hair, the general principle is to get a non-trivial solution which is compatible with a proper metric that hides singu-
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larity. So we propose a solution of the form

\[ \Phi(r) = \frac{a}{r} + \Phi_0, \quad (7.15) \]

where \( \Phi_0 \) is the asymptotic value of scalar field and 'a' is a constant which may be derived from the mass of black hole and scalar field. Non-trivial solution of scalar field have been proposed by many people earlier but never extended it to the concept of scalar hair [135, 145]. By substituting Eq. (7.15) in Eq. (7.14), we get

\[ -\partial_1 \Phi + g^{11}(\Phi - \Phi_0)\partial_1 \nu = 0. \quad (7.16) \]

From Eq. (7.16), we can determine the metric function which will be given in the Sec.7.3. The profile of Eq. (7.15) shows that the field has a finite value \( \Phi_h \) at the horizon and then falls to the asymptotic value \( \Phi_0 \). The variation of \( \Phi \) against \( r \) is shown in Fig. (7.1). The field has the highest magnitude at the horizon and falls to the asymptotic value \( \Phi_0 \). The profile of the scalar field shows that a trace of scalar field is hidden behind the event horizon. If we get a proper metric (a metric with horizon and temperature) in addition to the solution, then that shows the existence of a strong hair.

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The form of metric, which is compatible with the scalar solution, will be now determined. Since \( g^{11} = -e^{2\nu} \), Eq. (7.16) becomes

\[ \partial_1 \Phi + e^{2\nu}(\Phi - \Phi_0)\partial_1 \nu = 0. \quad (7.17) \]
By introducing a transformation of the type [146]

\[ \nu = \frac{1}{2} \log(1 + f), \]  \hspace{1cm} (7.18)

we get from Eq. (7.17)

\[ \frac{d\Phi}{(\Phi - \Phi_0)} = -\frac{1}{2} df, \]  \hspace{1cm} (7.19)

where \( f \) is a radial function. Now introduce a gauge transformation of the type, \( \Phi - \Phi_0 = \Phi_0 \). Then

\[ d\Phi = d\Phi_0, \]

\[ \frac{d\Phi}{\Phi} = -\frac{1}{2} df. \]  \hspace{1cm} (7.20)

Integrating Eq. (7.20), we get

\[ \log \Phi = -\frac{f}{2} + C_0. \]  \hspace{1cm} (7.21)
In the asymptotic limit, \( \nu = 0 \) and hence \( f = 0 \). Putting a new value, say \( \Phi_0 \), as the asymptotic value of \( \Phi \) in the new scale, we can obtain

\[
\log\left( \frac{\Phi}{\Phi_0} \right) = -\frac{f}{2}.
\]

(7.22)

Since the transformation is only of a gauge type, we get that, \( \frac{\Phi}{\Phi_0} = \frac{\Phi}{\Phi_0} \). Thus from Eqs. (7.18), (7.22), we find

\[
e^{2\nu} = 1 - 2\log\left( \frac{\Phi}{\Phi_0} \right).
\]

(7.23)

Eq. (7.23) represents the metric function which is compatible with the scalar solution, \( \Phi(r) = \frac{a}{r} + \Phi_0 \). In the asymptotic limit the metric function,

\[
e^{2\nu} = 1,
\]

(7.24)

coincides with those of Schwarzschild and RN black holes. Denoting the field at the horizon as \( \Phi_h = \Phi_0 e^{1/2} \), the radius of the horizon can be obtained as:

\[
r_h = \frac{a}{\Phi_0 (e^{1/2} - 1)}.
\]

(7.25)

In Eq. (7.23), \( \log\left( \frac{\Phi}{\Phi_0} \right) \) can be expanded as a series if \( \frac{\Phi}{\Phi_0} \leq 2 \) and it is true in the present case. Therefore,

\[
\log\left( \frac{\Phi}{\Phi_0} \right) = \left( \frac{\Phi}{\Phi_0} - 1 \right) - \frac{1}{2} \left( \frac{\Phi}{\Phi_0} - 1 \right)^2 + \frac{1}{3} \left( \frac{\Phi}{\Phi_0} - 1 \right)^3 \ldots.
\]

(7.26)

The metric function then may be written as a series as,

\[
e^{2\nu} = 1 - 2\left( \frac{\Phi}{\Phi_0} - 1 \right) + \left( \frac{\Phi}{\Phi_0} - 1 \right)^2 - \frac{2}{3} \left( \frac{\Phi}{\Phi_0} - 1 \right)^3 \ldots
\]

(7.27)
In Eq. (7.27), let \( \frac{\Phi}{\Phi_0} - 1 = \frac{a}{\Phi_0 r} = \frac{b}{r} \). The metric function gets modified as,

\[
e^{2\nu} = 1 - \frac{2b}{r} + \frac{b^2}{r^2} - \frac{2b^3}{3r^3} \ldots
\]  

(7.28)

The above mentioned metric is in unison with a recent work[147]. Eq. (7.28) represents a composite metric which would converge to different metrics by truncating appropriate terms.

### 7.3.1 Study of metric

Eq. (7.27) may be written in a concise form as,

\[
e^{2\nu} = 1 - 2\left[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left( \frac{\Phi}{\Phi_0} - 1 \right)^n \right].
\]  

(7.29)

When \( n = 1 \), we have the Schwarzschild like black hole:

\[
e^{2\nu} = 1 - 2\left( \frac{\Phi}{\Phi_0} - 1 \right) = 1 - \frac{2b}{r}.
\]  

(7.30)

In this case, \( \frac{\Phi}{\Phi_0} = \frac{3}{2} \). The radius of horizon is \( r_h = 2 \frac{a}{\Phi_0} = 2b \).

When \( n = 2 \), we have the extremal case similar to extremal RN black hole. The metric function is reduced to,

\[
e^{2\nu} = 1 - 2\left( \frac{\Phi}{\Phi_0} - 1 \right) + \left( \frac{\Phi}{\Phi_0} - 1 \right)^2 = [1 - \left( \frac{\Phi}{\Phi_0} - 1 \right)]^2
\]  

(7.31)

\[= (1 - \frac{b}{r})^2.\]

In the extremal case, \( \frac{\Phi}{\Phi_0} = 2 \), which gives the maximum value of \( \frac{\Phi}{\Phi_0} \). The radius of horizon is \( r_h = \frac{a}{\Phi_0} = b \). When \( n = 3 \), \( \frac{\Phi}{\Phi_0} = \frac{13}{8} \) and \( r_h = \frac{5}{3} \frac{a}{\Phi_0} = \frac{5}{3} b \). When \( n = 4 \), \( \frac{\Phi}{\Phi_0} = \frac{17}{10} \) and \( r_h = \frac{2}{3} \frac{a}{\Phi_0} = \frac{2}{3} b \).

It is obtained that the horizon's radius increases and decreases with
diminishing magnitude as \( n \) increases. We can thus introduce more types of black holes by putting \( n = 5, 6, \ldots \). But as the series progress, the series very quickly diminishes. The variation of scalar field \( \Phi \) against \( r \) for different black holes are shown in Fig. (7.2). The thick line represents the case with \( n = 1 \). The normal line graph represents the case with \( n = 2 \). The dashed line represents the case with \( n = 3 \) and the dotted line represents the case with \( n = 4 \). It may be thought that the signature 'a' in the scalar solution is derived from the mass of the black hole. But, the signature of scalar field in the metric is due to the asymptotic value of scalar tensor \( T_0^0 \). The asymptotic value of \( T_0^0 \) is given as (Eq. (7.7)),

\[
T_0^0 = \frac{1}{6} G_0^0(\infty) \Phi_0^2 - \frac{1}{12} \frac{\mu^4}{\delta^2}.
\]  
(7.32)

As \( G_0^0 \) vanishes in the asymptotic limit we get,

\[
T_0^0 = -\frac{1}{12} \frac{\mu^4}{\delta^2}.
\]  
(7.33)
The trace term corresponding to scalar field may be written as,

$$\eta(r) = -\int_{r_h}^{r_0} 4\pi r^2 T_0^0 dr. \quad (7.34)$$

Substituting Eq. (7.33) in Eq. (7.34), we get,

$$\eta = \frac{\pi \mu^4}{9 \delta^2} (r_0^3 - r_h^3). \quad (7.35)$$

\(\eta\) of Eq. (7.35) has its contribution in the making of signature of scalar field in the metric.

### 7.3.2 Stability of field

There exist a possibility that a transformation of 'r' coordinate may eliminate the scalar field, making it unstable. The technique to ascertain the stability of the field is to see whether the field has a finite value in a different coordinate at the horizon. With \(n = 1\), the metric would become, \(e^{2\nu} = 1 - 2b/r\). So if we define

$$dr_* = \frac{dr}{\sqrt{1 - 2b/r}}, \quad (7.36)$$

we get

$$r_* = \frac{\sqrt{r - 2b} + 2b(\sqrt{2b - r})arctan[\frac{\sqrt{r}}{\sqrt{2b - r}}]}{\sqrt{r - 2b}} + C_1, \quad (7.37)$$

where \(C_1\) is a constant integration. At the horizon, \(r_* = C_1\). So the field at the horizon in the new coordinate is

$$\Phi = \frac{a}{C_1} + \Phi_0, \quad (7.38)$$
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which is finite. Hence the field can not be eliminated at the horizon by a coordinate transformation, i.e., the field is stable. Stability of field shows that the hair does not fall out.

7.3.3 Mass of hairy black hole

Mass of a hairy black hole, in general, must be greater than a non-hairy black hole. The mass \( m_{rh} \) of a hairy black hole is related to the mass \( M_{rh} \) of a non-hairy black hole \[69\] through the relation,

\[
\begin{align*}
m_{rh} &= M_{rh} + 4\pi r^2 \int_{r_h}^r r[V(\Phi) - V(\Phi_\infty) + \frac{1}{2}e^{2\nu} \Phi'^2]dr.
\end{align*}
\] (7.39)

In Eq. (7.39), \( V(\Phi) = 0 \) at the horizon and \( V(\Phi_\infty) = \frac{1}{4} \frac{\mu^4}{\delta^2} \). As the distance from the center of black hole increases, \( m(r_h) \) increases, but never blows up since it attains a steady value as \( r \) increases. The variation of \( m(r_h) \) against \( r \) is shown in the Fig. (7.3).
7.3.4 Entropy

The black hole with a cavity is called a dressed black hole and without it is called a naked black hole. A black hole dressed with a scalar field can be stable only inside a cavity, whose entropy may be the sum of entropy of naked black hole and that of surface. A dressed black hole results in the phenomenon called back reaction and the surface entropy is the result of back reaction. Black hole entropy is nothing but the Noether charge, i.e., $\frac{\kappa}{2\pi}S = f_\Sigma Q$, where $f_\Sigma$ is the surface integral and $Q$ is the Noether charge[148]. The entropy of a stationary black hole can be expressed as $2\pi \int Q$ over any cross-section of the horizon [149]. The entropy of a black hole can be evaluated as a local quantity on the horizon using two dimensional gravity [150]. Surface gravity of the black hole is given as,

$$\kappa = \left. \frac{\partial_r g_{tt}}{2\sqrt{-g_{tt}g_{rr}}} \right|_{r=r_h} \quad (7.40)$$

With $g_{tt} = e^{2\nu} = 1 - 2 \log(\frac{\Phi}{\Phi_0})$ and $\Phi(r) = \frac{a}{r} + \Phi_0$, we get,

$$\kappa = \frac{a}{a r_h + \Phi_0(r_h)^2}. \quad (7.41)$$

Substituting for $r_h = \frac{a}{\Phi_0(e^{1/2}-1)}$ in Eq. (7.41), we find the temperature of black hole as,

$$T_{bh} = \frac{\Phi_0(e^{1/2} - 1)^2}{2\pi ae^{1/2}}. \quad (7.42)$$

The area of the horizon,

$$A_h = 4\pi r_h^2 = 4\pi \frac{a^2}{\Phi_0(e^{1/2}-1)^2}. \quad (7.43)$$
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exclusively depends on the parameter \( a \). Following the standard procedure, we can determine the entropy of black hole in the presence of scalar field as

\[
dA_h = \frac{4}{e^{1/2} \Phi_0} dS. \tag{7.44}
\]

On integrating Eq. (7.44),

\[
S = \Phi_0 e^{1/2} A_h + c = \Phi_h A_h + c, \tag{7.45}
\]

where \( c \) is a constant and \( S \) the entropy of black hole in the presence of scalar field. By applying the boundary condition that when \( \Phi_h = 0 \), \( S = S_0 \), the entropy of naked black hole. Then \( c = S_0 \). But by Hawking's theory, \( S_0 = \frac{A_h}{4} \). So the total entropy is written as,

\[
S = \Phi_h \frac{A_h}{4} + \frac{A_h}{4} = S_s + S_0. \tag{7.46}
\]

Eq. (7.46) clearly indicates that the scalar field contributes to the entropy of a black hole.

7.4 Thermodynamics

When the Hawking radiation is fully thermal, the thermal pressure is \( \frac{1}{3} \alpha T_{loc}^4 \), where \( T_{loc} = \frac{T_{bh}}{\sqrt{-g_{00}}} \). A dressed black hole (with e.m field, dilation field etc) possess a black hole temp, \( T_{bh} \leq \frac{h}{\sqrt{4 \pi A_h}} \) [151]. The stress-energy tensor of the radiation is a function of black hole temperature \( T_{bh} \). The hairy black hole is a thermodynamical system with a modified temperature.

The effective potential of the test particles moving in static and spherically symmetric background geometry is determined by the
Hamilton-Jacobi approach. By Hamilton-Jacobi equation,

\[ g^{k\lambda} \partial_k S \partial_{\lambda} S + \mu^2 = 0, \quad (7.47) \]

where \( \mu \) is the mass of scalar field and,

\[ S(t, r, \phi) = -Et + S(r) + L\phi. \quad (7.48) \]

\( E \) and \( L \) are the constant energy and angular momentum of the test particle. Substituting Eq. (7.48) in Eq. (7.47) and simplifying we get the action as

\[ S(r) = \mp \int \left[ E^2 - \left( \frac{L^2}{r^2} + \mu^2 \right) f \right]^{1/2} \frac{1}{f} dr, \quad (7.49) \]

where \( f = e^{2\nu} \). From the expression for the action, we can measure the temperature of black hole by the following method.

### 7.4.1 Temperature of different black holes

(a). In Eq. (7.29), with \( n = 1 \),

\[ e^{2\nu} = 1 - 2 \left( \frac{\Phi}{\Phi_0} - 1 \right) = 1 - \frac{2b}{r}. \]

This is SBH like. As \( r \to r_h \), \( S(r) \) is modified as,

\[ S(r) = \mp \int \frac{r h E dr}{r - r_h} = \mp \beta \log(r - r_h). \quad (7.50) \]

with \( \beta = Er_h \). Assuming that the scalar field gets reflected at the horizon, the scalar field in the neighborhood of the horizon can be written as [152],

\[ \Phi(r) = e^{-i\beta \log(r-r_h)} + Re^{i\beta \log(r-r_h)}, \quad (7.51) \]
with $R$ as the coefficient of reflection. The coefficient of reflection $R$ and the probability of reflection by horizon $P$ may be given as,

$$R = e^{-2\pi \beta}; P = |R|^2 = e^{-4\pi \beta}. \quad (7.52)$$

Using the thermodynamical relation,

$$P = e^{-E/T_{bh}}, \quad (7.53)$$

we get,

$$E/T_{bh} = 4\pi \beta, \quad (7.54)$$

which gives the black hole temperature as,

$$T_{bh} = \frac{1}{4\pi r_h}. \quad (7.55)$$

This is the temperature of Schwarzschild like black hole. Eq. (7.55) reveals that the temperature of a black hole is a matter related to the radius of horizon. Since, radius of horizon is a function of parameters, such as mass, charge (vector as well as scalar) and angular momentum, it can be seen that black hole temperature depends on these parameters.

(b). In Eq. (7.29), with $n = 2$, $e^{2\nu} = [1 - (\frac{\Phi}{\Phi_0} - 1)]^2 = [1 - \frac{b}{\tau}]^2$. This is extremal case in which no black hole temperature is observed.

(c). In Eq. (7.29), with $n = 3$, $e^{2\nu} = 1 - 2(\frac{\Phi}{\Phi_0} - 1) + (\frac{\Phi}{\Phi_0} - 1)^2 - \frac{2}{3}(\frac{\Phi}{\Phi_0} - 1)^3 = 1 - \frac{2b}{\tau} + \frac{b^2}{\tau^2} - \frac{2b^3}{3\tau^3}$. The action $S(r)$ as $r \to r_h$ is given as,

$$S(r) = \mp \int \frac{E r_h^3 dr}{(r - r_h)(r_c - r_h)(r_h - r_0)}. \quad (7.56)$$
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with, \( \beta = \frac{E_r^3}{(r_c-r_h)(r_h-r_0)} \) and \( (r_c+r_h+r_0) = -2b; (r_hr_c+r_c r_0+r_b r_0) = -b^2; r_h r_c r_0 = -\frac{3}{3} b^2. \) Now,

\[
S(r) = \mp \beta \log(r - r_h). \tag{7.57}
\]

By proceeding as in the previous case, the temperature of the black hole can be shown to be,

\[
T_{bh} = \frac{(r_c-r_h)(r_h-r_0)}{4\pi r_h^3}. \tag{7.58}
\]

Thus we see that the hairy black hole acts as a thermodynamical system with proper horizon and temperature.

7.5 conclusion

There is an argument that a regular horizon is possible only when the scalar solution is trivial [73, 74]. So when the solution is non-trivial, the event horizon will be a surface of singularity and hence it will not represent a black hole. A black hole exists only when the event horizon hides the naked singularity.

As a weak interpretation of scalar hair, a non-trivial solution of scalar field in terms of the existing conserved quantities is enough to show that there is hair [70, 71]. Whether a horizon naturally occurs, even when the solution is non-trivial, will be the primary objective of strong interpretation of scalar hair. Thus as a strong interpretation, in the presence of a scalar field, a black hole would have a signature different from mass, angular momentum and vector charge.

We have shown that a non-trivial scalar black hole solution for a massive self interacting conformal scalar field would be obtained in
the case of static (3+1) black hole. The metric proposes a horizon and temperature for the black hole. The horizon and surface temperature ensure a true black hole. The metric element has a term other than the existing conserved quantities. In our case, only a particular pair of scalar field and metric are found to be mutually compatible. We have also shown that the scalar field never vanishes in a transformed coordinate, making it stable. The hair does not fall out if the field is stable.

In the proposed metric, only one parameter, i.e., $b$ has appeared. This may invite some criticisms against the strong interpretation of scalar hair. But in the standard extremal case, same parameter does the job of mass and vector charge. Another argument against $b$ is that it may have been originated from the mass of black hole itself. But Eq. (7.35) clearly indicates that the origin of the parameter $b$ is from scalar field and hence we may conclude that scalar field can depict its signature in the proposed metric.