Chapter 6

Scalar hair for an AdS black hole

6.1 Introduction

*If the facts don’t fit the theory, change the facts.*, Albert Einstein.

The no-hair theorem in general relativity says that in the exterior of black hole the only information available regarding the black hole may be that of its mass, charge and angular momentum. All other informations about the matter which formed a black hole or infalling into it, disappear behind the black hole event horizon and are therefore permanently inaccessible to external observers. The statement that the black hole has no-hair means, there are no features other than mass, charge and angular momentum that distinguish a black hole from another one. If we construct two black holes with the same mass, charge and angular momentum,
the first being made out of ordinary matter and the second out of anti-matter, they would be completely indistinguishable.

Scalar hair in a black hole demands a non-trivial solution for the scalar field in the vicinity of a black hole. The profile of the field must be Gaussian type, as if a scalar source is present on the black hole. But, it is found that the scalar field becomes trivial if one demands a regular horizon at a finite distance from the centre [122]. A regular horizon means, a horizon which has a radius and temperature so that it would hide the singularity. A non-trivial solution seemed possible only for a black hole that exhibit naked singularity.

The question of scalar hair for a static black hole has been a matter of debate for quite some time. Situations of preservation of no-hair conjecture [123, 66, 67, 68, 69] and its violations [70, 71] had been reported earlier many times. Saa [73] deduced a theorem which shows that the static spherically symmetric exterior solution for the gravitational field equations in a wide class of scalar tensor theories will essentially reduce to the well known Schwarzschild solutions if one has to hide the naked singularity at the centre of a black hole by the event horizon [73, 74]. Bocharova and Bekenstein [124, 125] constructed solutions with regular horizon for a scalar field conformally coupled to Einstein's gravity. They are

\[ ds^2 = -(1 - \frac{r_0}{r})^2 dt^2 + (1 - \frac{r_0}{r})^{-2} dr^2 + r^2 d\Omega^2 \]
\[ \Phi = -\frac{r_0 \alpha^{-1/2}}{r - r_0}, \]

where \( \alpha = \frac{8\pi G}{6} \). Initially, divergence of field at the horizon was considered as a pathology of the solution. However, further analysis [126] suggested that the divergence of \( \Phi \) on the horizon might be innocu-
ous. In general, asymptotically flat, static, spherically symmetric nontrivial solutions of scalar field $\Phi$ coupled to Einstein's gravity do not possess a regular horizon [127]. As a strong interpretation, Bizon [128] and Weinberg [72] showed that a theory allows a hairy black hole, if there is a need to specify quantities other than the conserved charges defined at asymptotic infinity, in order to characterize completely a stationary black hole solution. Eq. (6.1) is characterized by the Arnowitt-Deser-Misner (ADM) mass $r_0/2$ [129], and scalar charge $Q = 4\pi r_0 \alpha^{-1/2}$. So Eq. (6.1) carries "hair" and violates the "no-hair conjecture".

But the divergence of the scalar field at the horizon is so severe that Eq. (6.1) doesn't satisfy Einstein's equation at the horizon, hence Eq. (6.1) need not be a black hole solution [69]. As a weak interpretation of scalar hair, non-trivial solution in terms of conserved charges was mooted [72]. Using that ideology, scalar hair was reported in asymptotically anti-de Sitter spacetime and asymptotically flat spacetime [70, 71]. There is no regular black hole solution when the scalar field is massless or has a convex potential (containing only mass term). Examples of black hole solution, such as, $\Phi \sim r^{-3/2} \cos(\sqrt{\frac{4\beta - 9}{2}} \ln r)$, where $\beta$ is a constant, in the symmetric and asymmetric double well potential was reported [70]. This unexpected discovery of black hole solutions by considering asymptotically AdS, rather than asymptotically flat spacetime were thoroughly analyzed [69]. Applying the principle of the conservation of the '$r'$ component of the total energy-momentum tensor ($T^\mu_{\mu} = 0$) in Einstein's equation [130], they showed that there were no non-trivial static and spherically symmetric black hole solutions in the asymptotically AdS with true cosmological constant. The asymptot-
ically AdS region corresponds to one where the effective cosmological constant is

\[ \Lambda_{\text{eff}} = \Lambda + 8\pi V(\Phi_\infty). \] (6.2)

Eq. (6.2) gives the idea that in the asymptotically flat case we must require \( V \) to go to 0 at infinity, while in asymptotically AdS case any non zero value of \( V \) at infinity can be absorbed into the effective cosmological constant. So the argument is that only the change in the true cosmological constant makes any sense, not the change in effective cosmological constant.

In this chapter, our aim is to find whether the scalar hair would occur in the asymptotically AdS space time with regular horizon and we took (2+1) non-rotating Bananas-Teitelboim-Zanelli (BTZ) black hole as a model [47, 131] which shares many of the features of its (3+1) dimensional counterparts and also in the Reissner-Nordström (RN) black hole. In contrast with (3+1) dimensional general relativity, the (2+1) dimensional model has only finite physical degrees of freedom. As a result, questions about quantum gravity can be explored in considerable detail [132, 133]. we report non-trivial black hole solution showing no divergence at the horizon and asymptotically falling to the vacuum value. The scheme of the chapter is as follows. In Sec. 6.2, non-trivial scalar hair solution is obtained for the BTZ and RN black holes by solving the scalar field equations. Mass of hairy black hole is also discussed in this section. In Sec. 6.3, the complete solution to scalar field equation is obtained. In Sec. 6.4, stability analysis is described by applying the theory of perturbation. In Sec. 6.5, we give the conclusion.
6.2 Solution with a minimal coupling

A non-trivial radial solution of a scalar field, whose source is a massive double well scalar potential, in the vicinity of the $BTZ$ black hole will be discussed here. We will restrict our consideration to the minimally coupled case. Consider the action

$$S = \int d^3x \sqrt{-g} \left\{ \frac{1}{16\pi} (R - 2\Lambda) - \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) \right\}, \quad (6.3)$$

where $\Phi$ is the real scalar field and $V(\Phi)$ is the potential. The metric of $BTZ$ black hole is

$$ds^2 = -f e^{-2\delta} dt^2 + f^{-1} dr^2 + r^2 d\phi^2. \quad (6.4)$$

Here we propose a regular horizon so that it hides the naked singularity. For the $BTZ$, $f = -m + \frac{r^2}{\Lambda}$, where $\Lambda = -\frac{1}{\ell^2}$, the cosmological constant. Let the functions $m$ and $\delta$ depend only on $r$. The Lagrangian density for the action of Eq. (6.3) can be obtained from

$$L = \sqrt{-g} \left\{ \frac{1}{16\pi} (R - 2\Lambda) - \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) \right\}. \quad (6.5)$$

In Eq. (6.5), $\Phi$ is a radial function. Using the Euler's equation

$$\frac{d}{dr} \left( \frac{\partial L}{\partial \Phi'} \right) - \frac{\partial L}{\partial \Phi} = 0, \quad (6.6)$$

where $\Phi' = \frac{\partial \Phi}{\partial r}$. We get the scalar field equation from Eq. (6.6) and $\delta'$ by varying the action as

$$[r e^{-\delta} f \Phi']' = e^{-\delta} r \frac{dV(\Phi)}{d\Phi}$$

$$\delta' = -2\pi \Phi'^2. \quad (6.7)$$
Eq. (6.7) governs the scalar field in the black hole spacetime and perturbation. Whether a non-trivial solution results from Eq. (6.7) will be the present investigation. Multiplying both sides of Eq. (6.7) by $(\Phi - \Phi_0)$, where $\Phi_0$ is the vacuum value or the asymptotic value of $\Phi$, we get:

$$(\Phi - \Phi_0)[r e^{-\delta} f \Phi'] = (\Phi - \Phi_0)e^{-\delta} r \frac{dV}{d\Phi}$$

$$\frac{d}{dr}[(\Phi - \Phi_0)(r e^{-\delta} f \Phi')] = (\Phi - \Phi_0)e^{-\delta} r \frac{dV(\Phi)}{d\Phi} + e^{-\delta} r f \Phi'^2. \quad (6.8)$$

Integrating both sides of Eq. (6.8), from $r_h$ to $r$, we have

$$[(\Phi - \Phi_0)(r e^{-\delta} f \Phi')]_{r_h}^r = \int_{r_h}^r dr e^{-\delta} r[(\Phi - \Phi_0)\frac{dV}{d\Phi} + f \Phi'^2]. \quad (6.9)$$

In the asymptotic limit, $\Phi \rightarrow \Phi_0$ and $\Phi' \rightarrow 0$. At $r = r_h$, $f = 0$. So the left hand side of Eq. (6.9) vanishes. Therefore

$$[(\Phi - \Phi_0)\frac{dV(\Phi)}{d\Phi} + f \Phi'^2] = 0. \quad (6.10)$$

We know that $f$ is positive outside the horizon of the AdS black hole. If $V(\Phi)$ represents a convex potential (mass term only), then $rac{dV(\Phi)}{d\Phi} > 0$. So Eq. (6.10) is zero only when $\Phi = \Phi_0$ and $\Phi' = 0$. That is, the magnitude of field remains constant throughout the space. But such a solution is trivial. Since we need a non-trivial solution, a double well potential is proposed to act in the Eq. (6.10). The double well potential has a mass term and a self interaction term with an asymptotic value. The potential function against $\Phi$ is
shown in Fig. (6.1). The double well potential function is given as

\[ V(\Phi) = -\frac{1}{2}\mu^2(\Phi - \Phi_0)^2 + \frac{1}{4}\lambda^2(\Phi - \Phi_0)^4 + \frac{1}{4}\mu^4/\lambda^2, \tag{6.11} \]

where, \( \mu \) is the mass and \( \lambda \) is the self interaction coefficient of scalar field. In the double well potential case, \( \frac{dV(\Phi)}{d\Phi} < 0 \) in the limit \( \Phi \) away from \( \Phi_0 \), so Eq. (6.10) can become zero even if \( \Phi \neq \Phi_0 \) and \( \Phi' \neq 0 \). On substituting Eq. (6.11) and \( f = \frac{r^2}{l^2} - m \) in Eq. (6.10) and putting, \( \Phi - \Phi_0 = x; d\Phi = dx; \frac{E'}{\lambda} = a \), we get

\[ \frac{dr}{\sqrt{r^2 - l^2m}} = \frac{dx}{\lambda dx \sqrt{a^2 - x^2}}. \tag{6.12} \]

Integrating Eq. (6.12) and rearranging we get

\[ \Phi = \frac{\mu}{\lambda} \sec h[-\mu l \arccos h(\frac{r}{r_h})] + \Phi_0, \tag{6.13} \]

with \( r_h = l\sqrt{m} \). The presence of \( ' - \mu l' \) inside the bracket will restrict us from getting a well defined solution. So we put, \( \mu = -\frac{1}{l} \),
thus defining a negative cosmological constant for the spacetime. So \( \Phi = \frac{k}{\lambda} \sec h[\arccosh(\frac{r}{r_h})] + \Phi_0 \). At the horizon, \( \Phi = \frac{k}{\lambda} + \Phi_0 \), which is finite and in the asymptotic limit, \( \Phi = \Phi_0 \). The profile of Eq. (6.13) is shown in Fig. (6.2), which brings the characteristic of scalar field. The bold line is for BTZ black hole and thin line is for RN black hole (will be shown in next sub-section). In the figure, we put \( \mu = 1, \lambda = 0.1, \Phi_0 = 0.1 \). It can be shown that the curve drops at the rate \( 1/r^2 \), i.e., the field depends inversely on \( r \). We can

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.2.png}
\caption{Variation of field variable against \( r \), with \( \mu = 1, \lambda = 0.1, \Phi_0 = 0.1 \).}
\end{figure}

now conclude that scalar hair is possible with a potential function of double well type (Fig. (6.1)). The potential is maximum at \( \Phi = \Phi_0 \) and is zero at \( \Phi = \mu/\lambda + \Phi_0 \). The effective cosmological constant is

\[ \Lambda_{eff} = \Lambda + 2\pi \mu^2 / \lambda^2 = \Lambda + \Lambda_{add}. \]  

Eq. (6.14) reveals that the cosmological constant has an origin in the scalar field. The term \( \Lambda_{add} \) may be assumed to be a trace of scalar field.
6.2 Solution with a minimal coupling

6.2.1 Scalar hair in Reissner-Nordström black hole

In the expression, \[ (\Phi - \Phi_0) \frac{dV(\Phi)}{d\Phi} + f \Phi' = 0 \], substitute, \( f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \). Now solve for \( \Phi \). We get

\[
\Phi = 4 \frac{\mu}{\lambda} \frac{e^{-\mu \sqrt{Q^2 - 2mr + r^2}}}{(-m + \sqrt{Q^2 - 2mr + r^2}) \mu m} + \Phi_0. \tag{6.15}
\]

At the horizon, \( \sqrt{Q^2 - 2mr + r^2} = 0 \). Since, \( \mu m \) is negligibly small, the field at the horizon is

\[
\Phi = 4 \frac{\mu}{\lambda} + \Phi_0. \tag{6.16}
\]

In the asymptotic limit, \( \Phi \to \Phi_0 \). From Eqs. (6.13) and (6.15), we can see that the scalar field solutions are expressed in terms of the prevailing parameters of the black hole. That shows that the hair is weak. To have a strong hair, the solutions must have conserved quantities other than mass, charge and angular momentum.

6.2.2 Mass of hairy black hole

Mass of hairy black hole must be in general greater than the mass of non-hairy black hole. Now let us compare \( m(r_h) \), the mass of the non-trivial static black hole of radius \( r_h \), with \( M(r_h) \), the mass of the corresponding naked black hole of the same radius. We have

\[
M(r_h) = \frac{r_h^2}{l^2}. \tag{6.17}
\]

The mass of the non-trivial black hole is [69]

\[
m(r_h) = M(r_h) + 2\pi \int_{r_h}^r \left[ V(\Phi) - V(\Phi_\infty) + (1/2) f \Phi'^2 \right] dr. \tag{6.18}
\]
In the above equation, \( V(\Phi) \) is zero at the horizon and \( V(\Phi_\infty) = \frac{1}{4}\mu^4/\lambda^2 \). We have, \( f = m(-1 + \frac{r^2}{r_h^2}) \); \( \Phi' = -\frac{\mu r_h^2}{\lambda r^4} \) and \( \Phi = \frac{\mu r_h^2}{\lambda r^4} \) (as can be shown from Eq. (6.13)). We get, \( 2\pi r[V(\Phi) - V(\Phi_\infty)] = -\frac{\pi \mu^4 r_h^2}{\lambda^2 r^4} + \frac{\pi \mu^4 r_h^2}{2\lambda^2 r^3} \) and \( \pi r f \Phi'^2 = \frac{\pi \mu^2 r_h^2}{\lambda^2 l^2} (-\frac{r^2}{r_h^2} + \frac{1}{r^2}) \). On integrating Eq. (6.18) we get

\[
m(r_h) = M(r_h) - \frac{\pi \mu^4 r_h^2}{\lambda^2} \log r/r_h - \frac{\pi \mu^4 r_h^2}{4\lambda^2} \left( \frac{1}{r^2} - \frac{1}{r_h^2} \right) + \frac{\pi \mu^2 r_h^2}{2\lambda^2 l^2} \left( \frac{1}{r^2} - \frac{1}{r_h^2} \right) + \frac{\pi \mu^2 r_h^2}{\lambda^2 l^2} \log r/r_h. \tag{6.19}
\]

with \( m(r_h) \) is the mass of black hole with scalar hair and \( M(r_h) \) the mass with out scalar hair. From the above assumption, \( \mu = -\frac{1}{l} \) we get

\[
m(r_h) = M(r_h) + \frac{\pi \mu^4 r_h^4}{4\lambda^2} \left( \frac{1}{r^2} - \frac{1}{r_h^2} \right). \tag{6.20}
\]

At the horizon, \( m(r_h) = M(r_h) \). As the distance from centre in-
From Eq. (6.14), \( \Lambda_{\text{eff}} = \Lambda + 2\pi \mu^4 / \lambda^2 = \Lambda + \Lambda_{\text{add}} \), where, \( \Lambda_{\text{add}} = -\frac{1}{l^2_{\text{add}}} \). Therefore, \( m(r_h) = M(r_h) + \frac{r_h^2}{8l^2_{\text{add}}} \). Since \( \frac{1}{l^2} \) is positive, \( m(r_h) > M(r_h) \). This is an essential condition to have scalar hair [69, 130]. It is seen that \( m(r_h) \) is a function of \( r \). The function \( m(r_h) \) in principle diverges, but under condition \( \mu = -\frac{1}{l} \), the mass never blows up. The profile of a mass function for non-trivial Schwarzschild black hole was given earlier [130]. The mass function of nontrivial BTZ black hole is depicted in Fig. (6.3).

### 6.3 Solution to scalar field equation

The solution to scalar field equation under the action of gravity and scalar potential will be now calculated. Defining [131, 134]

\[
dr_* = e^{\delta \frac{dr}{f}}, \quad (6.21)
\]

The metric of BTZ in the tortoise co-ordinate is obtained as

\[
ds^2 = -fe^{-2\delta} dt^2 + fe^{-2\delta} dr_*^2 + r(r_*)^2 d\phi^2, \quad (6.22)
\]

with

\[
\sqrt{-g} = fe^{-2\delta} r(r_*). \quad (6.23)
\]

The field equation of massive scalar field under the action of a scalar potential coupled to gravity is

\[
[\Box + \xi R] \Phi = \frac{dV(\Phi)}{d\Phi}. \quad (6.24)
\]

The mass term and interaction term of the scalar field have been included in the potential \( V(\Phi) \). The scalar curvature, \( R = -\frac{6}{l^2} \)
For minimal coupling, Eq. (6.24) is modified as

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} [\sqrt{-g} g^{\mu \nu} \partial_{\nu}] \Phi = \frac{dV(\Phi)}{d\Phi}.
\]  

(6.25)

On expanding Eq. (6.25) we get field equation as

\[
-\frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{r(r_*)} \frac{\partial \Phi}{\partial r(r_*)} + \frac{\partial^2 \Phi}{\partial r(r_*)^2} + \frac{f e^{-2\delta}}{r(r_*)^2} \frac{\partial^2 \Phi}{\partial \phi^2}
\]

(6.26)

\[-f e^{-2\delta} \{-\mu^2 (\Phi - \Phi_0) + \lambda^2 (\Phi - \Phi_0)^3\} = 0.\]

For separating the variables, express \( \Phi = T(t)R_k^p(r(r_*))Y_m(\phi). \) The radial field equation is

\[
\frac{d^2 R_k^p}{d^2 r(r_*)} + \frac{1}{r(r_*)} \frac{dR_k^p}{d r(r_*)} + [k^2 + \frac{f e^{-2\delta}}{r(r_*)^2} \beta^2
\]

(6.27)

\[-f e^{-2\delta} \{-\mu^2 (\Phi - \Phi_0) \frac{1}{\Phi} + \lambda^2 (\Phi - \Phi_0)^3 \frac{1}{\Phi}\} R_k^p = 0,\]

where \(-\frac{1}{T} \frac{\partial^2 T}{\partial t^2} = k^2\) and \(\frac{1}{Y} [\frac{\partial^2 Y}{\partial \phi^2}] = \beta^2.\)

The wave function \( \Phi = \frac{\mu r_A}{r} \) (as can be shown from Eq. (6.13)) and \((\Phi - \Phi_0)/\Phi \simeq 1, (\Phi - \Phi_0)^3/\Phi \simeq \Phi^2.\) So the Eq. (6.27) can be modified as

\[
\frac{d^2 R_k^p}{d^2 r(r_*)} + \frac{1}{r(r_*)} \frac{dR_k^p}{d r(r_*)} + [k^2 + \frac{f e^{-2\delta} \beta^2}{r(r_*)^2} + f e^{-2\delta} \mu^2 - \frac{f e^{-2\delta} \mu^2 r_A^2}{r(r_*)^2} ] R_k^p = 0.
\]  

(6.28)

In Eq. (6.28), \( R_k^p \) represents the radial field equation. The effective potential of scalar field from Eq. (6.28) is

\[
V_{eff} = -\frac{f e^{-2\delta} \beta^2}{r(r_*)^2} - f e^{-2\delta} \mu^2 + \frac{f e^{-2\delta} \mu^2 r_A^2}{r(r_*)^2}.
\]

(6.29)
At the horizon, $f = 0$, hence, $V_{eff} = 0$. From Eq. (6.13), $\Phi' = -\frac{\mu}{\lambda} \frac{r_h}{r^2}$.

Substituting the value of $\Phi'$ in Eq. (6.7), we get

$$\delta = \frac{\pi \mu^2 r_h^2}{\lambda r^{2\gamma}}. \quad (6.30)$$

As we go away from the horizon, $f \sim \frac{r^2}{r^2}$. Substituting the value of $f$ and $e^{-2\delta}$ in Eq. (6.29), we get

$$V_{eff} = \frac{-\beta^2}{\lambda} + \frac{2\pi \mu^2 \beta^2 m}{\lambda^2 r^2} + \frac{2\pi \mu^4 m}{\lambda^2} + \mu^4 - \frac{2\pi \mu^4 m}{\lambda^2 r^2}. \quad (6.31)$$

In the above equation, terms with $\mu^4$ have been eliminated, since they are negligibly small. The role of $\mu$ and $\lambda$ of the scalar field, even though evident in the calculation of $V_{eff}$, may be omitted.

Then, $V_{eff} = \frac{-\beta^2}{\lambda} + \frac{2\pi \mu^2 \beta^2 m}{\lambda^2 r^2} \simeq \frac{\alpha^2}{r^2}$. From Eq. (6.31) it can be shown that for values of $r > r_h$, $V_{eff}$ very soon rises to a positive value and then falls and finally reaches the asymptotic value. So for majority parts of the spacetime, $V_{eff}$ is inversely proportional to $r^2$. In the deSitter spacetime the effective potential is zero both at the horizon and in the asymptotic limit. The $V_{eff}$ rises only slowly as we go away from the horizon, reaches a maximum value and then falls to zero in the asymptotic limit[136]. As we approach the horizon, $f \rightarrow 0$ and hence Eq. (6.28) reduces to

$$\frac{d^2 R^p_k}{d^2 r(r_*)^2} + \frac{1}{r(r_*)} \frac{d R^p_k}{d r(r_*)} + k^2 R^p_k = 0. \quad (6.32)$$

This is a Bessel equation with root $k$ and order zero. As we go away from horizon, the potential function drops approximately to
the order $1/r^2$. So Eq. (6.28) away from the horizon is
\[ \frac{d^2 R_\alpha^c}{d^2r(r_*)^2} + \frac{1}{r(r_*)} \frac{dR_\alpha^c}{dr(r_*)} + \left[k^2 - \frac{a^2}{r^2}\right]R_\alpha^c = 0. \tag{6.33} \]

This is a Bessel equation of root $k$ and order $\alpha$. So throughout the spacetime, the radial field can be represented by a Bessel function. The above two equations have solutions which are normalizable.

\[ \int_0^\infty R_\alpha^c(r)R_\alpha^{c'}(r)r(r_*)dr(r_*) = \delta(k - k'). \tag{6.34} \]

Solution to Eq. (6.32) can be represented as
\[ R_\alpha^0(r(r_*)) = \sum_{n=0}^\infty [(-1)^n (\frac{kr}{2})^{2n} \frac{1}{\Gamma(n+1)n!}]. \tag{6.35} \]

Solution to Eq. (6.33) is given as
\[ R_\alpha^\alpha(r(r_*)) = \sum_{n=0}^\infty [(-1)^n (\frac{kr}{2})^{\alpha+2n} \frac{1}{\Gamma(\alpha+n+1)n!}]. \tag{6.36} \]

where $\Gamma$ is the usual Gamma function. Eq. (6.35) and Eq. (6.36) represent the full solution of scalar field which may be normalized and hence can be quantized.

### 6.4 Stability analysis

In Sec 6.2 we have found a scalar hair for non-rotating $BTZ$ and $RN$ black holes. A mere non-trivial solution is not enough to show that there is definite hair, since, no-hair conjecture still holds if the solution soon falls out. So we will now consider the stability of the scalar field solution. The first order perturbed equation for the scalar
6.4 Stability analysis

The scalar field is obtained from the Eq. (6.28) as [69]

\[ \ddot{\Phi} = -\hat{A}\Phi = \left[ \frac{d^2}{dr^2(r_*)} + \frac{1}{r(r_*)} \frac{d}{dr(r_*)} - V_{eff} \right] \Phi, \quad (6.37) \]

where \( dr_* = e^\delta \frac{dr}{f} \) and \( \hat{A} = -\left[ \frac{d^2}{dr^2(r_*)} + \frac{1}{r(r_*)} \frac{d}{dr(r_*)} - V_{eff} \right] \) is a Hermitian operator. If \( \Phi \) is a vector of the Hilbert space, then the inner product in the context of BTZ black hole is given as [137]

\[ \langle \Phi_1, \Phi_2 \rangle = \int_0^{2\pi} \int_0^{\infty} r(r_*) \Phi_1 \Phi_2 d\phi dr(r_*). \quad (6.38) \]

If, \( \langle \Phi, \hat{A}\Phi \rangle > 0 \), then the state function \( \Phi \) which represents the scalar field around the black hole is stable. The solution to scalar field equation, \( [\Box + \xi R] \Phi = \frac{dV(\Phi)}{d\Phi} \), can be expressed as, \( \Phi(t, r, \phi) = \Phi(r)e^{ikt}Y_m(\phi) \), where \( \Phi(r) \) is the radial part of the solution which is a Bessel function as given by Eq. (6.36). Using the radial part of the scalar field solution and substituting the operator \( \hat{A} \) in Eq. (6.38), we get

\[ \langle \Phi, \hat{A}\Phi \rangle = -2\pi \int_0^{\infty} dr(r_*) R_k^\alpha [rR_k^{\alpha''} - R_k^{\alpha'} - V_{eff} R_k^\alpha]. \quad (6.39) \]

If \( R_k^\alpha \) is positive definite, then \( R_k^{\alpha''} \) is also positive definite but \( R_k^{\alpha'} \) is negative definite. Then, \( rR_k^{\alpha''} \) is negative definite and \( -rV_{eff} R_k^\alpha \) is positive definite in the range 0 to \( \infty \). So the net term inside the bracket of Eq. (6.39) is negative definite. Again, if \( R_k^\alpha \) is negative definite, the net term inside the bracket is positive definite. In both cases, the expression \( \langle \Phi, \hat{A}\Phi \rangle \geq 0 \). That means, the scalar field equation is stable under the first order perturbation and hence the configuration is stable. This shows the existence of a stable hair.
As a second order perturbation, we have

$$\frac{d^2 R_\alpha^k}{dr(r_\ast)^2} + (k^2 - V_{eff}) R_\alpha^k = 0$$

(6.40)

with smooth real potentials independent of $k$ and of short range. If $k^2$ is negative, the perturbations diverge exponentially with time and then the solution is unstable [67]. The wave function $R_\alpha^k$ must approach zero at the horizon when the eigenmode is negative [67]. But in our case the eigenmode is positive and hence the wave function is not zero at the horizon (Eqs. (6.13) and (6.16)). As a result solution is stable against radial perturbations and the scalar hair does not fall out.

### 6.5 Conclusion.

There is a general feeling that anything that is added to a black hole will not induce any trace of it on the black hole except mass, angular momentum and charge and hence no-hair conjecture. As a strong interpretation, in the presence of a scalar field, black hole should possess a trace different from mass, angular momentum and charge inorder to have the notion of a scalar hair. The scalar potential at infinity acts as an added cosmological constant. That added cosmological constant can be treated as a signature of the scalar field, and hence scalar hair. But since, only the effective cosmological constant is affected and not the true cosmological constant, the above argument did not draw much attention. As a weak interpretation, a non-trivial solution in terms of the existing conserved quantities is enough to show that there is hair. Saa [73] and Banerjee [74] ar-
argued that a regular horizon is possible only when the scalar solution is trivial and when the solution is non-trivial, the horizon will be a surface of singularity. Torii [67] argued that the hair falls out easily, since for every non-trivial solution the eigenmode is negative.

In our case a non-trivial scalar black hole solution is obtained with a double well potential as the source, for $BTZ$ black hole and $RN$ black hole with regular horizons. The horizon is not singular and it hides singularity. Since the eigenmode is positive, the scalar field is finite at the horizon and falls to a minimum value in the asymptotic limit. The mass of black hole with hair is greater than that with out hair with a condition $\mu = -\frac{1}{l}$. All these conclusions show that scalar hair solution is possible for the $BTZ$ and $RN$ black holes.