Chapter 4

Generalized second law and entropy bound in a black hole

4.1 Introduction

*Things fall apart; the centre cannot hold and more anarchy is loosed upon the world.*

W. B. Keats, The Second coming.

The studies on black holes during the last 30 years have brought to light strong hints of a deep and fundamental relationship among gravitation, thermodynamics and quantum theory. The cornerstone of this relationship is the black hole thermodynamics, where it appears that laws of black hole mechanics are, in fact, simply the ordinary laws of thermodynamics, at least in a theoretical perspective. There is a huge increase, of the order of $10^{20}$, in
entropy of a star during its gravitational collapse to become a black hole. It is presumably associated with the gravitational microstates of the black hole through the number of ways in which a black hole of a given mass 'm' and area 'A' can be formed.

It has been proposed that black hole entropy can be due to quantum entanglement between the interior and exterior states of the black hole and also that entanglement entropy $S$ is equal to quantum corrections of Bekenstein-Hawking entropy, $S_{bh}$[99, 100, 101, 102]. In these arguments, the black hole entropy is related to the entanglement entropy in the $QFT$ in the same spacetime [103, 104]. Even now calculations of black hole entropy, $GSL$ and the conditions to conserve $GSL$ are live subjects in black hole physics. Appropriate vacuum state for the interior of a box with reflecting walls being lowered towards a Schwarzschild black hole is the Boulware state, which warrants finite but high value for the stress-tensor [105]. In a gedanken experiment, Jenson [106] showed that $GSL$ is valid when matter with negative gravitating energy is added to a near-extremal U(1)-charged static black hole in Einstein-Maxwell theory.

The analogy between laws of black hole mechanics and classical statistical mechanics will break down once the $GSL$ is violated. Unruh and Wald [43, 44] analyzed this situation by performing a gedanken experiment. They considered a thought experiment of lowering a box containing matter toward a black hole taking into account of the effect of “acceleration radiation” (the effective radiation a stationary observer near a black hole would observe) [21]. The resulting change in entropy of the black hole $\Delta S_{bh}$ in the round-trip process was shown to be greater than $S_s$, the original entropy of the contents of the box. The existence of Hawking radiation [19]
preserves the validity of the GSL because the thermal radiation is the state of matter and radiation which maximizes entropy at fixed energy and volume[44].

In order to make GSL valid, Bekenstein [50, 107] proposed a conjecture: There exists a universal upper bound on entropy $S$ for an arbitrary system of effective radius $R$ and energy $E$, which can be expressed in Planck units ($c = h = G = k_B = 1$) as $\frac{S}{E} \leq 2\pi R$.

If the Hawking radiation were fully thermal, the radiation pressure at the horizon would be infinitely high, since $P(r) = \frac{1}{3} \alpha \frac{T^4}{\chi(r)}$, where $\chi(r)$ is the red shift factor. This situation refrain us from bringing the box in the gedanken experiment to the event horizon. The concept that the thermal pressure of Hawking radiation assumes infinite value at the horizon is unwarranted as no physical pathology is believed to exist at the horizon. To overcome this problem, many have [45, 46, 47, 75] proposed stress-energy calculations and obtained a finite value for the temporal and radial components of stress-energy tensor at the horizon.

The information loss paradox in a black hole can be resolved by treating the Hawking radiation as not exactly thermal [108] and this concept will be used in this chapter. This implies that the pressure of Hawking radiation will have only a finite value at the horizon eventhough Boulware vacuum inside the box suggests high value of stress energy tensor to explain the thermal radiation. Hence the box containing matter can be brought to the horizon. The state equations of radiation in asymptotic limit are given as

$$\rho = \alpha T_r^4; s = \frac{4}{3} \alpha T_r^3,$$

(4.1)
where, $\rho$ is the energy density, $s$ is the entropy density, $T_r$ is the temperature of radiation and $\alpha$ is a constant. The asymptotic state equations of radiation when applied in the calculations of gedanken experiment, it is obtained that the GSL is violated. As the sanctity of $GSL$ cannot be questioned, the state equations of radiation (Eq. (4.1)) need to be modified.

In a gedanken experiment, a box filled with radiation is lowered on to the horizon. For an inertial observer (freely falling), outside the box, he sees a Hartle-Hawking vacuum. This vacuum has a finite positive energy density value at the horizon. But he sees a vacuum inside the box with a negative energy density which blows up at the horizon. This vacuum is called the Boulware vacuum. The interior of the box which is initially empty will acquire a negative energy density through the lowering process. This negative energy density is an outcome of Boulware vacuum. So in evaluating the buoyancy on the box by the Hawking radiation, the pressures due to the stress-energies of Boulware vacuum and Hartle-Hawking vacuum must be taken into consideration.

The knowledge that the Hawking radiation near the horizon is not fully thermal, leads us to the conjecture that the gravitational field near the horizon can influence the equations of state of radiation. The state equations of radiation near the Schwarzschild black hole were earlier studied [109]. In this chapter we discuss the acceptability of general state equations of radiation near the horizon of a Reissner-Nordström (RN) black hole. The spacetime around the $RN$ black hole is static and spherically symmetric so that the Ricci scalar $R$ is zero but $R_{ab} \neq 0$. But in a Schwarzschild black hole both $R$ and $R_{ab}$ equal to zero. The scheme of the chapter is as follows. In Sec.
4.2 Violation of GSL?

4.2. We describe the violation of GSL where ordinary equations of radiation are used. In Sec. 4.3, new equations of radiation and the upper bound are given. In Sec. 4.4, we give the conclusion.

4.2 Violation of GSL?

In a gedanken experiment, a box filled with matter or radiation is brought from infinity to the horizon and the bottom lid is opened so that the contents are released to the black hole. The box is then filled with Hawking radiation and is lifted back to infinity. In this process, we can determine the gain of entropy of the black hole as matter is swallowed and the loss of entropy as the Hawking radiation is lost. So, whether the loss of entropy is greater than the gain of entropy is the bone of contention in the study of GSL.

A RN black hole with mass $M$ and charge $Q$ is situated inside a spherical cavity with radius $r_0$ greater than $r_h$, negligible mass and perfect reflectability. Let us imagine that the black hole and Hawking radiation be in thermal equilibrium in the cavity. We fill a rectangular box of volume $aA$ ($a$ the height and $A$ the cross section area of the box) with thermal radiation of temperature $T_r$ at infinity. Now lower the box (Fig.4.1) adiabatically through a hole on the cavity to the horizon, release the contents, then slowly raise the box back to infinity. In general $T_r \gg T_{bh}$. The increase in the energy of the black hole in the above process is [43, 44]

$$\epsilon = E_r - W_\infty,$$  \hspace{1cm} (4.2)

where, $W_\infty$ is the work delivered to infinity and $E_r$ is the rest energy.
Generalized second law and entropy bound in a black hole

Figure 4.1: Gedankenexperiment: Black hole is kept inside a cavity and a box filled with radiation is brought to the horizon.

of radiation in the box. This increase in the energy of the black hole manifests as the increase in the entropy. We have

\[ E_r = \alpha a A T_r^4 \]

\[ W_{\infty} = W_1 - W_2, \tag{4.3} \]

where, \( E_r \) is the energy of radiation in the box, \( W_1 \) is the work delivered to infinity on account of the weight of box and radiation and \( W_2 \) the work delivered to the black hole on account of the buoyancy force of Hawking radiation. The entropy of radiation inside the box is

\[ S_r = \frac{4}{3} \alpha a A T_r^3. \tag{4.4} \]

Since the process of lowering and raising the box is adiabatic, \( S_r \) remains constant, since no heat exchange takes place in an adiabatic process.
4.2 Violation of GSL?

4.2.1 Calculation of $W_1$

$W_1$ is the energy delivered to infinity as the box is dropped on to the horizon under the action of the gravitational force of the black hole and may be given as

$$W_1 = E_r - E;$$

$$E = A \int_{l^+}^{l+a} \rho(x) \chi(x) dx. \quad (4.5)$$

When the box is brought to the horizon, the bottom lid of the box is opened so that the radiation in the box will be in contact with the Hawking radiation. Then, $E$ is the energy of the radiation inside the box after it has attained the thermal equilibrium with the Hawking radiation near the horizon, $l$ is the distance from the horizon to the bottom of box, $\chi(x)$ is the red shift factor and $x$ is the proper distance from the horizon to the box. Under thermodynamic equilibrium between acceleration radiation and radiation inside the box at a height $l$, the temperature of radiation becomes $T_0(l)$. Then we have

$$\rho(x) = \alpha T_{loc}^4$$

$$\chi(x) = [1 - \frac{2M}{r(x)} + \frac{Q^2}{r^2(x)}]^{1/2}, \quad (4.6)$$

where $T_{loc}$ is the temperature of acceleration radiation locally. $T_{loc}$ is related to the equilibrium temperature $T_0(l)$ as [110]

$$T_{loc} = \frac{T_0(l)}{\chi(x)}. \quad (4.7)$$
On the horizon, $T_0(l = 0) = T_{bh}$. For $l \ll r_h$ and writing $r = r_h + x$, $\chi(x)$ in Eq. (4.6) may be modified as

$$\chi(x) = \frac{2^{1/2}(M^2 - Q^2)^{1/4}}{r_h} x^{1/2}. \quad (4.8)$$

$\chi(x)$ will be zero at $x = 0$, i.e., on the horizon. Eq. (4.8) is a reasonably good approximation of the metric function inside the cavity in which the black hole is situated. On substituting Eqs. (4.6, 4.8) in the expression for $E$ in Eq. (4.5), we get

$$E = \frac{\alpha A r_h^2}{\sqrt{2l}} \frac{T_0^4}{(M^2 - Q^2)^{3/4}}. \quad (4.9)$$

Similarly, the entropy of the contents of the box is

$$S_r = \frac{4}{3} \alpha T_0^3 A \int_{l}^{l+a} \frac{dx}{\chi(x)^3} \int_{l}^{l+a} \frac{dx}{\chi(x)^3}$$

$$= \frac{4\alpha A r_h^3}{3\sqrt{2l}} \frac{T_0^3}{(M^2 - Q^2)^{3/4}}. \quad (4.10)$$

But in an adiabatic process entropy never changes. So

$$S_r = \frac{4}{3} \alpha A T_r^3. \quad (4.11)$$

From (4.10) and (4.11), the equilibrium temperature is obtained as

$$T_0 = (2l)^{1/6} a^{1/3} (M^2 - Q^2)^{1/4} \frac{T_r}{r_h}. \quad (4.12)$$

From Eq. (4.9), energy of the radiation after attaining thermal equilibrium is obtained as

$$E = (2l)^{1/6} \alpha a^{4/3} A (M^2 - Q^2)^{1/4} \frac{T_r^4}{r_h}. \quad (4.13)$$
But, $E_r = a A a T^4_r$, which is the energy of thermal radiation in the box. So
\[ E = \frac{(2l)^{1/6} a^{1/3} (M^2 - Q^2)^{1/4}}{r_h} E_r. \] (4.14)

Now work done to infinity on account of gravity is
\[ W_1 = E_r - E = E_r - \frac{(2l)^{1/6} a^{1/3} (M^2 - Q^2)^{1/4}}{r_h} E_r. \] (4.15)

The work $W_1$ depends on the distance $l$ from the horizon to the bottom of the box.

### 4.2.2 Calculation of $W_2$

The work done on the black hole on account of the buoyancy of Hawking radiation is [43, 44]
\[ W_2 = A \int^{l+a} P(x) \chi(x) dx, \] (4.16)

where, $P(x)$ is the pressure of Hawking radiation. If the Hawking radiation is fully thermal, then
\[ P(x) = \frac{1}{3} \alpha T^4_{\text{loc}} = \frac{1}{3} \alpha \frac{T^4_{\text{bh}}}{\chi^4(x)}. \] (4.17)

So, at the horizon, $P(x) \to \infty$. This makes $W_2 \to \infty$, which means the box cannot be dropped on to the horizon. If the pressure is finite, the box can be brought to the horizon. In classical gravity, the geometry is treated classically while matter fields are quantized. In examining the semiclassical perturbations of the $RN$ metric caused by the vacuum energy of the quantized scalar fields, we can treat the background electromagnetic field as a classical field. The right hand
side of the semiclassical Einstein equations will then contain both classical and quantum stress-energy contributions

\[ G^\mu_\nu = 8\pi [T^\mu_\nu + \langle T^\mu_\nu \rangle]. \]  

(4.18)

\( T^\mu_\nu \) represents the classical stress-energy tensor of scalar field and \( \langle T^\mu_\nu \rangle \) is its quantum counterpart. Now consider the situation where the black hole is in thermal equilibrium with the quantized field, so that the perturbed geometry continues to be static and spherically symmetric. To first order in \( \epsilon = \frac{\hbar}{M^2} \), the general form of the perturbed RN metric may be written as:

\[ ds^2 = -[1 + 2\epsilon \rho(r)] f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \]  

(4.19)

where, \( f = (1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}) \) and \( [1 + 2\epsilon \rho(r)] \) represents the perturbation due to the scalar field. In order to save the black hole from extinction due to evaporation, the black hole is assumed to be placed inside a massless reflecting spherical shell. Inside the shell the quantum field and Hawking radiation are in thermodynamic equilibrium and hence the black hole mass function \( m(r) \) contains classical mass and the quantum first-order perturbation. So [111]

\[ m(r) = M[1 + \epsilon \mu(r)]. \]  

(4.20)

This equation explains the back reaction. The metric perturbation functions, \( \rho(r) \) and \( \mu(r) \) are determined by solving the semi-classical Einstein’s equation expanded to first order in \( \epsilon \) [111].

\[ \frac{d\mu}{dr} = -\frac{4\pi r^2}{M \epsilon} \langle T^t_t \rangle, \]  

\[ \frac{dm}{dr} = \frac{4\pi r^2}{M \epsilon} \langle T^r_r \rangle. \]
The right hand side of Eq. (4.21) is divergent on the horizon unless \( [\langle T_r^r \rangle - \langle T_t^t \rangle] \) vanishes there. Not only that both \( \langle T_r^r \rangle \) and \( \langle T_t^t \rangle \) must be finite at the horizon. The expectation value of stress-energy tensor of a quantized massive scalar field in the \( RN \) spacetime is given as [111]

\[
\langle T_r^r \rangle \big|_{r_h} = \langle T_t^t \rangle \big|_{r_h} = \frac{6\pi^3 \epsilon}{105\tilde{m}^2} T_{bh}^4, \tag{4.22}
\]

where, \( \tilde{m} \) the mass of scalar field and \( \epsilon = \frac{\hbar}{M_T} \). Hence, a probable form of the Hawking pressure at the horizon is given by Eq. (4.22) multiplied by a constant \( \alpha \).

But, it has been shown that [105], a box which is static on the horizon suffers the pressure of acceleration radiation and it induces a Boulware state inside box. In particular, the Hartle-Hawking state has been used in the computation of the renormalized stress-energy tensor in Eq. (4.22) and it is valid only for a freely falling observer near the horizon, not for the stationary (that is, accelerating) box. Therefore the effect of acceleration radiation also needs to be taken into consideration in the calculation of work \( W_2 \).

Let the vacuum state inside the box is Boulware \((B - state)\) and that outside is Hartle-Hawking \((H - H)\). Considering the contributions of radiation pressures on the top and bottom of the box, the pressure on each side will be the difference of the pressure of \( H - H \) vacuum on outside and pressure of \( B - vacuum \) on the inside [105].
Generalized second law and entropy bound in a black hole

So we get

\[ P_{\text{net}} = P_{HH} - P_B \]  
\[ (4.23) \]

The pressure inside the box due to \( B - \text{vacuum} \) is given as [112]

\[ T^t_t(B) = kT^4_{bh} \left( \frac{T+}{r} \right)^6 \left[ \frac{A^t}{(1 - \frac{r+}{r})^2} + B^t_t \right] \]  
\[ (4.24) \]

where \( A^t_t \) and \( B^t_t \) are finite tensors and \( r = (r_+ + l) \), where \( l \) is the proper distance of the box from the horizon. Hence Eq. (4.24) is modified as

\[ T^t_t(B) = kT^4_{bh} \left( \frac{T+}{r_+ + l} \right)^6 \left[ A^t_t \left( \frac{r_+ + l}{l^2} \right)^2 + B^t_t \right] \]  
\[ (4.25) \]

So the net force to be applied to bring the box to the horizon is

\[ P_{\text{net}} = \alpha \frac{6\pi^3\epsilon}{105m^2} T^4_{bh} + \alpha kT^4_{bh} \left[ \frac{r_+^6}{(r_+ + l_T)^6} \left( A^t_t \left( \frac{r_+ + l_T}{l_T^2} \right)^2 + B^t_t \right) - \frac{r_+^6}{(r_+ + l_B)^6} \left( A^t_t \left( \frac{r_+ + l_B}{l_B^2} \right)^2 + B^t_t \right) \right] \]  
\[ (4.26) \]

where, \( l_T \) and \( l_B \) are the proper distances of the top and bottom of the box from the horizon. By using Eq. (4.16), we now get

\[ W_2 = \frac{8(M^2 - Q^2)^{1/4} \pi^3 \epsilon^3 \alpha^{3/2} A}{105\sqrt{2}m^2r_H} \alpha T^4_{bh} + \]
\[ \frac{4(M^2 - Q^2)^{1/4} \pi^{3/2} A \alpha kT^4_{bh}}{3\sqrt{2}r_h} \frac{r_+^6}{(r_+ + l_T)^6} \left( A^t_t \left( \frac{r_+ + l_T}{l_T^2} \right)^2 + B^t_t \right) - \frac{r_+^6}{(r_+ + l_B)^6} \left( A^t_t \left( \frac{r_+ + l_B}{l_B^2} \right)^2 + B^t_t \right) \]  
\[ (4.27) \]

The increase in the energy of the black hole in the gedanken experi-
4.2 Violation of GSL?

ment is obtained from Eqs. (4.2,4.3,4.15,4.27) as

\[
\varepsilon = \frac{(2l)^{1/6}a^{1/3}(M^2 - Q^2)^{1/4}}{r_h} E_r + \frac{8(M^2 - Q^2)^{1/4} \pi^3 \varepsilon a^{3/2} A}{105 \sqrt{2} m^2 r_h} \alpha T_{bh}^4 + \frac{4(M^2 - Q^2)^{1/4} a^{3/2} A \alpha k T_{bh}^4}{3 \sqrt{2} r_h} \left[ \frac{r_+^6}{(r_+ + l_T)^6} \left( A_t^t \frac{(r_+ + l_T)^2}{l_T^2} + B_t^t \right) - \frac{r_+^6}{(r_+ + l_B)^6} \left( A_t^t \frac{(r_+ + l_B)^2}{l_B^2} + B_t^t \right) \right](4.28)
\]

On the horizon, \( l \approx 0 \), for the \( H_H \) vacuum, but \( l_B \neq 0 \) for the \( B \) vacuum because of the thickness of the box. Hence

\[
\varepsilon = \frac{8(M^2 - Q^2)^{1/4} \pi^3 \varepsilon a^{3/2} A}{105 \sqrt{2} m^2 r_h} \alpha T_{bh}^4 + \frac{4(M^2 - Q^2)^{1/4} a^{3/2} A \alpha k T_{bh}^4}{3 \sqrt{2} r_h} \left[ \frac{r_+^6}{(r_+ + l_T)^6} \left( A_t^t \frac{(r_+ + l_T)^2}{l_T^2} + B_t^t \right) - \frac{r_+^6}{(r_+ + l_B)^6} \left( A_t^t \frac{(r_+ + l_B)^2}{l_B^2} + B_t^t \right) \right](4.29)
\]

The increase of the black hole entropy in the gedankenexperiment may be given as,

\[
\Delta S_{bh} = \frac{8(M^2 - Q^2)^{1/4} \pi^3 \varepsilon a^{3/2} A \alpha T_{bh}^3}{105 \sqrt{2} m^2 r_h} + \frac{4(M^2 - Q^2)^{1/4} a^{3/2} A \alpha k T_{bh}^3}{3 \sqrt{2} r_H} \left[ \frac{r_+^6}{(r_+ + l_T)^6} \left( A_t^t \frac{(r_+ + l_T)^2}{l_T^2} + B_t^t \right) - \frac{r_+^6}{(r_+ + l_B)^6} \left( A_t^t \frac{(r_+ + l_B)^2}{l_B^2} + B_t^t \right) \right](4.30)
\]

Since, \( T_{bh} \ll T_r \) we find, \( \Delta S_{bh} \ll S_r \). This is a violation of GSL, but GSL is more or less a universal law, hence must be conserved. In
the above calculations, we took, $\rho = \alpha T_{loc}^4$ and $s = \frac{4}{3} \alpha T_{loc}^3$, which are not true, near the horizon. These equations don't prevail, unless the Hawking radiation is fully thermal. So we would expect a modified state equations of radiation under gravity.

### 4.3 State equations of radiation

As Hawking radiation is not fully thermal, the buoyancy would be finite and hence the box can be brought to the horizon. Since the gravity is very strong near the horizon, the equations of radiation near the horizon would be affected. By the first law of thermodynamics, we have

$$d(\rho V) = T_{loc} ds - p dV,$$

(4.31)

where, $P$ is the pressure of thermal radiation, $\rho$ is the energy density, $s$ is the entropy density and $V$ is the volume of the box. The local temperature of Unruh radiation is $T_{loc}$. The above equation yields

$$\rho + p = s T_{loc},$$

$$dp = s dT_{loc}.$$  \hspace{1cm} (4.32)

For a static spacetime, the hydrostatic equilibrium equation, derived from $\nabla^a T_{ab} = 0$, for a perfect fluid stress-energy tensor [43] is

$$\nabla_a p = (\rho + p) \left[ \frac{\zeta^b}{\chi} \right] \nabla_b \left[ \frac{\zeta^a}{\chi} \right]$$

$$= -(\rho + p) \frac{1}{\chi} \nabla_a \chi,$$

(4.33)
where, $\zeta^a$ is a static Killing vector field. Since the Hawking radiation satisfies the hydrostatic equilibrium, from Eq. (4.31), we have

$$\frac{d[\chi(x)p]}{dx} = -\rho(x)\frac{d\chi(x)}{dx},$$

(4.34)

where, $\chi(x)$, is the metric function close to the horizon. In the flat space situation, the relation connecting $\rho$ and $s$ is given as

$$s_r = \frac{4}{3T_r} \rho_r. \quad (4.35)$$

The term $\frac{4}{3} \frac{1}{T_r}$ is the proportionality term connecting $\rho$ and $s$, which can be expressed as $C(\infty)$. This term is not a constant, but a parameter that depends on the distance from horizon and may be expressed as $C(l)$, where $l$ is the distance from the horizon to the bottom of the box. In the spacetime of black hole, red shift factor also must be taken into account. Therefore, we may propose that [109]

$$s = C(l)\rho(x)\chi(x). \quad (4.36)$$

This relation will converge to the flat space situation when there is no gravity. Substituting Eq. (4.36) in Eq. (4.32), we get

$$\rho(x) + p = C(l)\rho(x)\chi(x)T_{loc} = C(l)\rho(x)T_0;$$

$$p = \rho(x)[C(l)T_0 - 1]. \quad (4.37)$$

From Eqs. (4.34) and (4.37), we get the expressions of radiation in the context of $RN$ black hole as

$$\rho(x) = \rho_0 \frac{-C T_0}{C T_0 - 1}.$$
Generalized second law and entropy bound in a black hole

\[ s(x) = C(l) \rho_0 \chi^{\frac{1}{C T_{\infty} - 1}}. \]  

(4.38)

Eq. (4.38) represents the modified state equations of radiation and are more realistic in explaining the physical situation near the horizon. If \( \rho_0 \) is the energy density in the asymptotic limit and in the asymptotic limit \( \chi(\infty) = 1 \), then

\[ \rho(\infty) = \rho_0 \]
\[ s(\infty) = C(\infty) \rho_0 = \frac{4}{3} \frac{1}{T} \rho_0. \]

(4.39)

Eq. (4.38) converges to flat spacetime equations (Eq. (4.1)), as \( \chi(\infty) \to 1 \). The state equation of radiation in the context of Schwarzschild black hole had been utilized in calculating the entropy of self-gravitating radiation systems [113]. As we approach the horizon, \( \chi \to 0 \), hence the energy density increases but never becomes infinity because of the thickness of the box. From Eq. (4.38) and \( S_r = \frac{4}{3} \chi \alpha T_r^3 aA \), it can be shown that

\[ C(l) \chi^{\frac{1}{C T_{\infty} - 1}} = \frac{4}{3} \frac{1}{T}. \]

(4.40)

The R.H.S of Eq. (4.40) is a constant. As \( l \to 0 \), both \( \chi \) and \( \frac{1}{C T_{\infty} - 1} \to 0 \). \( C(l) \) increases as we approach the horizon and on the horizon, \( C(l \to 0) = \frac{4}{3} \frac{1}{T_{\text{bh}}} \).

4.3.1 Generalized second law

In calculating the entropy change of black hole, we have earlier considered the flat spacetime equations of radiation. Now we will eval-
uate $W_1$ with the new equations of radiation. We have

$$W_1 = E_r - E;$$

$$E = A \int_l^{l+a} \rho(x)dx$$

$$= A \rho_0 \int_l^{l+a} \chi^{-\xi}dx,$$  \hspace{2cm} (4.41)

where, $\xi = \frac{C\rho_0}{(C\rho_0 - 1)}$. By substituting Eq. (4.8) and Eq. (4.38) in Eq. (4.41), we get

$$E = \frac{1}{(2 - \xi)} \frac{2aA\rho_0}{\left[4(M^2 - Q^2)\right]^{\xi/4}} \left[r_H/\sqrt{a}\right]^\xi.$$  \hspace{2cm} (4.42)

The entropy may be calculated as

$$S_r = A \int_l^{l+a} sdx = A \int_l^{l+a} C(l) \rho(x)\chi(x)dx$$

$$= AC(l)\rho_0 \int_l^{l+a} \chi^{1-\xi}dx.$$  \hspace{2cm} (4.43)

Eq. (4.43) is evaluated using Eq. (4.8) near the horizon as $(l \ll r_H)$

$$S_r = \frac{1}{(3 - \xi)} \frac{2aAC(l)\rho_0}{\left[4(M^2 - Q^2)\right]^{\xi-1}} \left[r_h/\sqrt{a}\right]^{\xi-1}.$$  \hspace{2cm} (4.44)

But entropy can also be written as, $S_r = \frac{4}{3}aA\alpha T_r^3$. Equating this equation with Eq. (4.44) and evaluating for $\rho_0$, we get

$$\rho_0 = \frac{(3 - \xi)[4(M^2 - Q^2)]^{\xi-1}}{\frac{3}{2}C(l)(\frac{r_h}{\sqrt{a}})^{\xi-1}} \alpha T_r^3.$$  \hspace{2cm} (4.45)

In the asymptotic limit, $C(\infty) = \frac{4}{3} \frac{1}{T_r}$. We can calculate the asymptotic value of energy density $\rho_0$ by using the relation, $\xi(\infty) =$
Generalized second law and entropy bound in a black hole

\[
\frac{C(\infty)T_0}{C(\infty)T_0 - 1} \approx 0, \text{ considering the fact that } T_r \gg T_0. \text{ Substituting } C(\infty) \text{ and } \xi(\infty) \text{ in Eq. (4.45), we get}
\]

\[
\rho_0 = \frac{3}{2} \frac{r_h/\sqrt{a}}{\left[4(M^2 - Q^2)\right]^{1/4}} \alpha T_r^4. \quad (4.46)
\]

But \[\frac{r_h/\sqrt{a}}{\left[4(M^2 - Q^2)\right]^{1/4}}\] is a dimensionless constant and it may be absorbed in \[\frac{3}{2}\]. Now substitute \(\rho_0\) in Eq. (4.42)

\[
E = 3 \frac{\alpha T_r^4}{2 (2 - \xi)} \frac{2aA}{\left[4(M^2 - Q^2)\right]^{(\xi+1)/4}} \left[\frac{r_h/\sqrt{a}}{\xi+1}\right]^{\xi+1}. \quad (4.47)
\]

As \(l \to 0\), \(\xi(l \to 0) = \frac{4/(3\xi_0)}{\left(4/(3\xi_0)T_0 - 1\right)} \approx 1\). Energy of radiation near the horizon is obtained from Eq. (4.47) as

\[
E = 3 \frac{[r_h/\sqrt{a}]^2}{\left[4(M^2 - Q^2)\right]^{1/2}} aA \alpha T_r^4. \quad (4.48)
\]

The term \[\frac{[r_h/\sqrt{a}]^2}{\left[4(M^2 - Q^2)\right]^{1/2}}\] is a dimensionless constant. Had we taken the asymptotic expressions in calculating the energy of radiation near the horizon, the value would have been approximately zero. Now in Eq. (4.28)

\[
\varepsilon = 3[r_h/\sqrt{a}]^2 aA \alpha T_r^4 \left[\frac{8(M^2 - Q^2)^{1/4}a^{3/2}A \alpha T_r^4}{105\sqrt{2}\bar{m}^2 r_h} \frac{r_+^6}{(r_+ + l_T)^6} \right. \\
\left. + \frac{4(M^2 - Q^2)^{1/4}a^{3/2}A \alpha k T_{bh}^4}{3\sqrt{2}r_h} \frac{r_+^6}{(r_+ + l_T)^6} \right] \\
(A_t^t \left(\frac{r_+ + l_T}{l_T^2} + B_t^t\right) - \frac{r_+^6}{(r_+ + l_T)^6} (A_t^t \left(\frac{r_+ + l_B}{l_B^2} + B_t^t\right)). \quad (4.49)
\]
The entropy change of the black hole in the round trip process is

\[ \Delta S_{bh} = \frac{3[M^2 - Q^2]^{1/4} \pi a^3}{4} \frac{\alpha T}{T_{bh}} \frac{8(M^2 - Q^2)^{1/4} \pi^3 \epsilon a^{3/2} A T_{bh} T_{bh}}{105 \sqrt{2m^2} r_h} \]

\[ + \frac{4(M^2 - Q^2)^{3/2} a^{3/2} A \alpha k T_{bh}^3}{3 \sqrt{2} r_h} \frac{r_+^6}{(r_+ + l_T)^6} \]

\[ (A_t^t \frac{(r_+ + l_T)^2}{l_T^2} + B_t^t) - \frac{r_+^6}{(r_+ + l_B)^6} (A_t^t \frac{(r_+ + l_B)^2}{l_B^2} + B_t^t) \] \hspace{1cm} (4.50)

The entropy of thermal radiation is \( \frac{4}{3} \alpha T^3 \). Eq. (4.50) says that the increase in the entropy of the black hole is greater than \( \frac{4}{3} T^3 \). So the GSL is conserved.

### 4.3.2 Upper bound on \( S/E \)

The upper bound on the entropy was identified as a necessary condition to conserve the GSL. Hence it is desirable to look into the verification of the upper bound on \( S/E \). We have from Eqs. (4.42, 4.44)

\[ \frac{S}{E} = \frac{(2 - \xi)}{(3 - \xi)} C(l) [4(M^2 - Q^2)]^{1/4} \frac{\sqrt{a}}{r_h}. \] \hspace{1cm} (4.51)

\( RN \) Black hole temperature is given as

\[ T_{bh} = \frac{\sqrt{M^2 - Q^2}}{2\pi r_h^2}, \] \hspace{1cm} (4.52)

where, \( M \) is the mass, \( Q \) is the charge and \( r_h \) is the horizon radius of the black hole. From Eq. (4.51)

\[ \frac{3 - \xi}{2 - \xi} = \frac{E}{S} C(l) T_{bh} \frac{4\pi \sqrt{a} r_h}{\sqrt{2(M^2 - Q^2)^{1/4}}}. \] \hspace{1cm} (4.53)
Near the horizon, \( C(l \to 0) = \frac{4}{3} \frac{1}{T_{bh}} \). Eq. (4.53) is modified with the situation \( \frac{3-\xi}{2-\xi} > 1 \), as

\[
1 < \frac{E}{S} \frac{4 \pi \sqrt{\alpha r_h}}{3^{3/4} \sqrt{2 (M^2 - Q^2)^{1/4}}}
\]

\[
S \frac{E}{S} < \frac{4 \pi \sqrt{\alpha r_h}}{3^{3/4} \sqrt{2 (M^2 - Q^2)^{1/4}}},
\]

(4.54)

Dimensionally, this formula is of the Bekenstein form \([107, 50]\). The Bekenstein upper bound is \( S/E \leq 2\pi R \). Dimensionally, \( \frac{r_h}{\sqrt{M^2 - Q^2}} = L^{1/2} \equiv a^{1/2} \). Hence Eq. (4.54) may be written as \( S/E \leq 2\pi a \), where \( a \) is the dimension of the box.

### 4.4 Conclusion

Generalized second law must be valid in all situations. When evaluating the GSL, if the asymptotic state equations of radiation are considered, the GSL will be violated. Since the Hawking radiation is not fully thermal, the gedanken experiment could be conducted close to the horizon, as the buoyancy force of Hawking radiation is finite at the horizon. The gravity is so strong near the horizon that the state equations of radiation must have been affected by it. Here we have obtained the state equation of radiation near the horizon of a Reissner-Nordström black hole and found that the GSL is conserved. In the asymptotic limit, the equations converge to the usual expressions \( \alpha T_r^4 \) and \( \frac{4}{3} \alpha T_r^3 \). The parameter \( C(l) \) connecting the entropy and energy density is \( \frac{4}{3} \frac{1}{T_r} \) in the asymptotic limit and \( \frac{4}{3} \frac{1}{T_{bh}} \) near the horizon.

In the above calculation, the upper bound on \( S/E \) is analogous to
the one given by Bekenstein. The upper bound on $S/E$ is a necessary condition to have the conservation of $GSL$. The above procedure has a slight disadvantage that the Eq. (4.38), doesn't give the exact value of $\rho$ and $s$ on the horizon because of the wrong coordinate. The correct equation will be obtained only in the absence of coordinate singularity and will be initiated somewhere else.