Chapter III

**Edge Product Graphs**

Rosa called a function $f$, a $\beta$-valuation of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct [30]. The notation of a sum graph was introduced by Harary [11] in 1990. A graph $G(V, E)$ is said to be a sum graph if there exists a bijective labeling from the vertex set $V$ to a set of positive integers $S$ such that $(x \times y) \in E$ if and only if $f(x) + f(y) \in S$.

In the chapter, we define the edge function, edge product function, edge product graph and unit edge product graph for a given graph $G$. The properties of edge product graphs are also discussed. The concepts are then extended to prove few theorems.

**Definition 3.1.1.**

Let $G(V, E)$ be a simple and connected graph. Let $P$ be a set of positive integers with $|E| = |P|$. Then any bijection $f : E \rightarrow P$ is called an edge function of the graph $G$.

**Definition 3.1.2.**

The function $F(v) = \prod \{f(e) / \text{edge } e \text{ is incident on } v\}$ on $V$ is called an edge product function of the edge function $f$.

**Definition 3.1.3.**

The graph $G(V, E)$ is said to be an edge product graph if there exists an edge function $f : E \rightarrow P$ such that the edge function $f$ and the corresponding edge product function $F$ of $f$ on $V$ have the following two conditions.
1. \( F(v) \in P \) for every vertex \( v \in V \)

2. If \( f(e_1) \times f(e_2) \times \ldots \times f(e_p) \in P \) for some edges \( e_1, e_2, \ldots, e_p \in E \) then the edges \( e_1, e_2, \ldots, e_p \) are all incident on \( v \in V \)

Example: Let \( G(V, E) \) be a given graph where \( V = \{ u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9 \} \) the vertex set and \( E = \{ u_1u_2, u_2u_3, u_3u_4, u_2u_7, u_3u_5, u_3u_6, u_8u_9 \} \) the edge set. Define the edge function \( f : E \to P \) by \( f(u_1u_2) = 30, f(u_2u_3) = 7, f(u_3u_4) = 3, f(u_2u_7) = 4, f(u_3u_5) = 20, f(u_3u_6) = 2 \) and \( f(u_8u_9) = 840 \).

The edge product function \( F \) of \( f \) is defined by \( F(u_1) = 30, F(u_2) = 840, F(u_3) = 840, F(u_4) = 3, F(u_5) = 20, F(u_6) = 2, F(u_7) = 4, F(u_8) = 840 \) and \( F(u_9) = 840 \).

Hence the given graph \( G \) is an edge product graph.

![Figure 3.1](image-url)

**Definition 3.1.4.**

Let \( G(V, E) \) be a graph. An edge \( e \in E \) bridge then the graph is denoted by \( G(e) \).
**Definition 3.1.5.**

For an edge product graph $G(V, E)$ there exists an edge function $f : E \rightarrow P$ such that an element $1 \in P$ then the graph $G$ is called an unit edge product graph.

**Theorem 3.1.6.**

If the graph $(K_{(1,q)}(e))$ is an edge product graph with an edge $e \in E$ is a bridge and $f(e) = 1$. Then $(K_{(1,q)}(e))$ is an unit edge product graph.

**Proof:**

Let the graph $G(V, E) = (K_{(1,q)}(e))$ with $V = \{v_1, v_2, \ldots, v_{(q-1)}, v_q\}$ be the vertex set and $E = \{uv_1, uv_q\} \cup \{v_1v_i : 2 \leq i \leq (q-1)\}$ be the edge set. The set of all elements of $P = \{1, 2, \ldots, (q-1), (q-1)!\}$.

Consider the edge $e = uv_1$.

The edge product function $f : E \rightarrow P$ is defined by $f(uv_1) = 1$; $f(uv_q) = (q-1)!$.

$f(v_1v_i) = i$ for $2 \leq i \leq (q-1)$

The edge product function $F$ of $f$ is defined by $F(v_i) = i$ for $2 \leq i \leq (q-1)$ and $F(u) = F(v_1) = F(v_q) = (q-1)!$.

All the edges of $G$ are incident on the vertex $v_1$ except the only one edge $uv_q$. Hence the value $f(uv_q) = (q-1)!$ can be obtained by multiplying either the functions $f(v_1v_2), f(v_1v_3), \ldots, f(v_1v_{(q-1)})$ or the functions $f(uv_1), f(v_1v_2), f(v_1v_3), \ldots, f(v_1v_{(q-1)})$.

Thus, in both the cases the corresponding edges are all incident on the vertex $v$.

Hence the graph $(K_{(1,q)}(e))$ is an unit edge product graph.
Example: Let \( G(V, E) \) be the given graph. Let \( V = \{ v_1, v_2, v_3, v_4, v_5, v_6 \} \) be the vertex set and \( E = \{ v_1v_2, v_1v_3, v_1v_4, v_1v_6, v_5v_6 \} \) be the edge set of the graph \( G \). Define the edge function \( f : E \rightarrow P \) by \( f(v_1v_2) = 2, f(v_1v_3) = 3, f(v_1v_4) = 4, f(v_1v_6) = 1 \) and \( f(v_5v_6) = 24 \). The edge product function \( F \) of \( f \) is defined by \( F(v_1) = 24, F(v_2) = 2, F(v_3) = 3, F(v_4) = 4, F(v_5) = 24 \) and \( F(v_6) = 24 \). Thus the graph \( G(V, E) = K_{1, 3}(e) \) is an unit edge product graph.
Note: In the above example, the element 1 is among the labels.

The following theorem gives a characterization for a unit edge product graph.

**Theorem 3.1.7.**

The graph $G$ is an unit edge product graph then $G$ is $K_{1,q}(e)$ for some $q \in \mathbb{N}$.

**Proof:**

Consider $G(V, E)$ is an edge product graph without isolated vertices. The mapping $f : E \rightarrow P$ is an edge function and $F$ is its corresponding edge product function.

Then there exists an edge $e = uv$ such that $f(e) = 1$. If $e_1$ is any other edge, then $f(e_1) = f(e_1) \times f(e) \in P$.

Therefore, the edges $e$ and $e_1$ are incident on a vertex and all other edges are adjacent to the edge $e$. Let $w$ be any vertex other than the vertices $u$ and $v$.

Then we have the edge product function $F(w) = F(w) \times f(e) \in P$.

If the vertices $w_1, w_2, \ldots, w_r$ are adjacent to the vertex $w$ then the edges $ww_1, ww_2, \ldots, ww_r$ and $e$ are adjacent on a vertex.

Hence, the vertex $w$ is a pendent vertex which is adjacent to either $u$ or $v$.

Suppose $\text{deg } u > 1$. Then the vertices $u_1, u_2, \ldots, u_q$ are pendent vertices which are adjacent to vertex $u$ other than $v$ and the vertices $v_1, v_2, \ldots, v_p$ are pendent vertices which are adjacent to vertex $v$ other than $u$. Then we get $F(u) = f(uu_1) \times f(uu_2) \times \ldots \times f(uu_q) \times f(uv) \in P$. Let $F(u) = f(e_i)$ where $e_i = uu_i$, for some $i$. Then $F(u) = f(uu_i) = f(uu_1) \times f(uu_2) \times \ldots \times f(uu_q)$. That is, $f(uu_2) \times \ldots \times f(uu_q) = 1$. Hence $F(v) = F(v) \times$
f(uu_2) \times \ldots \times f(uu_q) \in P. Then we have the edge e is adjacent to all other edges of G and any vertex other than u and v as a pendent vertex adjacent to either u or v.

Thus, the graph G is $K_{(1, q)}(e)$ for some q.

*Note:* In all the other edge product graphs 1 is not among the edge labels. It is shown that the unit product of edge labels is also not possible in the edge product graphs which are non unit. If f is an edge function of a non unit edge product graph G, then $f(e_1) \times f(e_2) \times \ldots \times f(e_p)$ is not one for any sub collection \{e_1, e_2, \ldots , e_q\} of the edge set of the graph G.

### 3.2. Theorems related to non unit edge product graphs

In this section, we consider a graph $G(V, E)$ which is not an unit edge product graph. The following theorem proves that a connected edge product graph can have only one edge.

**Theorem 3.2.1.**

Let $G(V, E)$ be an edge product graph. Then $K_2$ is a component of G.

**Proof:**

Let $G(V, E)$ be an edge product graph. Let $f : E \rightarrow P$ be an edge function and $F$ be an edge product function of $f$. Let $p$ be the largest element in the set of positive integers $P$, the vertices $u, v, w \in V$ and the edge $e \in E$.

For the bijection $f : E \rightarrow P$, there exists an edge $e$ joining the vertices $u$ and $v$ such that $f(e) = p$. To prove that both the vertices $u$ and $v$ are pendent vertices.

Suppose the vertex $u$ is adjacent to a vertex $w$ other than $v$. 
Then \( F(u) \geq f(uv) \times f(uw) > f(uv) = p \) which is a contradiction to our hypothesis that the element \( p \) is the largest element in \( P \). Hence the vertex \( u \) is a pendant vertex. Similarly the vertex \( v \) is also a pendant vertex.

Therefore, the vertices \( u \) and \( v \) form a component of \( K_2 \) in \( G \).

*Note:* The graph \( K_2 \) is an edge product graph.

**Theorem 3.2.2.**

Let \( G \) is an edge product graph of the graph \( G(V, E) \); where \( f : E \to P \) the edge function and \( F \) the edge product function of \( f \) of the graph \( G \). Let \( u \in V \) be any vertex and \( e_1, e_2, \ldots, e_p \) for \( p > 1 \) be a collection of edges incident on \( u \). Let \( \ell_1, \ell_2, \ldots, \ell_q \), be another collection of edges, none of them incident on the vertex \( u \) such that \( f(e_1) \times f(e_2) \times \ldots \times f(e_p) = f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_q) \). Let the edges \( \ell_1, \ell_2, \ldots, \ell_q \) are all incident on a vertex, say \( v \), with \( v \not= u \) such that \( (\text{deg } u, \text{deg } v) \in \{(p,q), (p,q+1), (p+1,q), (p+1, q+1)\} \). Then, either the vertices \( u \) and \( v \) are adjacent and \( (\text{deg } u, \text{deg } v) \not= (p, q) \) or the vertices \( u \) and \( v \) are non-adjacent and \( (\text{deg } u, \text{deg } v) = (p, q) \).

**Proof:**

Let \( G \) is an edge product graph of the graph \( G(V, E) \); where \( f : E \to P \) the edge function and \( F \) is its corresponding edge product function of \( f \). Let \( e_1, e_2, \ldots, e_p \) be a collection of edges incident on a vertex \( u \) for \( p > 1 \). Then, there may arise two cases to \( \text{deg } u \).

*Case (1) when \( \text{deg } u = p \)*

Suppose the collection of edges \( e_1, e_2, \ldots, e_p \) are incident on the vertex \( u \). Then, \( f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_q) = f(e_1) \times f(e_2) \times \ldots \times f(e_p) = F(u) \in P \)
Thus, the edges $\ell_1, \ell_2, \ldots, \ell_q$ are all incident on a vertex, say $v$. But no one of
the edge $\ell_i$, ($i = 1, 2, \ldots, q$) is incident on the vertex $u$, $u \neq v$.

Let $\deg v = (q + k)$ and $\ell_1, \ell_2, \ldots, \ell_q, \ell_{(q+1)}, \ell_{(q+2)}, \ldots, \ell_{(q+k)}$ be the edges incident on $v$. Then we have

$$F(v) = f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_q) \times f(\ell_{(q+1)}) \times f(\ell_{(q+2)}) \times \ldots \times f(\ell_{(q+k)}) \in \mathbb{P}$$

$$= f(e_1) \times f(e_2) \times \ldots \times f(e_p) \times f(\ell_{(q+1)}) \times f(\ell_{(q+2)}) \times \ldots \times f(\ell_{(q+k)}) \in \mathbb{P}$$

Hence, the edges $e_1, e_2, \ldots, e_p, \ell_{(q+1)}, \ell_{(q+2)}, \ldots, \ell_{(q+k)}$ are incident on a
vertex. But $e_1, e_2, \ldots, e_p$ are incident on $u$ and $\ell_{(p+1)}, \ell_{(p+2)}, \ldots, \ell_{(p+k)}$ are incident
on $v$ and there can be at most one edge incident on both the vertices $u$ and $v$.

Therefore, $k = 0$ or $k = 1$.

When $k = 0$, either $u$ and $v$ are not adjacent and $\deg u = p$, $\deg v = q$ or $u$ and $v$ are
adjacent with one $e_i = uv$ for $1 \leq i \leq p$ and $\deg u = p$, $\deg v = (q + 1)$. When $k = 1$, the
vertices $u$ and $v$ are adjacent with $uv = \ell_{(q+1)}$ and $\deg u = p$, $\deg v = (q + 1)$.

Case (2) when $\deg u > p$

Suppose $\deg u = (p + r)$ with $r > 0$.

Let $e_1, e_2, \ldots, e_p, e_{(p+1)}, e_{(p+2)}, \ldots, e_{(p+r)}$ be the edges incident on the vertex $u$.

Then $F(u) = f(e_1) \times f(e_2) \times \ldots \times f(e_p) \times f(e_{(p+1)}) \times f(e_{(p+2)}) \times \ldots \times f(e_{(p+r)}) \in \mathbb{P}$

$$= f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_q) \times f(e_{(p+1)}) \times f(e_{(p+2)}) \times \ldots \times f(e_{(p+r)}) \in \mathbb{P}$$

Hence, the edges $\ell_1, \ell_2, \ldots, \ell_q, e_{(p+1)}, e_{(p+2)}, \ldots, e_{(p+r)}$ are all incident on a vertex, say
$v$. say. But the edges $\ell_1, \ell_2, \ldots, \ell_q$ are not incident on the vertex $u$ and the edges $e_{(p+1)}, e_{(p+2)}, \ldots, e_{(p+r)}$ are incident on $u$. Thus, we have $v \neq u$.

There can be at most one edge incident on both the vertices $u$ and $v$ with $r = 0$ or $r = 1$.

We obtain that $r = 1$ but we assume that $r > 0$. 

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Therefore, the vertices u and v are adjacent with $e_i = uv$ for some $i$, $1 \leq i \leq (q + 1)$ (not necessarily $e_{(q + 1)}$) and $\deg u = (p+1)$ and $\deg v = (q+1)$.

**Definition 3.2.3.**

Let $G$ be an edge product graph. Let $f : E \to P$ be an edge function and $F$ be an edge product function of $f$. Let $v \in V$ be any vertex of $G$ such that $F(v) \in P$. Let $\deg v$ be $q$ and the vertices $v_1, v_2, \ldots, v_q$ are adjacent to the vertex $v$. If we obtain a graph $G'(V', E')$ from the given graph $G$ with vertex set $V' = [V - \{v\}] \cup \{v_1', v_2', \ldots, v_q'\}$ and the set $E' = [E - \{vv_1, vv_2, \ldots, vv_q\}] \cup \{v_1v_1', v_2v_2', \ldots, v_qv_q'\}$. The new graph $G'$ which we obtain from the given graph $G$ is known as flowering the vertex $v$. The following figure shows the process of flowering the vertex $v$.

![Figure 3.4](image-url)

**Remark 3.2.4.**

1. The edge sets $E$ and $E'$ have the same number of edges.
2. Define $f' : E' \rightarrow P$ by $f'(e) = f(e)$ for all $e \in E \cap E'$ and $f'(v_iv_i') = f(vv_i)$ for $1 \leq i \leq q$. Thus the mapping $f' : E' \rightarrow P$ is an edge function and $F'(u) = F(u)$ for all $u \in V \cap V'$, is its edge product function and $F'(v_i') \in P$ for $1 \leq i \leq q$, where $F(v) \in P$.

**Theorem 3.2.5.**

Let $G(V, E)$ be a connected graph and $G(V, E)$ be an edge product graph with an edge function $f : E \rightarrow P$ and the edge product function $F$. If the vertex $v$ is the only one vertex such that $F(v) \in P$, then the vertex $v$ is adjacent to a pendent vertex.

**Proof:**

Let $G(V, E)$ be a connected edge product graph where $V = \{v_1, v_2, \ldots, v_q\}$ the vertex set. The vertices $v_i, (i = 1, 2, \ldots, q)$ are all adjacent to a vertex $v$, then the degree of the vertex $v$ is $q$. That is, $\deg v = q$. Now flowering the vertex $v$, we obtain a new graph $G' (V', E')$ where the vertex set $V' = [V - \{v\}] \cup \{v_1', v_2', \ldots, v_q'\}$ and the edge set $E' = [E - \{vv_1, vv_2, \ldots, vv_q\}] \cup \{v_1v_1', v_2v_2', \ldots, v_qv_q'\}$.

The mapping $f : E \rightarrow P$ is an edge function of the graph $G$, gives an edge function $f' : E \rightarrow P$ of the graph $G'$ such that the edge product function $F'$ satisfies $F'(u) = F(u)$ for all $u \in V \cap V'$. Hence the functions $F'(u) \in P$ for all $u \in V \cap V'$ and $F'(v_i') \in P$ for $1 \leq i \leq q$.

Therefore, the graph $G' (V', E')$ is an edge product graph and $K_2$ is a component of the graph $G'$. Since the given graph $G(V, E)$ is a connected graph, one of its edge $v_iv_i'$ is a $K_2$ component of $G'$.

Thus the vertex $v_i$ is a pendent vertex which is adjacent to $v$ in $G$. 
Theorem 3.2.6.

Let $G(V, E)$ be a connected graph and $G$ has no pendent vertices. Let the mapping $f : E \to P$ be an edge function and $F$ be the edge product function of $f$ in $G$. For the vertices $v_1, v_2, \ldots, v_q$ of $G$, $F(v_i) \in P$ for $1 \leq i \leq q$. Then the induced sub graph $G$ with the vertex set $\{v_1, v_2, \ldots, v_q\}$ is not $K_q$.

Proof:

Consider a connected graph $G(V, E)$ which has no pendent vertices. The mapping $f : E \to P$ is an edge function and $F$ is an edge product function of $f$ in $G$. Then the graph $G$ is an edge product graph.

Let $v_1, v_2, \ldots, v_q$ be the vertices in $G$. Flowering the vertices in $G$, we obtain a new graph $G'$, which is also an edge product graph. Since $G$ is a connected graph, one of its edges is a $K_2$ component of $G'$. Also $G$ has no pendent vertex; only the edge $v_i v_j$ will be a $K_2$ component of $G'$. Then the induced sub graph of $G$ with the vertex set $\{v_1, v_2, \ldots, v_q\}$ has the edge $v_i v_j$ and is not $K_q$.

Corollary 3.2.7:

Let $G(V, E)$ be a connected graph and $G$ has no pendent vertices. The mapping $f : E \to P$ is an edge function and $F$ is the corresponding edge product function of $G$. There exists vertices $u$ and $v$ such that $F(u), F(v) \in P$, then the vertices are adjacent.

Corollary 3.2.8:

Let $G(V, E)$ be a connected graph. Let $f : E \to P$ be an edge function and $F$ be its corresponding edge product function of $G$. Then there exist vertices $v_1, v_2, \ldots, v_q$ of $G$
such that $F(v_i) \in P$ for $1 \leq i \leq q$. If any $v_i$ for $1 \leq i \leq q$, is not adjacent to the pendent vertex, then the induced sub graph of $G$ with vertex set $\{v_1, v_2, \ldots, v_q\}$ is not $K_q$.

**Theorem 3.2.9:**

Let $G$ be an edge product graph with edge function $f: E \rightarrow P$ and its edge product function $F$. Let $w$ be a non pendent vertex and $e = uv \in E$ be such that $F(w) = f(e)$. Then, either the induced sub graph of $G$ with vertex set $\{u, v\}$ forms a $K_2$ component in $G$ or the induced sub graph of $G$ with vertex set $\{u, v, w\}$ is $K_3$ or $P_2$ with one of the vertex $u, v$ as a pendent vertex in $G$.

**Proof:**

The given graph $G(V, E)$ is an edge product graph with $f : E \rightarrow P$ is an edge function and $F$ is the edge product function of $f$. A collection of $p$ vertices $w_1, w_2, \ldots, w_p$ are all adjacent to a vertex $w$,

we have $f(e) = F(w) = f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_p) \in P$, where $\ell_i = ww_i$ for $1 \leq i \leq p$.

Suppose the vertex $u$ is not adjacent to $w$, but the vertex $u$ is adjacent to the vertices $u_1, u_2, \ldots, u_q$ other than the vertex $v$. Then we have

$F(u) = f(e_1) \times f(e_2) \times \ldots \times f(e_q) \times f(e), \text{ where } e_i = uu_i \text{ for } 1 \leq i \leq q$

$= f(e_1) \times f(e_2) \times \ldots \times f(e_q) \times f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_p) \in P$

Therefore, the $(p + q)$ edges $e_1, e_2, \ldots, e_q, \ell_1, \ell_2, \ldots, \ell_p$ are all incident on a vertex. But the $p$ edges $e_1, e_2, \ldots, e_q$ are incident on $u$ and the $q$ edges $\ell_1, \ell_2, \ldots, \ell_p$ is incident on $w$. Since the vertices $u$ and $w$ are not adjacent, we get $q = 1$ and the vertex $u$ is a pendent vertex.

Suppose the vertices $u$ and $v$ is not adjacent to the vertex $w$. Then $u$ and $v$ are pendent vertices which form a $K_2$ component in the graph $G$. If one of the vertex $u$ and
v is adjacent to w and the other one vertex is not adjacent to w, then the second is a pendent vertex which forms $P_2$.

Suppose the vertices $u$ and $v$ are adjacent to $w$, then the induced subgraph with vertex set \{u, v, w\} is $K_3$.

**Theorem 3.2.10.**

Let $G$ be an edge product graph. The mapping $f: E \rightarrow P$ is an edge function and $F$ is an edge product function of $f$ in $G$. Let a collection of edges $\ell_1, \ell_2, \ldots, \ell_p$ ($p > 1$) be incident on a vertex, say $w$. Let $ww_i = \ell_i$ for $1 \leq i \leq p$ and there exists an edge $e = uv$ be such that $f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_p) = f(e)$. Then, either the induced subgraph of $G$ with vertex set \{u, v\} forms a $K_2$ component in $G$ or the induced subgraph of $G$ with vertex set \{u, v, w\} is $K_3$ or $P_2$ or $P_1$ with one of the vertex $u$, $v$ as a pendent vertex in $G$.

**Proof:**

Let us consider the given graph $G$ be an edge product graph with edge function $f : E \rightarrow P$ and $F$ be its corresponding edge product function. The proof follows from the following two cases.

**Case (1) when vertex $u$ is not adjacent to $w$**

Suppose vertex $u$ is adjacent to the vertices $u_1, u_2, \ldots, u_q$ other than the vertex $v$. Then we have

\[
F(u) = f(e_1) \times f(e_2) \times \ldots \times f(e_q) \times f(e) \in P,
\]

where $e_i = uu_i$ for $1 \leq i \leq q$

\[
= f(e_1) \times f(e_2) \times \ldots \times f(e_q) \times f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_p) \in P
\]

Therefore, the $(p + q)$ edges $e_1, e_2, \ldots, e_q$, $\ell_1, \ell_2, \ldots, \ell_p$ are all incident on a vertex. But the $p$ edges $e_1, e_2, \ldots, e_q$ are incident on the vertex $u$ and the $q$ edges $\ell_1, \ell_2, \ldots, \ell_p$
are incident on the vertex \( w \). Since we consider that the vertices \( u \) and \( w \) are not adjacent, then we get \( q = 1 \) and \( u \) is a pendent vertex.

Case (2) when the vertex \( u \) is adjacent to \( w \) and the edge \( uw \neq \ell_i \) for \( 1 \leq i \leq p \)

Consider vertex \( u \) is adjacent to the vertices \( u_1, u_2, \ldots, u_q \) other than the vertices \( v \) and \( w \). Then we get the following function

\[
F(u) = f(e_1) \times f(e_2) \times \ldots \times f(e_q) \times f(e) \times f(ww) \in P \quad \text{where} \quad e_i = uu_i \quad \text{for} \quad 1 \leq i \leq q
\]

Thus, the \( (p + q) \) edges \( e_1, e_2, \ldots, e_q, \ell_1, \ell_2, \ldots, \ell_p \) and the edge \( uw \) are incident on a vertex. But \( e_1, e_2, \ldots, e_q \) are incident on a vertex \( u \) and the edges \( \ell_1, \ell_2, \ldots, \ell_p \) are incident on \( w \).

Therefore, the only edge \( uw \) is incident on both \( u \) and \( w \). Then we have \( q = 1 \) and the vertex \( u \) is adjacent only to \( v \) and \( w \).

The next two cases are the vertex \( u \) is adjacent to \( w \) with an edge \( uw = \ell_i \) for some \( i, 1 \leq i \leq p \) and the vertex \( u \) coincides with \( w \).

Consider the vertices \( u \) and \( v \) are not adjacent to \( w \), then they form a \( K_2 \) component in \( G \). Suppose one of the vertex \( u \) or \( v \), say \( u \), is adjacent to \( w \) with an edge \( uw \neq \ell_i \) for \( 1 \leq i \leq p \), then the \( \deg u = 2 \) and the vertex \( v \) is a pendent vertex. Then the induced sub graph with vertex set \( \{u, v, w\} \) form a path \( P_2 \).

Suppose vertex \( u \) is adjacent to \( w \) with an edge \( uw = \ell_i \) for some \( i \). But vertex \( v \) is not adjacent to \( w \), then the induced sub graph with vertex set \( \{u, v, w\} \) form a \( P_2 \) path with \( v \) as a pendent vertex in \( G \). If the vertices \( u \) and \( v \) are adjacent to \( w \) then the induced sub graph with vertex set \( \{u, v, w\} \) form \( K_3 \) (if \( uw \neq \ell_i \) for \( 1 \leq i \leq p \), then \( \deg u = 2 \), otherwise \( \deg u \geq 2 \)).
Theorem 3.2.11.

Let $G$ be an edge product graph with edge function $f : E \rightarrow P$ and edge product function $F$ of $f$. Let the edges $e_1, e_2, \ldots, e_p$ be incident on $u$ and the edges $\ell_1, \ell_2, \ldots, \ell_q$ be incident on $v$. It there exist proper sub collections $e_1, e_2, \ldots, e_r$ of $e_1, e_2, \ldots, e_p$ and $\ell_1, \ell_2, \ldots, \ell_s$ of $\ell_1, \ell_2, \ldots, \ell_q$ such that $f(e_1) \times f(e_2) \times \ldots \times f(e_r) = f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_s)$ then, the vertices $u$ and $v$ are adjacent and $r = (p - 1), s = (q - 1)$.

**Proof:**

Let $F(u) = f(e_1) \times f(e_2) \times \ldots \times f(e_r) \times f(e_{(r+1)}) \times f(e_{(r+2)}) \times \ldots \times f(e_p) \in P$

$$= f(\ell_1) \times f(\ell_2) \times \ldots \times f(\ell_s) \times f(e_{(r+1)}) \times f(e_{(r+2)}) \times \ldots \times f(e_p) \in P$$

where $r < p$. Therefore the edges $\ell_1, \ell_2, \ldots, \ell_s, e_{(r+1)}, e_{(r+2)}, \ldots, e_p$ are all incident on a vertex. But the $s$ edges $\ell_1, \ell_2, \ldots, \ell_s$ are incident on $v$ and the edges $e_{(r+1)}, e_{(r+2)}, \ldots, e_p$ are incident on $u$. Then we get $p = (r + 1)$ and $e_{(r+1)} = uv$.

That is, $r = (p - 1)$ and $e_{(r+1)} = uv$. Similarly, we get $s = (q - 1)$ and $\ell_{(s+1)} = uv$.

Therefore, we obtain that $r = (p - 1), s = (q - 1)$ and $\ell_q = e_p = uv$.

In the next chapter, we discuss more about Edge Product Number of Graphs.