Chapter VII

Edge Product Number of Fan and Basket

In this chapter we define fan, basket and some different types of graphs and also investigate the edge product number of that graphs.

7.1. Edge Product Number of Fan

Definition 7.1.1.

If each vertex \(v_1, v_2, \ldots, v_q\) \((q \geq 1)\) of an arc is connected to another vertex \(v\), the resulting graph is a Fan with \(q\) blades. It is denoted by \(F_q\). Thus \(F_q\) has \((q + 2)\) vertices and \((2q + 1)\) edges.

Theorem 7.1.2:

If \(F_q\) is a fan then \(\text{EPN}(F_q) = 1\) for \(q\) is odd and for all \(q \geq 3\).

Proof:

Consider the graph \(G = F_q \cup K_2\) where \(q = (2k + 1)\) for some \(k \geq 1\). Let \(V = \{v, v_0, v_1, v_2, \ldots, v_{2k + 1}, w_1, w_2\}\) be the vertex set and \(E = \{vv_0, vv_{2k + 1}\} \cup \{vv_i : 1 \leq i \leq 2k\} \cup \{v_{2i}v_{2i + 1} : 1 \leq i \leq k\} \cup \{w_1w_2\}\) be the edge set of \(G\).

Let \(P = \{2^1, 2^2, 2^k, 2^{2k + 1}, \ldots, 2^{3k}, 2^{2k + 4k + 1}, \ldots, 2^{2k^2 + 5k + 1}, 2^{2k^2 + 8k + 2}\}\).

The mapping \(f: E \rightarrow P\) is an optimal edge function and \(F\) be its corresponding optimal edge product function of \(f\). The optimal edge function \(f\) is defined by

\[
\begin{align*}
f(vv_0) &= 2^{3k + 1} \\
f(vv_{(2k+1)}) &= 2^{4k + 1} \\
f(vv_i) &= 2^i \text{ for } 1 \leq i \leq 2k \\
f(v_{2i}v_{(2i-1)}) &= 2^{2k + (k - i) + 1} \text{ for } 1 \leq i \leq k
\end{align*}
\]
\[ f(v_{2i+1}) = 2^{2k^2+4k+1+(k-i)} \text{ for } 1 \leq i \leq k \text{ and } f(w_1w_2) = 2^{2k^2+8k+2} \]

The optimal edge product function \( F \) is defined by

\[
F(v_0) = 2^{3k+1} \times 2^{2k^2+5k+1} = 2^{2k^2+8k+2} = 2^{4k+1} \times 2^{2k^2+4k+1} = F(v_{2k+1})
\]

\[
F(v_{2i}) = 2^{3k+1-i} \times 2^{2i} \times 2^{2k^2+5k+1-i} \text{ for } 1 \leq i \leq k
\]

\[
F(v_{2i+1}) = 2^{3k+1-i} \times 2^{2i+1} \times 2^{2k^2+5k+1-i} \text{ for } 1 \leq i \leq (k-1)
\]

\[
F(v) = 2^{3k+1} \times 2^{4k+1} \times 2^{1+2+\ldots+2k} = 2^{7k+2+k(2k+1)} = 2^{2k^2+8k+2}
\]

\[
F(w_1) = F(w_2) = 2^{2k^2+8k+2}
\]

The mapping \( f : E \to P \) is a bijection. The edge products of all vertices are the rearrangements of \( 2^{(2k^2+8k+2)} \) for all \( k \geq 1 \). That is, \( F(v) = 2^{(2k^2+8k+2)} \) for all \( k \geq 1 \), for every \( v \in V \).

Also the edge product function \( F \) satisfies \( F(v) = F(v_0) = F(v_{2i}) = \ldots = F(v_{2i+1}) = F(w_1) = F(w_2) \). The ranges of \( F \) are all elements of \( P \). Thus \( F \) is into \( P \).

Hence, \( \text{EPN}(F_q) \geq 1 \).

Thus, we get the desired result that \( \text{EPN}(F_q) = 1 \) for \( q \) is odd and for all \( q \geq 3 \).

**Example:** (1) Let \( F_3 \) be a fan. The graph \( F_3 \cup K_2 \) is the edge product graph and \( \text{EPN}(F_3) = 1 \) is shown below.
(2) If $F_5$ is a fan then the graph $F_5 \cup K_2$ is an edge product graph and $\text{EPN}(F_5) = 1$ is illustrated in the following figure.

7.2. Edge Product Number of Basket

Definition 7.2.1.

If each vertex $v_1, v_2, \ldots, v_q$ ($q \geq 3$) of a cycle is connected to another vertex $v$, the resulting graph is a wheel with $q$ spokes. It is denoted by $W_q$. If one spoke is
removed from $W_q$, that is the vertices $v$ and $v_q$ are not joined, the resulting graph is a Basket. It is denoted by $B_q$.

**Theorem 7.2.2:**

If $B_q$ is a basket then $\text{EPN}(B_q) = 1$ for $q = 4$ and for all $q \geq 6$

**Proof:**

Consider $G = B_q \cup K_2$ is the graph for all $q \geq 6$. Let $V = \{v, v_1, v_2, \ldots, v_q, w_1, w_2\}$ be the vertex set and $E = \{vv_i: 1 \leq i \leq (q-1)\} \cup \{v_(i+1): 1 \leq i \leq (q-1)\} \cup \{v_qv_1\} \cup \{w_1w_2\}$ be the edge set of the graph $G$.

Let $P = \{2^i: 1 \leq i \leq (2k - 1)\} \cup \{2(k^2 - k), 2k^2, 2(k^2 + k - 3), \ldots, 2(k^2 + k), 2(k^2 + k + 2), 2(k^2 + 2k - 3), 2(k^2 + 2k - 1), 2(k^2 + 2k), 2(4k - 3), \ldots, 2(3k - 1), 2(2k^2 + 3k - 2), 2(2k^2 - 2)\}$

Let $f : E \to P$ be an optimal edge function of the graph $G$ and $F$ be its corresponding optimal edge product function.

**Case (1)** when $q$ is odd

If $q = (2k + 1)$ for some $k \geq 3$. Then the edge function $f$ is defined by

$f(vv_i) = 2^i$ for $1 \leq i \leq (2k - 2)$

$f(vv_{2k}) = 2^{2k - 1}$

$f(vv_{(2k - 1)}) = 2^{4k - 2}$

$f(v_{2k}v_{(2k - 1)}) = 2^{2k - k}$

$f(v_{(2i + 1)}v_{2i}) = 2^{k^2 + k - 1 - i}$ for $1 \leq i \leq (k - 1)$

$f(v_{2i}v_{(2i - 1)}) = 2^{k^2 + 2k + 1 - i}$ for $1 \leq i \leq (k - 1)$

$f(v_1v_{(2k + 1)}) = 2^{2k^2 + k - 1}$

$f(v_{(2k + 1)}v_{2k}) = 2^{k^2 + 2k - 1}$

$f(w_1w_2) = 2^{2k^2 + 3k - 2}$
The mapping \( f : E \rightarrow P \) is a bijection. The edge products of all vertices are rearrangements of \( 2^{(2k^2 + 3k - 2)} \) for all \( k \geq 3 \). That is, the edge product function \( F \) satisfies \( F(v) = 2^{(2k^2 + 3k - 2)} \) for every \( v \in V \). Also \( F(w_1) = F(w_2) = f(w_1w_2) = F(v) \). Thus the range of the function \( F \) are all elements of \( P \). Hence \( \operatorname{EPN}(B_q) \geq 1 \).

Therefore we have \( \operatorname{EPN}(B_q) = 1 \) for \( q \) is odd and for all \( q \geq 6 \).

**Example:** The following figure 7.3 shows that \( \operatorname{EPN}(B_9) = 1 \) for the edge product graph \( B_9 \cup K_2 \).

![Figure 7.3](image)

**Case (2) when \( q \) is even**

If \( q = 2k \) for some \( k \geq 4 \). Then the edge function \( f \) is defined by

\[
\begin{align*}
f(vv_i) &= 2^i \quad \text{for} \quad 1 \leq i \leq (2k - 3) \\
f(v_1v_{2k}) &= 2^{4k - 3} \\
f(vv_{2k - 1}) &= 2^{2k - 1} \\
f(vv_{2k - 2}) &= 2^{3k - 3} \\
f(v_{2k - 1}v_{2k - 2}) &= 2^{2k - 1}
\end{align*}
\]
\[ f(v_{2i+1}v_{2i}) = 2^{4k-3-i} \text{ for } 1 \leq i \leq (k-2) \]

\[ f(v_{2i}v_{2i-1}) = 2^{2k^2-4k+1-i} \text{ for } 1 \leq i \leq (k-1) \]

\[ f(v_{2k}v_{2k-1}) = 2^{2k^2-4k+1} \]

\[ f(w_1w_2) = 2^{2k^2-2} \]

The mapping \( f: E \to P \) is a bijection. The edge products are all rearrangements of \( 2^{(2k^2-2)} \) for some \( k \geq 4 \). Thus the edge product function \( F \) satisfies \( F(v) = 2^{(2k^2-2)} \) for every \( v \in V \). Also \( F(v) = F(w_1) = F(w_2) = f(w_1w_2) \). The ranges of \( F \) are all elements of \( P \). Hence, \( \text{EPN}(B_q) \geq 1 \).

Thus, we get \( \text{EPN}(B_q) = 1 \) for \( q \) is even and \( q \geq 6 \).

**Example:** (1) Let \( B_6 \) is a basket on 7 vertices and \( G = B_6 \cup K_2 \) be the edge product graph. The following figure shows that \( \text{EPN}(B_6) = 1 \).

![Diagram of edge product graph]

Figure 7.4
(2) Let $B_8$ be a basket on 9 vertices and $B_8 \cup K_2$ be the edge product graph. The following figure shows that $\text{EPN}(B_8) = 1$.

![Figure 7.5](Image)

(3) The following figure 7.6 shows that the graph $B_4 \cup K_2$ is the edge product graph and $\text{EPN}(B_4) = 1$ where $B_4$ is a basket on 5 vertices.

![Figure 7.6](Image)
Remark 5.2.3: A basket on 3 and 5 vertices, that is, $B_3$ and $B_5$ have edge product number two. That is, $\text{EPN}(B_3) = 2$ and $\text{EPN}(B_5) = 3$ is given in the following figures.

Figure 7.7

Figure 7.8
7.3. Edge Product Number of Special Kinds of Graphs

**Definition 7.3.1.**

Let $d$ and $p$ be non-negative integers such that $k = (d + p) \geq 2$. Let $M_{(d,p)}$ denote a graph with vertex set $V = \{u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_p\}$ and the edge set $E = \{e_1, e_2, \ldots, e_{(2k-1)}, e_{2k}, f_1, f_2, \ldots, f_d, \ell_1, \ell_2, \ldots, \ell_p, r_1, r_2, \ldots, r_p\}$

where $e_1 = u_1u_2, e_2 = u_2u_3, \ldots, e_{(2k-1)} = u_{(2k-1)}u_{2k}, e_{2k} = u_{2k}u_1$;

$f_1 = u_1u_{(k+1)}, f_2 = u_2u_{(k+2)}, \ldots, f_d = u_du_{(k+d)}$;

$\ell_1 = v_1u_{(d+1)}, \ell_2 = v_2u_{(d+2)}, \ldots, \ell_p = v_pu_{(d+p)}$;

$r_1 = v_1u_{2k}, r_2 = v_2u_{(2k-1)}, \ldots, r_p = v_pu_{(2k-p+1)}$

**Theorem 7.3.2.**

For the graph $M_{(d,p)}$, $\text{EPN}(M_{d,p}) = 1$ for every odd positive integer $d$ and non-negative integer $p$.

**Proof:**

Consider a graph $G = M_{(d,p)} \cup K_2$ for $d$ and $p$ are non-negative integers. For every odd positive integer $d$, there exists a positive integer $s$ such that $d = (2s - 1)$. Take $a = (p + s)$. Let $V = \{u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_p\} \cup \{w_1, w_2\}$ be the vertex set and set $E = \{e_1, e_2, \ldots, e_{(2k-1)}, e_{2k}, f_1, f_2, \ldots, f_d, \ell_1, \ell_2, \ldots, \ell_p, r_1, r_2, \ldots, r_p\} \cup \{w_1w_2\}$ be the edge set of $G$. Let the elements of $P = \{2^a, \ldots, 2^a + \text{E}[M_{d,p}|-1]\}$.

The mapping $f: E \to P$ is an optimal edge function of the graph $G$ and $F$ is the corresponding optimal edge product function of $f$. Then the edge function $f$ is defined by
The mapping \( f : E \rightarrow P \) is a bijection. The edge product function \( F \) satisfies \( F(v) = 2^{(8p + 12s - 6)} \) for every vertex \( v \in V \). Also \( F(v) = f(w_1w_2) \). Therefore \( F(v) = F(w_1) = F(w_2) \). Hence \( \text{EPN}(M_{d, p}) \geq 1 \) for every odd positive integer \( d \) and non-negative integer \( p \). Therefore, \( \text{EPN}(M_{d, p}) = 1 \).

**Example:** (1) If \( M_{1, 3} \) is a graph and \( G(V, E) = M_{1, 3} \cup K_2 \) is the edge product graph. Then \( \text{EPN}(M_{1, 3}) = 1 \) is given below.

![Figure 7.9](image-url)
The graph $M_{(3,0)}\cup K_2$ is the edge product graph and $\text{EPN}(M_{3,0}) = 1$ is shown in the following graph.

![Graph Diagram]

**Figure 7.10**

**Definition 5.3.3.**

Let $C_{2k}$ be a cycle with vertex set $V = \{u_1, u_2, \ldots, u_{2k}\}$ and the edge set $E = \{e_1, e_2, \ldots, e_{2(k-1)}, e_{2k}\}$ where $e_1 = u_1u_2$, $e_2 = u_2u_3$, $\ldots$, $e_{(2k-1)} = u_{(2k-1)}u_{2k}$, $e_{2k} = u_{2k}u_1$. Let $g$ be a mapping from $E(C_{2k})$ to the set of positive integers defined by

$$g(e_i) = \begin{cases} 
2^{k-1 + (i-1)/2} & \text{for } i = 1, 3, \ldots, k, (k+4), \ldots, (2k-1) \\
2^{4k-2} & \text{for } i = 2 \\
2^{k-1 + (i)/2} & \text{for } i = (k+1) \\
2^{2k-2 + (i-3)/2} & \text{for } i = (k+2) \\
2^{2k-2 + (i-2)/2} & \text{for } i = 4, 6, \ldots, (k-1), (k+3), \ldots, (2k) 
\end{cases}$$

for $k$ is odd and
for $k$ is even. Let $S_k$ denote a graph with vertex set $V(C_{2k}) \cup \{v_1, v_2, \ldots, v_k\}$ and the edge set $E(C_{2k}) \cup \bigcup_{i=1}^{k} \{v_iu_{i1}, v_iu_{i2}\}$ where $v_iu_{i1}, v_iu_{i2}$ are vertices of $C_{2k}$ such that $f(u_{i1}) = 2^{(3k-3+i)}$ and $f(u_{i2}) = 2^{(5k-1-i)}$

**Theorem 7.3.4:**

For the graph $S_k$, $EPN(S_k) = 1$ for every odd positive integer $k \geq 2$.

**Proof:**

Consider a graph $G = (S_k \cup K_2)$. The vertex set of $G$ is $V = V(S_k) \cup \{w_1, w_2\}$ and the edge set of $G$ is $E = E(S_k) \cup \{w_1w_2\}$ where $v_iu_{i1}, v_iu_{i2}$ are vertices of $C_{2k}$. A mapping $f : E(S_k) \to P$ where $P = \{2^{(k-1)}, \ldots, 2^{(5k-2)}, 2^{(8k-4)}\}$ is an optimal edge function of the graph $G$ and $F$ be its corresponding optimal edge product function of $f$. The edge function $f : E(S_k) \to P$ is defined by

$$
g(e) = \begin{cases} 2^{(k-1+i/2)} & \text{for } i = 1,3,(k+1) \\
2^{(4k-2)} & \text{for } i = 2 \\
2^{(2k-2+i/2)} & \text{for } i = (k+2),2k \\
2^{(2k-6+i/2)} & \text{for } i = 4,6,\ldots,k,(k+4),\ldots,(2k-2) \\
2^{(3k-3-(i/2))} & \text{for } i = 5,7,\ldots,(k-1),(k+3),\ldots,(2k-1) \\
\end{cases}
$$

The mapping $f : E \to P$ is a bijection. The edge product function $F$ satisfies $F(v) = 2^{(8k-4)}$ for every $v \in V$. Also $F(v) = f(w_1w_2)$. That is, $F(v) = F(w_1) = F(w_2)$. Hence $EPN(S_k) \geq 1$ for every odd positive integer $k \geq 2$. Therefore we get the desired result that $EPN(S_k) = 1$ for all $k \geq 2$. 

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Example: The graph \( G = (S_4 \cup K_2) \) is the edge product graph and the following figure 7.11 shows that \( \text{EPN}(S_4) = 1 \).

![Graph Diagram](image)

Figure 7.11

Note: Let \( G(V, E) \) be a graph. The number of vertices in the vertex set \( V \) is called the order of \( G \) and the number of edges in the edge set \( E \) is called the size of \( G \).

Theorem 7.3.5.

If \( G \) be a 3 - regular graph of order 10 and size 15 then \( \text{EPN}(G) = 2 \).

Proof:

Let \( G(V, E) \cup K_2 \) be the graph, where \( G \) be a 3 - regular graph of order 10 and size 15 with \( V = \{v_0, v_1, v_2, \ldots, v_9, w_1, w_2\} \) be the vertex set and the edge set be \( E=\{v_0v_i; 1 \leq i \leq 3\} \cup \{v_iv_{(i+1) \mod 10}; 1 \leq i \leq 8\} \cup \{v_1v_4, v_5v_8, v_6v_9, v_7v_9\} \cup \{w_1w_2\} \).

Consider \( \text{EPN}(G) = 1 \). Let \( f : E \to P \) be an optimal edge function of the graph \( G \), where \( P = \{a_i, b_i, c_i, d_i, e_i, f_i, x, y, z : 1 \leq i \leq 2\} \) and \( F \) be the corresponding optimal edge product function of \( f \).

The edge product function \( F \) of \( f \) is defined by
\[ F(v_0) = f(v_1v_0) \times f(v_2v_0) \times f(v_3v_0) = a_1 \times x \times c_1 \]
\[ F(v_1) = f(v_0v_1) \times f(v_2v_1) \times f(v_4v_1) = a_1 \times d_1 \times f_1 \]
\[ F(v_2) = f(v_0v_2) \times f(v_1v_2) \times f(v_3v_2) = x \times d_1 \times b_1 \]
\[ F(v_3) = f(v_0v_3) \times f(v_2v_3) \times f(v_4v_3) = c_1 \times b_1 \times e_1 \]
\[ F(v_4) = f(v_1v_4) \times f(v_3v_4) \times f(v_5v_4) = f_1 \times e_1 \times z \]
\[ F(v_5) = f(v_4v_5) \times f(v_6v_5) \times f(v_8v_5) = z \times f_2 \times e_2 \]
\[ F(v_6) = f(v_5v_6) \times f(v_7v_6) \times f(v_9v_6) = f_2 \times a_2 \times d_2 \]
\[ F(v_7) = f(v_6v_7) \times f(v_8v_7) \times f(v_9v_7) = a_2 \times c_2 \times y \]
\[ F(v_8) = f(v_5v_8) \times f(v_7v_8) \times f(v_9v_8) = e_2 \times c_2 \times b_2 \]
\[ F(v_9) = f(v_6v_9) \times f(v_7v_9) \times f(v_8v_9) = d_2 \times y \times b_2 \]

**Figure 7.12**

Suppose \( F(v_0) = F(v_1) = F(v_2) = \ldots = F(v_8) = F(v_9) = f(v_1v_2) \). Then we have \( a_1 \times x \times c_1 = a_1 \times d_1 \times f_1 = x \times d_1 \times b_1 = \ldots = e_2 \times c_2 \times b_2 = d_2 \times y \times b_2 \).

\[ F(v_0) = F(v_1) \Rightarrow a_1 \times x \times c_1 = a_1 \times d_1 \times f_1 \Rightarrow x \times c_1 = d_1 \times f_1 \]

\[ \Rightarrow x = \left( \frac{d_1 \times f_1}{c_1} \right) \]

\[ F(v_2) = F(v_3) \Rightarrow x \times d_1 \times b_1 = c_1 \times b_1 \times e_1 \Rightarrow x \times d_1 = c_1 \times e_1 \]

\[ \Rightarrow x = \left( \frac{c_1 \times e_1}{d_1} \right) \]
Therefore \((x \times x) = \left( \frac{d_1 \times f_1}{c_1} \right) \times \left( \frac{e_1 \times e_1}{d_1} \right) \Rightarrow x^2 = (f_1 \times e_1)\)

\[F(v_6) = F(v_7) \Rightarrow f_2 \times a_2 \times d_2 = a_2 \times c_2 \times y \Rightarrow f_2 \times d_2 = c_2 \times y \Rightarrow y = \left( \frac{f_2 \times d_2}{c_2} \right)\]

\[F(v_8) = F(v_9) \Rightarrow e_2 \times c_2 \times b_2 = d_2 \times y \times b_2 \Rightarrow e_2 \times c_2 = d_2 \times y \Rightarrow y = \left( \frac{e_2 \times c_2}{d_2} \right)\]

Therefore \((y \times y) = \left( \frac{f_2 \times d_2}{c_2} \right) \times \left( \frac{e_2 \times c_2}{d_2} \right) \Rightarrow y^2 = (f_2 \times e_2)\]

But \(F(v_4) = F(v_5) \Rightarrow f_1 \times e_1 \times z = z \times f_2 \times e_2 \Rightarrow f_1 \times e_1 = f_2 \times e_2 \Rightarrow x^2 = y^2 \Rightarrow x = y\)

This is a contradiction since the mapping \(f : E \rightarrow P\) is a bijection. Thus for a 3-regular graph of order 10 and size 15 has more than one edge product number. That is, \(EPN(G) \geq 2\). Therefore \(EPN(G) = 2\) where \(G\) is a 3-regular graph of order 10 and size 15.

**Example:** Let \(G(V, E)\) be a 3-regular graph of order 10 and size 15. Then the graph \(G \cup 2K_2\) is the edge product graph and the following figure 7.13 shows that \(EPN(G) = 2\).

![Figure 7.13](image-url)