CHAPTER 4

Instabilities in displacement through porous media
4.1 INTRODUCTION

If a fluid contained in a porous medium is displaced by another fluid of lesser viscosity, then it is frequently observed that the displacing fluid has a strong tendency to protrude in form of fingers (instabilities) into more viscous fluid. This phenomenon is called fingering.

In Figure-1, fingering process has been shown between oil-water flows into a porous medium. In petroleum engineering, the fingering process is a well-known phenomenon occurring in displacement of oil by water by flooding that is a common oil recovery technique.

In the statistical treatment of fingering [8] only average cross-sectional area occupied by the fingers was observed while the size and shape of the individual fingers are neglected as in figure 2. Scheidegger and Jhonson [9] discussed the statistical behaviour in homogeneous porous media with capillary pressure. Verma [10] has examined the behaviour of fingering in a displacement process through heterogeneous porous media.
In this chapter, we have numerically discussed the phenomenon of instabilities in a displacement process involving two immiscible liquids by considering two cases. In the first case, the phenomenon is considered without involving the magnetic fluid whereas the second case with special reference to magnetic fluid. Numerical solution of governing non-linear partial differential equation for both the cases has been obtained by ADM. The numerical results are obtained at various time levels.

**PROBLEM 1: FINGERING IN A HOMOGENEOUS MEDIUM**

**4.2 Statement of the problem**

We consider that there is a uniform water injection into an oil saturated porous medium of homogeneous physical characteristics, such that the injecting water cuts through the oil formation and give rise to protuberance. This furnishes a well developed fingers flow. Since the entire oil at the initial boundary (x=0) is displaced through a small distance due to the water injection. Therefore, we assume, further that complete water saturation exists at the initial boundary.

**4.3 Formulation of the problem 1**

The seepage of water \( (v_w) \) and oil \( (v_o) \) are given by Darcy’s Law,

\[
\begin{align*}
    v_w &= - \left( \frac{k_w}{\delta_w} \right) K \left[ \frac{\partial P_w}{\partial x} \right] \\
    v_o &= - \left( \frac{k_o}{\delta_o} \right) K \left[ \frac{\partial P_o}{\partial x} \right]
\end{align*}
\]  (4.3.1) (4.3.2)
Where \( K \) is the permeability of the homogeneous medium, \( K_w \) and \( K_0 \) are relative permeability of water and oil, which are functions of \( S_w \) and \( S_o \) (\( S_w \) and \( S_o \) are the saturation of water and oil) respectively, \( P_w \) and \( P_o \) are pressure of water and oil, \( \delta_w \) and \( \delta_o \) are constant kinematic viscosities, \( \alpha \) is the inclination of the bed and \( g \) is acceleration due to gravity.

Regarding the phase densities are constant, the equations of continuity of the two phase are:

\[
P \left( \frac{\partial s_w}{\partial t} \right) + \frac{\partial v_w}{\partial x} = 0 \quad (4.3.3)
\]

\[
P \left( \frac{\partial s_o}{\partial t} \right) + \frac{\partial v_o}{\partial x} = 0 \quad (4.3.4)
\]

Where \( p \) is porosity of the medium. From the definition of phase saturation, it is evident that,

\[
S_w + S_o = 1 \quad (4.3.5)
\]

The capillary pressure \( P_c \) is defined as

\[
P_c = -\beta_0 S_w \quad (4.3.6)
\]

\[
P_c = P_o - P_w \quad (4.3.7)
\]

Where \( \beta_0 \) is a constant quantity.

At this state, for definiteness of mathematical analysis, we assume standard relationship due to Scheidegger and Jhonson [14], between phase saturation and relative permeability as

\[
K_w = S_w \quad (4.3.8)
\]

\[
K_0 = S_o = 1 - S_w \quad (4.3.9)
\]

The equation of motion for saturation can be obtained by substituting the values of \( V_w \) and \( V_0 \) from equation (4.3.1) and (4.3.2) into the equation (4.3.3) and (4.3.4) respectively, we get,

\[
P \left( \frac{\partial s_w}{\partial t} \right) = \frac{\partial}{\partial x} \left[ \frac{K_w}{\delta_w} \frac{\partial P_w}{\partial x} \right] \quad (4.3.10)
\]
These are the general flow equations of the phase in homogeneous medium, when effects due to pressure discontinuity and gravity term in inclined porous medium are considered.

Eliminating \( \frac{\partial P_w}{\partial x} \) from equations (4.3.10) and (4.3.7), we obtain

\[
P \left( \frac{\partial s_w}{\partial t} \right) = \frac{\partial}{\partial x} \left( \frac{K_w}{\delta_w} \right) K \left[ \frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} \right] \tag{4.3.11}
\]

(4.3.11)

Combining equation (4.3.11) and (4.3.12) and using equation (4.3.5), we get,

\[
\frac{\partial}{\partial x} \left[ \frac{K_w}{\delta_w} \left( \left( K_w + K_o \right) \frac{\partial P_o}{\delta_w} - K_w \frac{\partial P_c}{\delta_w} \right) \right] = 0 \tag{4.3.12}
\]

(4.3.12)

Integrating above equation with respect to \( x \), we have

\[
K \left( \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right) \frac{\partial P_o}{\delta_w} - K \frac{K_w}{\delta_w} \frac{\partial P_c}{\delta_w} = -V \tag{4.3.13}
\]

(4.3.13)

Where \( V \) is constant of integrating which can be evaluated from later on. Simplification of (4.3.13) gives

\[
\frac{\partial P_o}{\partial x} = \frac{-V}{K \left( \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right) + \frac{\partial P_c}{\delta_w} \left( \frac{\delta_w}{\delta_o} \right)} \tag{4.3.14}
\]

(4.3.14)

Using above equation in (4.3.12), we have

\[
P \left( \frac{\partial s_w}{\partial t} \right) + \frac{\partial}{\partial x} \left[ \frac{-V}{K \left( \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right) + \frac{\partial P_c}{\delta_w} \left( \frac{\delta_w}{\delta_o} \right)} \right] = 0 \tag{4.3.14}
\]

(4.3.14)

The value of pressure of oil \( (P_o) \) can be written as in [15] of the form

\[
P_o = \frac{P_o + P_w}{2} + \frac{P_o - P_w}{2} = \bar{P} + \frac{1}{2} P_c \tag{4.3.15}
\]

(4.3.15)

Where \( \bar{P} \) is the mean pressure which is constant, therefore (4.3.15) implies

\[
\frac{\partial P_o}{\partial x} = \frac{1}{2} \frac{\partial P_c}{\partial x} \tag{4.3.15}
\]

(4.3.15)

Using above equation in (4.3.13), we get,
\[ V = \frac{K}{2} \left[ \left( \frac{K_w}{\delta_w} \right) - \left( \frac{K_e}{\delta_{e0}} \right) \frac{\partial P_e}{\partial x} \right] \]

Substituting the value of \( V \) from above equation in equation (4.3.14), we get,

\[ P \frac{\partial S_w}{\partial t} + \frac{1}{2} \frac{\partial }{\partial x} \left[ K \left( \frac{K_w}{\delta_w} \right) \left( \frac{dP_e}{dS_w} \right) \frac{\partial S_w}{\partial x} \right] = 0 \]

Substituting the value of \( K_w \) and \( P_e \) from (4.3.6) and (4.3.8), we get,

\[ P \frac{\partial S_w}{\partial t} - \frac{\beta_0}{2} \frac{K}{S_w} \frac{\partial}{\partial x} \left[ S_w \frac{\partial S_w}{\partial x} \right] = 0 \quad (4.3.16) \]

A set of suitable boundary conditions associated to problem (4.3.16) are

\[ S_w(0, t) = 1 ; S_w(x, 0) = 0 ; S_w(L, t) = 0 \quad (4.3.17) \]

Equation (4.3.16) is reduced to dimensionless form by setting

\[ X = \frac{x}{L} , \quad T = \frac{\frac{K_B}{\beta_{0l}}} {2\delta_w L^2 P}, \quad S_w(x, t) = S^*_w(x, t) \]

So that \( \frac{\partial S_w}{\partial T} = \frac{\partial}{\partial x} \left( S_w \frac{\partial S_w}{\partial x} \right) \quad (4.3.18) \)

In equation (4.3.18) and (4.3.19) the asterisk are dropped for simplicity.

Equation (4.3.18) is desired nonlinear differential equation of motion for the flow of immiscible liquid in homogeneous medium.

The problem is solved by using Adomain Decomposition Method. The numerical values are shown by table. Curves indicate the behaviour of saturation of water corresponding to various time periods.

**4.4 Solution of the problem using Adomain Decomposition Method:**

\[ \frac{\partial S_w}{\partial T} = \frac{\partial}{\partial x} \left( S_w \frac{\partial S_w}{\partial x} \right) \]

\[ \therefore (S_w)_T = (S_w(S_w)_x)_x \quad (4.4.1) \]

Taking the initial condition \( s_w(x, 0) = s_w_0 = f(x) \)

Applying the operator \( J \) on both the sides of equation (4.4.1) using initial condition,
\[ s_w(x, T) = f(x) + \left[ \phi_1(s_w(x, T)) \right] + \left[ \phi_2(s_w(x, T)) \right] \] (4.4.2)

Where \( \phi_1(s_w(x, T)) = s_w(s_w)_{xx} \) and \( \phi_2(s_w(x, T)) = (s_w)_x^2 \)

Following Adomain decomposition method, the solution is represented as infinite series like,

\[ s_w(x, T) = \sum_{n=0}^{\infty} s_{w_n}(x, T) \] (4.4.3)

The nonlinear operator \( \phi_1(s_w) \) & \( \phi_2(s_w) \) are decomposed in these forms,

\[ \phi_1(s_w(x, T)) = \sum_{n=0}^{\infty} A_n, \quad \phi_2(s_w(x, T)) = \sum_{n=0}^{\infty} B_n \] (4.4.4)

Where \( A_n \) and \( B_n \) are so called Adomain polynomials and have the form,

\[ A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left[ \sum_{k=0}^{\infty} \lambda^k s_{w_k} \right] \right]_{\lambda=0} \]

\[ B_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left[ \left( \sum_{k=0}^{\infty} \lambda^k s_{w_k} \right) \left( \sum_{k=0}^{\infty} \lambda^k s_{w_{kxx}} \right) \right] \right]_{\lambda=0} \]

The first three components of these polynomials are,

\[ A_0 = s_{w_0}(s_{w_0})_{xx} \]

\[ A_1 = s_{w_0}s_{w_1xx} + s_{w_1}s_{w_0xx} \]

\[ A_2 = s_{w_2}s_{w_0xx} + s_{w_1}s_{w_1xx} + s_{w_0}s_{w_{2xx}} \]

\[ A_3 = s_{w_3}s_{w_0xx} + s_{w_1}s_{w_2xx} + s_{w_2}s_{w_1xx} + s_{w_0}s_{w_{3xx}} \]
Similarly, $B_0 = (s_{w0x})^2$

$B_1 = (s_{w1x})^2$

$B_2 = (s_{w2x})^2$

$B_3 = (s_{w3x})^2$

Other polynomials can be generated in like manner, substituting the decomposition series (4.4.3) and (4.4.4) into equation (4.4.2) yields the following recursive formula,

\[ S_{w0}(x, T) = f(x) \]

\[ s_{wn+1}(x, T) = j(A_n) + j(B_n); n \geq 0 \]

Let $S_{w0}(x, T) = f(x) = \frac{e^x - 1}{e - 1}$

\[ S_{w1} = j(A_0) + j(B_0) \]

\[ = j\left( s_{w0} (s_{w0})_{xx} \right) + j\left( (s_{w0x})^2 \right) \]

\[ = f_1 T \quad \text{Where } f_1 = ff_{xx} + f_x^2 = \frac{2e^{2x} - e^x}{(e-1)^2} \]

\[ S_{w2} = j(A_1) + j(B_1) \]

\[ = j(s_{w0} s_{w1xx} + s_{w1} s_{w0xx}) + j\left( (s_{w1x})^2 \right) \]

\[ = f_2 T \quad \text{Where } f_2 = ff_{1xx} + f_1 f_{xx} + f_{ix}^2 \]

\[ = \frac{10e^{3x+1} - 10e^{2x+1} + e^{x+1} - 18e^{3x} + 11e^{2x} - e^x + 16e^{4x}}{(e-1)^4} \]

\[ S_{w}(x, T) = \frac{e^x - 1}{e - 1} + \frac{2e^{2x} - e^x}{(e-1)^2} T + \frac{10e^{3x+1} - 10e^{2x+1} + e^{x+1} - 18e^{3x} + 11e^{2x} - e^x + 16e^{4x}}{(e-1)^4} T^2 + \ldots \]
4.5 Results:

The following table shows the approximate solution for saturation of injected liquid for different values of x at different time using adomain decomposition method.