Chapter 2

SYNTHESIS OF SHARP TRANSITION LINEAR PHASE DIGITAL LOWPASS FIR FILTERS

2.1 Introduction

For many digital signal processing applications, FIR filters are preferred over their IIR counterparts as the former can be designed with exact linear phase, guaranteed stability, free of phase distortion, absence of limit cycles and low coefficient sensitivity. However, FIR filters require especially in applications demanding narrow transition bandwidth, considerably more arithmetic operations than their IIR equivalents. Since the FIR filter length is inversely proportional to transition bandwidth its complexity becomes prohibitively high for sharp filters, which causes serious implementation problems [12]. First, very large number of multipliers renders real time high speed implementation impractical. Second, the roundoff noise power generated by a filter with large number of nontrivial coefficients will be unacceptable unless the word length of the registers and arithmetic units are sufficiently high. Finally, filters with a large number of nontrivial coefficients have high coefficient sensitivity. As a consequence, very sharp filters will have high hardware complexity, high coefficient sensitivity and high roundoff noise unless the filter coefficient vector is sparse.

Several methods have been proposed in the literature for reducing the arithmetic complexity of sharp transition FIR filters [7], [13]-[15]. One of the techniques employed is the concept of interpolation [14]. One of the computationally efficient realizations for narrow band FIR filters based on this concept is the Interpolated FIR filter or IFIR filter which is a FIR–FIR filter cascade realization. In this realization the first FIR structure called the shaping filter has a very sparse impulse response resulting in a greatly reduced number of arithmetic operations. The second FIR
structure called the interpolator attenuates the undesired spectral images of the desired passband of
the shaping filter below the prescribed stopband level and is usually a very simple filter of short
length. Interpolating the impulse response of low pass filter has the effect of reducing its passband
width by the interpolation ratio. By replacing every delay by M delays, the transition bandwidth is
reduced by factor of M but the passband width is also reduced by the same factor. Hence this
technique generally is suitable only for narrow passband design.

The total number of multipliers required for the implementation of FIR filters is a widely
used performance criterion. Another approach, leading to computationally efficient FIR filters is
the multiple use of the same filter. The resulting overall filter requires significantly fewer distinct
multipliers than equivalent direct-form designs at the expense of an increased overall filter order.
By appropriately cascading these filters large stopband attenuation can be obtained. Subfilter
approach for designing efficient FIR filters is described in [16]. There are basically two subfilter
approaches, the first one using different subfilters and the second one using identical subfilters. In
the first approach, the best results are obtained by forming the subfilters by replacing the basic
delay in a conventional prototype filter transfer function by a multiple delay and then by properly
designing and combining the subfilters. The subfilters have different basic delays. The transition
bandwidths of the prototype filters are very wide compared to that of the overall filter. Hence the
required filter orders and correspondingly the number of multipliers required are very low
compared to an equivalent direct-form minimax design. The subfilters that result when replacing
the basic delays in the prototype filters by multiple delays have sparse impulse responses with the
number of non-zero impulse response coefficient values being equal to that of the prototype filter,
thus high-order filters are realized using very few multipliers. In the second technique, the overall
filter is designed by interconnecting a number of identical subfilters with the aid of a few
additional adders and multipliers. The pass band and stopband edges of the subfilter are the same as those for the overall filter but the passband and the stopband ripples are significantly larger, resulting in a significant reduction in the filter order. Hence, by increasing the number of subfilters, the overall filter order can be reduced to any desired level. However, the realization which employs the minimum number of multipliers may sometimes require a slightly longer word length to achieve a specified noise performance or may result in a slightly longer group delay when compared to another realization which requires a slightly larger number of multipliers. A problem of practical interest is to find the lowest order filter, which satisfies certain maximum ripple requirements.

One of the important problems in digital filtering that has been considered by a number of authors in the past is the design of linear phase FIR filters with a very flat passband response and an equiripple stopband response. The design of linear-phase FIR filters with equiripple stopbands and with a prescribed degree of flatness of passbands is dealt in [17]. The design is based entirely on an appropriate use of well known Remez exchange algorithm for the design of weighted chebyshev FIR filters. Techniques for reduction of passband ripple in FIR filters have been recently investigated in [18] using optimization techniques.

This chapter deals with the formulation of various lowpass FIR digital filter models and their design. Models are proposed for linear phase, sharp transition, lowpass FIR filters with minimum passband ripple, good stopband attenuation with least filter order. The filter models are formulated using sinusoidal functions of frequency to evaluate the impulse response coefficients in closed form. Filter transfer function is evolved in frequency and time domain. The approach is simple, versatile and analytical without extensive computations. Expressions for filter design parameters $k_p, k_t, k_s$ and impulse response coefficients are derived. Magnitude response of the
proposed lowpass filter is obtained. The approach can be extended to design sharp transition arbitrary passband highpass, bandpass and bandstop filters.

Novel slope equalization technique is introduced and the technique is applied to the various lowpass digital filter models formulated. The filter design parameters $k_p$, $k_s$ and $k_s$ are evaluated by equalizing the slopes of the pseudo-magnitude response function at both the ends of the transition region. This allows the proposed function to be continuous thus reducing the effects due to Gibb's phenomenon and hence decreases passband ripple and increases stopband attenuation.

The filter models proposed in this chapter lay stress on achieving a sharp transition from the passband to the stopband. Sharper this passband, more oscillatory will be the frequency response near the edge of the passband, a trait described as Gibb's phenomenon [8], [12]. The filter models proposed in this chapter achieve a tradeoff between the transition region width and the Gibb's phenomenon. In addition, emphasis is laid upon a good passband i.e. low passband attenuation and good stopband i.e. large stopband attenuation. Thus a three fold compromise for the satisfactory performance of the filter in all the three bands namely passband, transition band and stopband is essential in addition to a trade off between Gibb's phenomenon and sharpness of transition of the filter. The first model namely, Class I FIR filter with nonmonotonic passband has maximum passband attenuation at the extremities of the passband and a negative excursion that is maximum at the extremity of the stopband to obtain a sharp transition. All the three regions of the response are formulated in terms of sinusoidal function to achieve twin objectives of reducing Gibb's phenomenon and evaluating the closed form expression for the impulse response coefficients. The three parameters $k_p$, $k_s$ and $k_s$ are evaluated with the above stated objectives in mind. The class II filter proposed has a monotonic passband and reduced number of filter design parameters. Its stopband is also monotonic but for these minor differences, its formulation is
similar to class I filter model in many respects. The class III filter model proposed with and without slope equalization has equiripple passband and stopbands. The formulation of the transition region is similar to that of the Class I filter. This filter achieves better filter performance than its predecessors. This model is refined further by applying slope equalization technique which avoids a discontinuity at the edges of the passband and stopband reducing the effects of Gibbs phenomenon and thus further improving the filter performance. The class IV filter model has equiripple passband and stopband and in addition it introduces a linear transition region. In this model the synthesis of the filter is greatly simplified with fewer filter design parameters which reduces the complexity of the design compared to Class III filter. The parameter $k_p$ is uniquely determined from the transition region width. The parameter is independent of passband width $\omega_p$. With the application of slope equalization technique Gibbs phenomenon is found to reduce considerably. The class V filter model has equiripple passband and stopband and has a linear transition region. It has variable density ripples in the passband and stopband. This technique is employed to further reduce the effects of Gibb's phenomenon.

2.2 Review of Gibb's Phenomenon

A causal FIR filter obtained by simply truncating the impulse response of the ideal filter, exhibits an oscillatory behavior in its magnitude response which is more commonly referred to as the Gibb's phenomenon. The oscillatory behavior of the magnitude response is on both sides of cutoff frequency at which the ideal response is discontinuous and the peak ripple moves closer to the discontinuity [12]. As the order of the filter is increased, the number of ripples in both passband and stopband increases and the ripples are squeezed into a narrower interval about the discontinuity. However the overshoots, which occur on both sides of the transition region remain
the same independent of the filter order and are approximately 18% of the difference between the passband and stopband magnitudes of the ideal filter [8]. The presence of the oscillatory behavior in the Fourier transform of a truncated Fourier series representation of an ideal filter response is basically due to two reasons. First, the impulse response of an ideal filter is infinitely long and not absolutely summable and as a result the filter is unstable. A stable filter realization needs a finite impulse response obtained by truncation of the impulse response. Second, the rectangular window used for truncation and obtaining an FIR response has an abrupt transition to zero which is the reason behind the appearance of the Gibbs phenomenon in the magnitude response of the windowed or truncated ideal filter impulse response sequence. The Gibbs phenomenon can be reduced by either using a window that tapers smoothly to zero at each end or by providing a smooth transition from the passband to the stopband. Use of a tapered window causes the height of the side lobe to diminish with a corresponding increase in the main lobe width resulting in a wider transition at the discontinuity.

2.3 Proposed Slope Equalization Technique

In this section, a novel slope equalization technique is proposed [23]. In the filter design without slope equalization the filter response functions are discontinuous at the passband edge, which leads to ripple at the points of discontinuity. The points of discontinuity are (i) at the end of passband region and start of transition region (ii) at the end of the transition region and start of the stopband region. The slopes of the magnitude response are equalized at these points of discontinuity i.e. \( \omega_p \) and \( \omega_z \) shown in Fig.2.1 which makes the function continuous at these points reducing the overshoots due to Gibb’s phenomenon. This leads to reduction of the peak ripple at
Fig. 2.1. Illustration of novel slope equalization technique: (a) Lowpass FIR filter model with equiripple magnitude response (b) Expanded view of passband (c) Expanded view of stopband.
the points of discontinuities and hence reduces passband ripple and increases stopband attenuation. In Fig. 2.1, the slopes are equalized at $\omega_p$ i.e., slope A equal to slope B and at $\omega_s$ i.e. slope C equal to slope D. The filter is modeled over three regions i.e., passband, transition and stopband regions using trigonometric functions. Filter design parameters of the filter model are evaluated with slope equalization.

2.4 Expressions for Impulse Response Coefficients of a Digital Filter

Let $h(n)$ be the impulse response sequence of an $N$-point, linear phase FIR digital filter where, $0 \leq n \leq N-1$. The linear phase condition implies that the impulse response satisfies the symmetry condition [8], [19].

\[ h(n) = h(N-1-n), \quad \text{where} \quad n = 0, 1, 2 \ldots N-1. \] (2.1)

The frequency response for the linear-phase FIR filters for order $N$ is given by [8].

\[ H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \] (2.2)

\[ H(e^{j\omega}) = e^{-j\left(\frac{N-1}{2}\right)\omega} H_{pm}(\omega) \] (2.3)

where the pseudo magnitude response $H_{pm}(\omega)$ is

\[ H_{pm}(\omega) = h\left(\frac{N-1}{2}\right) + 2 \sum_{n=0}^{\frac{N-1}{2}-1} h(n) \cos\left[\left(\frac{N-1}{2} - n\right)\omega\right] \quad \text{for \ N odd and} \] (2.4)

\[ H_{pm}(\omega) = 2 \sum_{n=0}^{\frac{N-1}{2}} h(n) \cos\left[\left(\frac{N-1}{2} - n\right)\omega\right] \quad \text{for \ N even} \] (2.5)

The impulse response sequence (coefficients) $h(n)$ of the filter obtained from this frequency response are,
\[ h(n) = \frac{1}{2\pi} \int_{\pi}^{\pi} H_{pm}(\omega) \cos \left( \left( \frac{N-1}{2} - n \right) \omega \right) d\omega, \]  
\[ \text{for } n = 0, 1, 2, \ldots, \frac{N-1}{2}, \text{ for } N \text{ odd and } \]
\[ n = 0, 1, 2, \ldots, \frac{N}{2} - 1, \text{ for } N \text{ even} \]  

2.5 Class I FIR Lowpass Filter with Non-Monotonic Response without Slope Equalization

2.5.1 Filter Model and Design

In this section, the formulation of a linear phase, sharp transition, lowpass FIR filter model with non-monotonic passband response and its design is presented. The filter model magnitude response \( H_{pm}(\omega) \) is as shown in Fig.2.2. For the proposed lowpass filter model, the various regions of the filter response are modeled using trigonometric functions of frequency as follows.

In the passband region, the frequency response is
\[ H_{pm}(\omega) = 1 + k_p \delta_p \sin (\omega - \omega_0), \quad 0 \leq \omega \leq \omega_p \]  

where \( \omega \) is the frequency variable, \( H_{pm}(\omega) \) is the pseudo-magnitude of the filter response, \( \delta_p \) is the passband ripple, \( k_p \) is a filter design parameter, \( \omega_p \) is the passband edge at which \( H_{pm}(\omega) \) is maximum and \( \omega_0 \) is a frequency at which \( H_{pm}(\omega) \) is unity.

In the transition region, which also spans part of the passband \([\omega_p, \omega_z]\) as well part of the stopband \([\omega_s, \omega_z]\) the frequency response is given by
\[ H_{pm}(\omega) = \left( 1 + \frac{\delta_p}{2} \right) \cos \left[ k_i (\omega - \omega_p) \right], \quad \omega_p \leq \omega \leq \omega_z \]  

(2.9)
Fig. 2.2. Illustration of proposed Class I lowpass filter model with non-monotonic magnitude response
where \( k_t \) is a filter design parameter, \( \omega_z \) is the frequency in the stopband at which \( H_{pm}(\omega) \) is zero and \( \omega_s \) is the stopband-edge.

In the stopband region beyond \( \omega_z \), the frequency response is given by,

\[
H_{pm}(\omega) = -\frac{k_s \delta_s}{2} \sin (\omega - \omega_z) , \quad \omega_z \leq \omega \leq \pi
\]  

(2.10)

where \( \delta_s \) is the stopband attenuation and \( k_s \) is a filter design parameter.

At \( \omega = 0 \), \( H_{pm}(0) = 1 - \frac{\delta_p}{2} = \left[1 - k_p \delta_p \sin (\omega_0)\right] \)  

(2.11)

Simplifying, we obtain, \( k_p = \frac{1}{2 \sin \omega_0} \)  

(2.12)

At \( \omega = \omega_p \), \( H_{pm}(\omega_p) = 1 + \frac{\delta_p}{2} = 1 + k_p \delta_p \sin (\omega_p - \omega_0) \)  

(2.13)

Simplifying we obtain,

\[
\omega_p = \omega_0 + \sin^{-1}\left[\frac{1}{2k_p}\right]
\]  

(2.14)

Substituting (2.12) in (2.14) we obtain,

\[
\omega_p = 2\omega_0
\]  

(2.15)

Thus, \( \omega_0 \) is the mid-band frequency.

At \( \omega = \omega_z \), \( H_{pm}(\omega_z) = 0 = \left(1 + \frac{\delta_p}{2}\right) \cos k_t (\omega_z - \omega_p) \)  

(2.16)

Simplifying (2.16), \( \omega_z = \omega_p + \frac{\pi}{2k_t} = 2\omega_0 + \frac{\pi}{2k_t} \)  

(2.17)

At \( \omega = \omega_c \), \( H_{pm}(\omega_c) = \left(1 - \frac{\delta_p}{2}\right) = \left(1 + \frac{\delta_p}{2}\right) \cos k_t (\omega_c - \omega_p) \)  

(2.18)
\begin{align}
\text{Simplifying (2.18), } \omega_c &= \omega_p + \frac{1}{k_t} \cos^{-1} \left[ \frac{1 - \frac{\delta_p}{2}}{1 + \frac{\delta_p}{2}} \right] \\
\text{At } \omega = \pi, H_{pm}(\pi) &= \frac{-\delta_s}{2} = -\frac{k_s \delta_s}{2} \sin(\pi - \omega_c) \\
\text{Simplifying (2.20), } k_s &= \frac{1}{\sin \omega_c} \\
\text{At } \omega = \omega_s, H_{pm}(\omega_s) &= \frac{\delta_s}{2} = \left(1 + \frac{\delta_p}{2}\right) \cos k_t (\omega_s - \omega_p) \\
\text{Simplifying (2.22), } \omega_s &= \omega_p + \frac{1}{k_t} \cos^{-1} \left[ \frac{\frac{\delta_s}{2}}{1 + \frac{\delta_p}{2}} \right] \\
\text{Using (2.19) and (2.23) we obtain,} \\
k_t &= \frac{1}{(\omega_s - \omega_c)} \cos^{-1} \left[ \frac{\frac{\delta_s}{2}}{1 + \frac{\delta_p}{2}} \right] - \cos^{-1} \left[ \frac{1 - \frac{\delta_p}{2}}{1 + \frac{\delta_p}{2}} \right] \\
\text{(2.24)}
\end{align}

\textbf{2.5.2 Expressions for Impulse Response Coefficients}

Referring to filter design theory of section 2.4, the impulse response coefficients \( h(n) \) for the lowpass filter are obtained by evaluating the integral below.

\begin{align}
\text{for } h(n) &= \frac{1}{\pi} \int_{0}^{\pi} H_{pm}(\omega) \cos k\omega \, d\omega \\
\text{for } h(n) &= \frac{1}{\pi} \left[ \int_{0}^{\alpha_p} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_s} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_s}^{\pi} H_{pm}(\omega) \cos k\omega \, d\omega \right] \\
\text{(2.25)}
\end{align}
\[ h(n) = \frac{1}{\pi} \left[ \int_0^{\omega_p} \left[ 1 + k_p \delta_p \sin (\omega - \omega_o) \right] \cos k \omega \ d\omega + \int_{\omega_p}^{\omega} \left( 1 + \frac{\delta_p}{2} \right) \cos k_t (\omega - \omega_p) \cos k \omega \ d\omega \right. \\
+ \left. \int_{\omega_s}^{\frac{\pi}{2}} - \frac{k_s \delta_s}{2} \sin (\omega - \omega_s) \cos k \omega \ d\omega \right] \]

where \( n = 0, 1, \ldots, \frac{N-1}{2} \) for \( N \) odd,
\( n = 0, 1, \ldots, \frac{N}{2} - 1 \) for \( N \) even and \( k = \frac{N-1}{2} - n \)

Evaluating (2.27), the expressions obtained for the impulse response coefficients \( h(n) \) for the lowpass filter are

\[ h(n) = h\left( \frac{N-1}{2} + k \right) = h\left( \frac{N-1}{2} - k \right) \]

\[ = \frac{\sin (k \omega_p)}{k \pi} + \frac{k_p \delta_p}{\pi (k^2 - 1)} \left[ k \sin (k \omega_p) \sin \omega_o + \cos (k \omega_p) \cos \omega_o - \cos \omega_o \right] \]

\[ + \frac{\left( 1 + \frac{\delta_p}{2} \right)}{\pi (k_t^2 - k^2)} \left[ k \cos k \omega_s + k \sin k \omega_p \right] \]

\[ + \frac{k_s \delta_s}{2 \pi (k^2 - 1)} \left[ \cos (k \omega_s) + \cos (k \pi) \cos \omega_s - k \sin (k \pi) \sin \omega_s \right] \]

(2.28)

Eq. (2.28) is valid for \( N \) even where \( k \) is a non-integer. For \( N \) odd, (2.28) is valid except for \( k = 0 \), \( k = 1 \) and \( k = k_t \).

For \( N \) odd, \( k = 0 \) we obtain,

\[ h(n) = h\left( \frac{N-1}{2} \right) = \frac{\omega_p}{\pi} + \frac{\left( 1 + \frac{\delta_p}{2} \right)}{\pi k_t} \cdot \frac{k_s \delta_s}{2 \pi} (1 + \cos \omega_s) \]

(2.29)
For $N$ odd, $k=1$ we obtain,

$$
\begin{align*}
\text{h}(n) &= \left(\frac{N-1}{2} + 1\right) = \left(\frac{N-1}{2} - 1\right) = \frac{\sin \omega_p}{\pi} + \frac{k_p \delta_p}{4 \pi} \left[\cos \omega_o - \cos (3 \omega_0)\right] - \frac{\delta_p \omega_p}{4 \pi} \\
+ \left(1 + \frac{\delta_p}{2}\right) \frac{\pi(k_t^2 - 1)}{\pi(k_t^2 - 1)} \left[k_t \cos \omega_z + \sin \omega_p\right] + \frac{k_t \delta_z}{4 \pi} \left(\pi - \omega_z\right) \sin \omega_z \\
&= \left(\frac{N-1}{2} - k_t\right)
\end{align*}
$$

(2.30)

For $N$ odd, $k= k_t$ we obtain,

$$
\begin{align*}
\text{h}(\frac{N-1}{2} + k_t) &= \frac{\sin (k_t \omega_p)}{k_t \pi} + \frac{k_p \delta_p}{\pi(k_t^2 - 1)} \left[k_t \sin (k_t \omega_p) \sin \omega_o + \cos (k_t \omega_p) \cos \omega_o - \cos \omega_o\right] \\
+ \left(1 + \frac{\delta_p}{2}\right) \frac{\pi(k_t^2 - 1)}{2 \pi} \left[(\omega_z - \omega_p) \cos (\omega_p k_t) - \frac{\sin (\omega_p k_t)}{k_t}\right] \\
+ \frac{k_t \delta_z}{2 \pi(k_t^2 - 1)} \left[\cos (k_t \omega_z) + \cos (k_t \pi) \cos \omega_z - k_t \sin (k_t \pi) \sin \omega_z\right]
\end{align*}
$$

(2.31)

### 2.5.3 Filter Synthesis Results

Design Example: A lowpass linear phase sharp transition FIR filter is designed for the desired filter specifications: Passband edge $\omega_p$ is $0.666 \pi$, transition bandwidth $(\omega_z - \omega_o)$ is $0.01 \pi$, maximum passband ripple $\delta_p$ is $\pm 0.1$ dB ($0.2$ dB) and minimum stopband attenuation $\delta_z$ is $40$dB using the proposed Class I filter design approach.

The lowpass filter is designed using MATLAB [20][21] with program MLP-1. The filter specifications obtained by measurement of the magnitude response of the filter using Signal Processing toolbox are passband edge $\omega_p = 0.6667 \pi$, cutoff edge $\omega_c = 0.6668 \pi$, stopband edge
\( \omega_s = 0.6767 \pi \), transition bandwidth \((\omega_s - \omega_c) = 0.01 \pi\), passband ripple \( \delta_p = 0.088 \text{dB} \) and stopband attenuation \( \delta_s = 32.2 \text{dB} \) for a filter order of 701. Results appropriate the desired filter specifications closely but the desired filter specifications of stopband attenuation is not achieved for order 701 and with slight improvement in stopband attenuation for higher filter order. Hence this filter model and design is treated as a developmental one. The magnitude, impulse and phase response of the proposed lowpass filter obtained is shown in Fig.2.3. Also Table 2.1 depicts the performance of the filter. It is observed that for conventional FIR sharp transition filters the peak passband ripple due to Gibb's phenomenon is about 18%. In the proposed Class I filter design the peak passband ripple is 0.99% for the filter order 701 and decreases for higher filter order. Filter synthesis and design steps are shown in Appendix A 6.1.1
Table 2.1
Variation of passband ripple and stopband attenuation with filter order for transition band width of 0.01π and passband edge of 0.666π without slope equalization for proposed Class I FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passband Ripple in dB</td>
<td>0.454</td>
<td>0.129</td>
<td>0.105</td>
<td>0.103</td>
<td>0.088</td>
<td>0.035</td>
</tr>
<tr>
<td>Stopband Attenuation in dB</td>
<td>25.72</td>
<td>28.88</td>
<td>30.7</td>
<td>31.57</td>
<td>32.2</td>
<td>33.2</td>
</tr>
</tbody>
</table>
Fig. 2.3. (a) Magnitude response of the proposed Class I lowpass filter (b) magnified view of the passband (c) Impulse response sequence (d) Phase response.
2.6 Class II FIR Lowpass Filter with Monotonic Response without Slope Equalization

2.6.1 Filter Model and Design
In this section, the formulation of linear phase, sharp transition lowpass FIR filter model with monotonic passband response and its design is presented [22]. The passband and the transition regions are modified in this design. The filter model magnitude response $H_{pm}(\omega)$ is as shown in Fig.2.4. For the proposed lowpass filter model, the various regions of the filter response are modeled using trigonometric functions of frequency as follows.

In the passband region, the frequency response is

$$H_{pm}(\omega) = 1 - k_p \sin \omega, \quad 0 \leq \omega \leq \omega_c$$  \hspace{1cm} (2.32)

where as before, $\omega$ is the frequency variable, $H_{pm}(\omega)$ is the pseudo-magnitude of the filter response, $k_p$ is a filter design parameter, $\delta_p$ is the passband ripple, $\omega_c$ is the cut-off edge, $H_{pm}(0) = 1$ and $H_{pm}(\omega_c) = 1 - \delta_p$.

In the transition region and part of the stopband $[\omega_s, \omega_z]$, the frequency response is

$$H_{pm}(\omega) = (1 - \delta_p)[1 - \sin_t(\omega - \omega_c)], \quad \omega_c \leq \omega \leq \omega_z$$  \hspace{1cm} (2.33)

where $H_{pm}(\omega_z) = 0$ and $H_{pm}(\omega_s) = \frac{\delta_t}{2}$ where $\delta_t$ is the stopband attenuation, $k_t$ is a filter design parameter, $\omega_s$ is the stopband-edge and $\omega_z$ is the frequency in the stopband region at which $H_{pm}(\omega)$ is zero.

In the stopband region, the frequency response is

$$H_{pm}(\omega) = \frac{-k_s \delta_z}{2} \sin \left( \omega - \omega_z \right), \quad \omega_z \leq \omega \leq \pi$$  \hspace{1cm} (2.34)
Fig. 2.4. Illustration of proposed Class II lowpass filter model with monotonic magnitude response.
where $k_z$ is a filter design parameter.

At $\omega = \omega_z$, $H_{pm}(\omega_z) = 0 = (1-\delta_p)[1-\sin k_t(\omega_z - \omega_c)]$  \hspace{1cm} (2.35)

Simplifying (2.35), $\omega_z = \omega_c + \frac{\pi}{2k_t}$  \hspace{1cm} (2.36)

At $\omega = \omega_c$, $H_{pm}(\omega_c) = 1 - \delta_p = 1 - k_p \sin \omega_c$  \hspace{1cm} (2.37)

Simplifying (2.37), $k_p = \frac{\delta_p}{\sin \omega_c}$  \hspace{1cm} (2.38)

At $\omega = \pi$, $H_{pm}(\pi) = \frac{-\delta_p}{2} = \frac{-k_z \delta_z}{2} \sin (\pi - \omega_z)$  \hspace{1cm} (2.39)

Simplifying (2.39), $k_z = \frac{1}{\sin \omega_z}$  \hspace{1cm} (2.40)

At $\omega = \omega_z$, $H_{pm}(\omega_z) = \frac{\delta_z}{2} = (1-\delta_p) \left[1 - \sin k_t(\omega_z - \omega_c)\right]$  \hspace{1cm} (2.41)

Simplifying (2.41),

$$\omega_z = \omega_c + \sin^{-1}\left[1 - \frac{\delta_z}{2(1-\delta_p)}\right]$$  \hspace{1cm} (2.42)

From (2.42), we have, $k_t = \frac{\sin^{-1}\left[1 - \frac{\delta_z}{2(1-\delta_p)}\right]}{\omega_z - \omega_c}$  \hspace{1cm} (2.43)

It is not possible to model the frequency response equalizing the slopes at $\omega = \omega_c$. The slope of the response on the passband side can be shown to be low and equalizing the slope on the transition region side for sharp transition is not possible. Therefore slope equalization technique cannot be applied for this design.
2.6.2 Expressions for Impulse Response Coefficients

Referring to filter design theory of section 2.4, the impulse response coefficients \( h(n) \) for the lowpass filter are obtained by evaluating the integrals below.

\[
h(n) = \frac{1}{\pi} \left[ \int_0^x H_{pm}(\omega) \cos k\omega \, d\omega \right]
\]

(2.44)

\[
h(n) = \frac{1}{\pi} \left[ \int_0^{\omega_c} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_c}^{x} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_s}^{x} H_{pm}(\omega) \cos k\omega \, d\omega \right]
\]

(2.45)

\[
h(n) = \frac{1}{\pi} \left[ \int_0^{\omega_c} (1-k_p \sin \omega) \cos k\omega \, d\omega + \int_{\omega_c}^{\omega_s} (1-\delta_p)(1-\sin k_c) \cos k\omega \, d\omega \right.
\]
\[+ \int_{\omega_s}^{x} \frac{k_{s\delta^2}}{2} \sin (\omega - \omega_s) \cos k\omega \, d\omega \]

(2.46)

where \( n=0,1,\ldots, \frac{N-1}{2} \) for \( N \) odd

\[n=0,1,\ldots, \frac{N}{2} - 1 \text{ for } N \text{ even and } k = \frac{N-1}{2} - n\]

Evaluating (2.46), the expressions obtained for the impulse response coefficients \( h(n) \) for the lowpass filter are

\[
h(n) = h\left(\frac{N-1}{2} + k\right)
\]

\[= h\left(\frac{N-1}{2} - k\right) = \frac{\sin \left( k \omega_c \right)}{k\pi} + \frac{k_p}{(k^2 - 1)\pi} [1 - k \sin \left( k \omega_c \right) \sin \omega_c - \cos \left( k \omega_c \right) \cos \omega_c]
\]
\[+ \frac{(1-\delta_p)}{\pi} \left[ \frac{\sin \left( k \omega_s \right) - \sin \left( k \omega_c \right)}{k} + k \sin \left( k \omega_s \right) - \frac{k \cos \left( k \omega_c \right)}{(k_t^2 - k^2)} \right]
\]

32
\[ + \frac{k_z \delta_z}{2(k_t^2 - 1)\pi} \left[ \cos(k \omega_z) + \cos(k \pi) \cos \omega_z - k \sin(k \pi) \sin \omega_z \right] \]  

(2.47)

Eq. (2.47) is valid for \( N \) even. For \( N \) odd, (2.47) is valid except for \( k = 0, k = 1 \) and \( k = k_t \)

For \( N \) odd, \( k = 0 \) we obtain

\[ h \left( \frac{N-1}{2} \right) = \frac{1}{\pi} \left[ (\cos \omega_c - 1) k_p + \omega_c \right] \left[ (\omega_z - \omega_c) - \frac{1}{k_t} \right] - \frac{k_z \delta_z}{2} \left( 1 + \cos \omega_z \right) \]

(2.48)

For \( N \) odd, \( k = 1 \) we obtain

\[ h \left( \frac{N-1}{2} + 1 \right) = h \left( \frac{N-1}{2} \right) = \frac{k_p}{4\pi} \left( \cos(2\omega_c) - 1 \right) + \frac{\sin \omega_c}{\pi} \]

\[ + \frac{(1-\delta_p)}{\pi} \left[ \left( \sin \omega_z - \sin \omega_c \right) + \frac{\sin \omega_c - k \cos \omega_c}{(k_t^2 - 1)} \right] + \frac{k_z \delta_z}{4\pi} \left[ (\pi - \omega_c) \sin \omega_z \right] \]

(2.49)

For \( N \) odd, \( k \) \( = k_t \) we obtain

\[ h \left( \frac{N-1}{2} + k_t \right) = h \left( \frac{N-1}{2} - k_t \right) \]

\[ = \frac{\sin \left( k_t \omega_c \right)}{\pi} + \frac{k_p}{(k_t^2 - 1)\pi} \left[ 1 - k_t \sin(k_t \omega_c) \sin \omega_c - \cos(k_t \omega_c) \cos \omega_c \right] \]

\[ + \frac{(1-\delta_p)}{\pi} \left[ \frac{\sin(k_t \omega_z) - \sin(k_t \omega_c)}{k_t} \right] - \frac{(1-\delta_p)}{2\pi} \left[ \frac{\cos(k_t \omega_c)}{k_t} + (\omega_z - \omega_c) \sin(k_t \omega_c) \right] \]

\[ + \frac{k_z \delta_z}{2(k_t^2 - 1)\pi} \left[ \cos(k_t \omega_z) + \cos(k_t \pi) \cos \omega_z - k \sin(k_t \pi) \sin \omega_z \right] \]

(2.50)

2.6.3 Filter Synthesis Results

Design Example: A lowpass linear phase sharp transition FIR filter is designed for the desired filter specifications: Cutoff edge \( \omega_c \) is 0.666 \( \pi \), transition bandwidth \( (\omega_s - \omega_c) \) is 0.01 \( \pi \), maximum
passband ripple $\delta_p$ is $\pm 0.1 \text{dB} \ (0.2 \text{ dB})$ and minimum stopband attenuation $\delta_s$ is $40 \text{dB}$ using the proposed Class II filter design approach.

The filter is designed using MATLAB with program MLP-2. The filter specifications obtained by measurement of the magnitude response of the filter using Signal Processing toolbox are cutoff edge $\omega_c = 0.6668 \pi$, stopband edge $\omega_s = 0.6767 \pi$, transition bandwidth ($\omega_s - \omega_c$) = 0.01 $\pi$, passband ripple $\delta_p = 0.204 \text{ dB}$ and stopband attenuation $\delta_s = 46.3 \text{ dB}$ for a filter order of 701. It is observed that the desired filter specifications are obtained as per design example with very good stopband attenuation. It is also observed that passband ripple is higher compared to all proposed filter model designs. The magnitude and impulse response of the proposed lowpass filter obtained is shown in Fig.2.5. Table 2.2 depicts the performance of the filter. In the proposed Class II filter the peak passband ripple is 2.3% for the filter order 701 and decreases for higher filter order as compared to about 18% peak passband ripple in conventional FIR filters. Filter synthesis and design steps are given in Appendix A6.1.2.
Table 2.2
Variation of passband ripple and stopband attenuation with filter order for transition bandwidth of $0.01\pi$ and passband edge of $0.666\pi$ without slope equalization for proposed Class II FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passband Ripple in dB</td>
<td>0.538</td>
<td>0.34</td>
<td>0.262</td>
<td>0.236</td>
<td>0.204</td>
<td>0.165</td>
</tr>
<tr>
<td>Stopband Attenuation in dB</td>
<td>31.95</td>
<td>45.10</td>
<td>45.26</td>
<td>45.79</td>
<td>46.3</td>
<td>46.98</td>
</tr>
</tbody>
</table>
Fig. 2.5. (a) Magnitude response of the proposed Class II lowpass filter (b) Linear plot (c) magnified view of the passband (d) Impulse response sequence.
2.7 Class III FIR Lowpass Filter with Equiripple Response

2.7.1 Filter Model and Design without slope equalization

In this section, the formulation of a linear phase, sharp transition, lowpass FIR filter model with equiripple passband and stopband with its design is presented [23]. The filter model magnitude response $H_{pm}(\omega)$ is as shown in Fig. 2.6. In the proposed lowpass filter model, the various regions of the filter response are modeled using trigonometric functions of frequency as follows.

In the passband region, the frequency response is

$$H_{pm}(\omega) = 1 + \frac{\delta_p}{2} \cos k_p \omega, \quad 0 \leq \omega \leq \omega_p$$  \hspace{1cm} (2.51)

where as usual $\omega$ is the frequency variable, $H_{pm}(\omega)$ is the pseudo-magnitude of the filter response, $\delta_p$ is the passband ripple, $k_p$ is a filter design parameter in the passband and $\omega_p$ is the passband edge.

Transition region spans part of the passband $[\omega_p, \omega_c]$ as well as part of the stopband $[\omega_s, \omega_z]$ where $\omega_c$ is the cutoff edge and $\omega_z$ is the stopband edge and $\omega_s$ is the frequency in the stopband region at which $H_{pm}(\omega)$ is zero. In the transition region, the frequency response is given by

$$H_{pm}(\omega) = A \cos k_t (\omega - \omega_0), \quad \omega_p \leq \omega \leq \omega_z \quad \text{and} \quad \omega_s < \omega_p$$  \hspace{1cm} (2.52)

where $k_t$ is a filter design parameter in the transition region, $A$ is amplitude parameter and is chosen greater than 1, $\omega_0$ is the frequency at which $H_{pm}(\omega)$ equals $A$. It may be noted that, $\omega_0$ is a fictitious frequency parameter used to shape the response in the region $\omega_p \leq \omega \leq \omega_z$ which does not include $\omega_0$. The formulation according to (2.52) yields some design flexibility later on.
Fig. 2.6. Illustration of proposed Class III lowpass filter model with equiripple magnitude response.
when slope equalization technique is applied to the design.

In the stopband region, the frequency response is given

\[ H_{pm}(\omega) = \frac{\delta_s}{2} \sin k_s (\omega - \omega_s), \quad \omega_s \leq \omega \leq \pi \]  

(2.53)

where \( \delta_s \) is the stopband attenuation, \( k_s \) is a filter design parameter in the stopband region.

The number of ripple cycles in the passband region \( [0, \omega_p] \) is

\[ k_r + \frac{1}{4} \]  

(2.54)

where \( k_r = 0, 1, 2, \ldots \) is a non-negative integer.

For the Class III lowpass filter design, the passband region \( [0, \omega_p] \) of the filter model possess

\[ \left( k_r + \frac{1}{4} \right) \) number of ripples cycles and is characterized by filter design parameter \( k_p \)

Therefore, \( k_p \omega_p = 2\pi \left( k_r + \frac{1}{4} \right) \)

(2.55)

i.e. \( k_p = \frac{2\pi \left( k_r + \frac{1}{4} \right)}{\omega_p} \)

(2.56)

It may be noted that a large value of \( k_r \) lead to a large value of \( k_p \) which in turn yields \( \omega_c \equiv \omega_p \)

and \( \omega_s \equiv \omega_z \) which guarantee a steep transition.

At \( \omega = 0, H_{pm}(0) = 1 + \frac{\delta_p}{2} \)

(2.57)

At \( \omega = \omega_p, H_{pm}(\omega_p) = 1.0 = A \cos k_i (\omega_p - \omega_0) \)

(2.58)

From (2.58) we obtain, \( A = \frac{1}{\cos k_i (\omega_p - \omega_0)} \)

(2.59)
The design strategy envisages the use of a part of the first quarter cycle of a cosine wave to simulate the transition region. To make the transition sharper, only the base of the waveform near zero crossing is used. The amplitude of the cosine wave is A and this wave is shifted in frequency by \( \omega_0 \). Accordingly A>1 and/or \( \omega_0 \leq \omega_p \) is chosen. To reduce the transition region width a frequency compression factor, \( k_t \gg 1 \) is used. It is noted that the choice of \( \omega_0 < \omega_p \) makes \( \omega_0 \) fictitious, since only the portion limited by \([\omega_p, \omega_z]\) of the cosine wave is used.

From (2.59) we obtain

\[
\omega_0 = \omega_p - \left( \frac{1}{k_t} \right) \cos^{-1} \left( \frac{1}{A} \right)
\]  

(2.60)

At \( \omega = \omega_0 \), \( H_{pm}(\omega_0) = A \)

(2.61)

At \( \omega = \omega_c \), \( H_{pm}(\omega_c) = 1 - \frac{\delta_p}{2} = A \cos k_t (\omega_c - \omega_0) \)

(2.62)

Simplifying (2.62), \( \omega_c = \omega_0 + \frac{1}{k_t} \cos^{-1} \left( \frac{1 - \frac{\delta_p}{2}}{A} \right) \)

(2.63)

At \( \omega = \omega_s \), \( H_{pm}(\omega_s) = \frac{\delta_s}{2} = A \cos k_t (\omega_s - \omega_0) \)

(2.64)

Simplifying (2.64),

\[
\omega_s = \omega_0 + \frac{1}{k_t} \cos^{-1} \left( \frac{\delta_s}{2A} \right)
\]  

(2.65)

Using (2.63) and (2.65) , \( k_t = \frac{1}{(\omega_s - \omega_c)} \left[ \cos^{-1} \left( \frac{\delta_s}{2A} \right) - \cos^{-1} \left( \frac{1 - \frac{\delta_p}{2}}{A} \right) \right] \)

(2.66)

At \( \omega = \omega_z \), \( H_{pm}(\omega_z) = 0 = A \cos k_t (\omega_z - \omega_0) \)

(2.67)
Simplifying (2.67), \( \omega_z = \omega_0 + \frac{\pi}{2k_1} \) \hspace{1cm} (2.68)

At \( \omega = \pi \), \( H_{pm}(\pi) = -\frac{\delta_p}{2} = -\frac{\delta_p}{2} \sin k_1 (\pi - \omega_z) \) \hspace{1cm} (2.69)

Simplifying (2.69), \( k_1 = \frac{\pi(4L+1)}{2(\pi - \omega_z)} \) \hspace{1cm} (2.70)

The lowpass filter design has \((L+1/4)\) number of ripples in the stopband i.e. in the region \([\omega_z, \pi]\)
where \(L\) is a non-negative integer.

2.7.1.1 Expressions for Impulse Response Coefficients

Referring to filter design theory of section 2.4, the impulse response coefficients \(h(n)\)
for the lowpass filter are obtained by evaluating the integral below.

\[
h(n) = \frac{1}{\pi} \left[ \int_0^\pi H_{pm}(\omega) \cos k\omega \, d\omega \right]
\]

\[
h(n) = \frac{1}{\pi} \left[ \int_0^{\omega_p} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_z} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_z}^{\pi} H_{pm}(\omega) \cos k\omega \, d\omega \right]
\]

\[
h(n) = \frac{1}{\pi} \left[ \int_0^{\omega_p} \left[ 1 + \frac{\delta_p}{2} \cos k_p \omega \right] \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_z} A \cos k_1 (\omega - \omega_0) \cos k\omega \, d\omega \\
+ \int_{\omega_z}^{\omega_p} \frac{\delta_p}{2} \sin k_1 (\omega - \omega_z) \cos k\omega \, d\omega \right]
\]

where \(n = 0, 1, \ldots, \frac{N-1}{2}\) for \(N\) odd

\(n = 0, 1, \ldots, \frac{N}{2} - 1\) for \(N\) even \(\quad\) and \(k = \frac{N-1}{2} - n\)
Evaluating (2.73), the expressions obtained for the impulse response coefficients $h(n)$ for the lowpass filter are

$$h(n) = h\left(\frac{N-1}{2} + k\right) = h\left(\frac{N-1}{2} - k\right) = \frac{\sin(k\omega_p)}{k\pi} + \frac{\delta_p k_p \cos(k\omega_p)}{2\pi(k_p^2 - k^2)}$$

$$+ \frac{1}{\pi(k_i^2 - k^2)} \left[ A k_i \cos(k_\omega) - k_i \left(\sqrt{A^2 - 1}\cos(k\omega_p) + k \sin(k\omega_p)\right)\right]$$

$$+ \frac{\delta_s}{2\pi(k_i^2 - k^2)} \left[ k_s \cos(k_\omega) \cos(k\pi) - k_s \cos(k\omega_p) + k \sin(k_\omega) \sin(k\pi)\right]$$

(2.74)

Eq. (2.74) is valid for $N$ even where $k$ is a non-integer. For $N$ odd (2.74) is valid except for $k = 0$, $k = k_p$, $k = k_s$ and $k = k_i$.

For $N$ odd, $k = 0$ we obtain

$$h(n) = h\left(\frac{N-1}{2}\right) = \frac{\omega_p}{\pi} + \frac{\delta_p}{2\pi k_p} + \frac{1}{\pi k_i} \left[A - \sqrt{A^2 - 1}\right] + \frac{\delta_s}{2\pi k_s} \left[\cos(k_\omega) - 1\right]$$

(2.75)

For $N$ odd, $k = k_s$ we obtain,

$$h\left(\frac{N-1}{2} + k_s\right) = h\left(\frac{N-1}{2} - k_s\right) = \frac{\sin(k_s\omega_p)}{k_s\pi} + \frac{\delta_p k_p \cos(k_s\omega_p)}{2\pi(k_p^2 - k_s^2)}$$

$$+ \frac{1}{\pi(k_i^2 - k_s^2)} \left[ A k_i \cos(k_\omega) - k_i \left(\sqrt{A^2 - 1}\cos(k_s\omega_p) + k_s \sin(k_s\omega_p)\right)\right]$$

$$+ \frac{\delta_s}{4\pi} (k_\omega + k_s \sin(k_s\omega_p) + \frac{\delta_s}{2\pi} \left[\cos(2k_s\pi) - k_s^2 \right]$$

(2.76)

For $N$ odd, $k = k_p$ we obtain,

$$h(n) = h\left(\frac{N-1}{2} + k_p\right)$$
\[ h\left( \frac{N-1}{2} + k \right) = h\left( \frac{N-1}{2} - k \right) = \frac{\sin k_i \omega_p}{k_i \pi} + \frac{\delta_p k_p \cos \left( k_i \omega_p \right)}{2\pi k_i} \]

\[ + \frac{A}{2\pi} (\omega_z - \omega_p) \cos(k_i \omega_0) - \left( \frac{A}{2\pi} \right) \frac{\sin(k_i \omega_0) + \sin(2k_i \omega_p - k_i \omega_0)}{2k_i} \]

\[ + \frac{\delta_s}{2\pi(k_s^2 - k_t^2)} \left[ k_s \cos k_s (\pi - \omega_z) \cos(k_i \pi) - k_s \cos(k_i \omega_z) + k_s \sin k_s (\pi - \omega_z) \sin(k_i \pi) \right] \]

(2.77)

For \( N \) odd, \( k = k_i \), we obtain,

\[ h\left( \frac{N-1}{2} - k \right) = \frac{1}{k_p \pi} + \frac{\delta_p \omega_p}{4\pi} + \frac{1}{\pi(k_i^2 - k_p^2)} \left( Ak_i \cos k_i \omega_z + k_p \right) \]

\[ + \frac{\delta_s}{2\pi(k_s^2 - k_p^2)} \left[ k_s \cos k_s (\pi - \omega_z) \cos(k_i \pi) - k_s \cos(k_i \omega_z) + k_s \sin k_s (\pi - \omega_z) \sin(k_i \pi) \right] \]

(2.78)

### 2.7.1.2 Filter Synthesis Results

Design Example: A lowpass linear phase sharp transition FIR filter is designed for the desired filter specifications: Passband edge \( \omega_p \) is 0.666 \( \pi \), transition bandwidth \( (\omega_s - \omega_c) \) is 0.01 \( \pi \), maximum passband ripple \( \delta_p \) is \( \pm 0.1 \text{dB} \) (0.2 dB) and minimum stopband attenuation \( \delta_s \) is 40dB using the proposed Class III filter design approach.

The filter is designed using MATLAB with program MLP-3. The filter specifications obtained by measurement of the magnitude response of the filter using Signal Processing toolbox are passband edge \( \omega_p = 0.6667 \pi \), cutoff edge \( \omega_c = 0.6668 \pi \), stopband edge \( \omega_s = 0.6767 \pi \), transition bandwidth \( (\omega_s - \omega_c) = 0.01 \pi \), passband ripple \( \delta_p = 0.079 \text{dB} \) and stopband attenuation
$\delta_s = 35$ dB for a filter order of 701. But it is found that stopband attenuation is less than the desired value of 40 dB, which can be achieved with higher filter order. The magnitude, impulse and phase response of the proposed lowpass filter obtained is shown in Fig.2.7. Also Table 2.3 depicts the performance of the filter. For conventional FIR sharp transition filters the peak passband ripple due to Gibb's phenomenon is about 18%. In the proposed Class III filter design the peak passband ripple is 0.92% for the filter order 701 and decreases for higher filter order. Filter synthesis and design steps are given in Appendix A6.1.3.
Table 2.3

Variation of passband ripple and stopband attenuation with filter order for transition bandwidth of 0.01π and passband edge of 0.666π without slope equalization for proposed Class III FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passband ripple in dB</td>
<td>0.44</td>
<td>0.119</td>
<td>0.1094</td>
<td>0.1056</td>
<td>0.079</td>
<td>0.0351</td>
</tr>
<tr>
<td>Stopband Attn. in dB</td>
<td>26.79</td>
<td>31.0</td>
<td>33.22</td>
<td>34.47</td>
<td>35.0</td>
<td>37.0</td>
</tr>
</tbody>
</table>
Fig. 2.7. (a) Magnitude response of the proposed Class III lowpass filter without slope equalization (b) magnified view of the passband (c) Impulse response sequence (d) Phase response.
2.7.2 Filter Design with Slope Equalization

Slope equalization technique is applied to the equiripple filter model of section 2.7.1, which further improves the performance of the filter. The filter design parameters $k_p, k_i$ and $k_s$ are evaluated by equalizing the slopes of the pseudo-magnitude response function at both the ends of the transition region. This allows the proposed function to be continuous thus reducing the effects due to Gibb’s phenomenon and hence reduces passband ripple and increases stopband attenuation of the filter.

At $\omega = \omega_p, H_{pm}(\omega_p) = 1 = 1 + \frac{\delta_p}{2} \cos k_p \omega_p$ (2.79)

Slope at $\omega_p = \frac{d}{d\omega} [H_{pm}(\omega)]_{\omega=\omega_p}$

$= \frac{d}{d\omega} \left[ 1 + \frac{\delta_p}{2} \cos k_p \omega_p \right] = -\frac{\delta_p}{2} k_p \sin k_p \omega_p$ (2.80)

Simplifying (2.79), $\cos k_p \omega_p = 0$ i.e. $\sin k_p \omega_p = \pm 1$

Since the slope of the magnitude response is negative at $\omega = \omega_p$, we choose the positive sign i.e. $\sin k_p \omega_p = 1$ (2.81)

Substituting (2.81) in (2.80),

Slope at $\omega_p = -\frac{k_p \delta_p}{2}$ (2.82)

Also at $\omega = \omega_p$, $H_{pm}(\omega_p) = 1 = A \cos k_i (\omega_p - \omega_0)$ (2.83)

Simplifying (2.83), $\omega_0 = \omega_p - \frac{1}{k_i} \cos^{-1} \left( \frac{1}{A} \right)$ (2.84)

Slope at $\omega_p = \frac{d}{d\omega} [H_{pm}(\omega)]_{\omega=\omega_p} = \frac{d}{d\omega} [A \cos k_i (\omega - \omega_0)]$
\[ k_t = -A k_t \sin k_t (\omega_p - \omega_0) \]  

(2.85)

Using (2.83) in (2.85) and simplifying, we obtain,

Slope at \( \omega_p \) = \(-A k_t \sqrt{1 - \frac{1}{A^2}}\)  

(2.86)

Equalization of slopes at \( \omega_p \) by equating (2.82) and (2.86) yields,

\[ k_p = \frac{2 k_t (\sqrt{A^2 - 1})}{\delta_p} \]  

(2.87)

The filter design parameter \( k_t \) is obtained as before from (2.66) for a given \( A > 1 \), as

\[ k_t = \frac{1}{(\omega_z - \omega_c)} \left[ \cos^{-1} \left( \frac{\delta_z}{2A} \right) - \cos^{-1} \left( \frac{1 - \delta_p}{2} \right) \right] \]  

(2.88)

At \( \omega = \omega_z \), \( H_{pm}(\omega_z) = 0 = A \cos k_t (\omega_z - \omega_0) \)  

(2.89)

Simplifying (2.89) we obtain, \( \sin k_t (\omega_z - \omega_0) = \pm 1 \). Since the slope of the response is negative at \( \omega = \omega_z \), we choose the positive sign i.e., \( \sin k_t (\omega_z - \omega_0) = 1 \)  

(2.90)

Using (2.52), Slope at \( \omega_z \) = \( \frac{d}{d\omega} [H_{pm}(\omega)]_{\omega=\omega_z} \)

\[ = \frac{d}{d\omega} [A \cos k_t (\omega - \omega_0)] = -A k_t \sin k_t (\omega_z - \omega_0) \]  

(2.91)

Substituting (2.90) in (2.91), Slope at \( \omega_z = -A k_t \)  

(2.92)

Using (2.53), Slope at \( \omega_z \) = \( \frac{d}{d\omega} [H_{pm}(\omega)]_{\omega=\omega_z} = \frac{d}{d\omega} \left[ -\frac{\delta_z}{2} \sin k_s (\omega - \omega_z) \right] \)

\[ = -\frac{\delta_z}{2} k_s \cos k_s (\omega_z - \omega_z) = -\frac{\delta_z}{2} k_s \]  

(2.93)
Equalizing the slopes at $\omega_s$ by equating (2.92) and (2.93),

$$-A k_i = -\frac{\delta_s}{2} k_s$$  \hspace{1cm} (2.94)

i.e.  \hspace{1cm} $k_s = \frac{2A k_i}{\delta_s}$ \hspace{1cm} (2.95)

The expressions for frequency domain parameters $\omega_x, \omega_s$ and $\omega_c$ are identical to those in section 2.7.1. The expressions for impulse response coefficients are same as in section 2.7.1.1.

### 2.7.2.1 Filter Synthesis Results

For the same design example, as in section 2.7.1.2, the filter is designed using MATLAB with program MLP-4. The filter specifications obtained by measurement of the magnitude response of the filter using Signal Processing toolbox. It is found that passband edge, cutoff edge, stopband edge and transition bandwidth are same as in Class III without slope equalization technique but with reduced passband ripple of 0.068 dB and improved stopband attenuation of 35.82 dB for the same filter order of 701. The ripple due to Gibb's phenomenon is reduced with slope equalization technique and hence passband ripple reduces and stopband attenuation of the filter increases. The magnitude and impulse response of the proposed lowpass filter obtained is shown in Fig.2.8. Also Table 2.4, Fig. 2.9 and Fig. 2.10 depict the performance of the filter. For conventional FIR sharp transition filters the peak passband ripple due to Gibb's phenomenon is about 18%. In the proposed Class III filter with slope equalization the passband ripple is 0.7% for the filter order 701 and decreases for higher filter order. Filter synthesis and design steps are given in Appendix A6.1.3.
Table 2.4

Variation of passband ripple and stopband attenuation with filter order for transition bandwidth of 0.01\(\pi\) and passband width of 0.666\(\pi\) with slope equalization for proposed Class III FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passband Ripple in dB</td>
<td>0.399</td>
<td>0.092</td>
<td>0.072</td>
<td>0.069</td>
<td>0.068</td>
<td>0.02</td>
</tr>
<tr>
<td>Stopband Attenuation in dB</td>
<td>27.2</td>
<td>31.15</td>
<td>33.35</td>
<td>35.09</td>
<td>35.82</td>
<td>37.59</td>
</tr>
</tbody>
</table>
(a)

(b)
Fig. 2.8. (a) Magnitude response of proposed Class III for lowpass filter with slope equalization (b) linear plot (c) magnified view of the passband (d) Impulse response sequence.
Fig. 2.9. Variation of lowpass filter order with passband ripple for constant transition width of 0.01 $\pi$ and passband edge 0.666 $\pi$ with and without slope equalization.

Fig. 2.10. Variation of lowpass filter order with stopband attenuation for constant transition width of 0.01 $\pi$ and passband edge 0.666 $\pi$ with and without slope equalization.
2.8 Class IV FIR Lowpass Filter with Equiripple Response and Linear Transition

2.8.1 Filter Model and Design

In this section, the formulation of a linear phase, sharp transition, lowpass FIR filter model with equiripple passband and stopband response and linear transition region and its design is presented [24]. The filter model magnitude response $H_{pm}(\omega)$ is as shown in Fig.2.11. Synthesis of the filter is greatly simplified with fewer filter design parameters. The filter design parameter $k_p$ is uniquely determined from the transition bandwidth and the parameter is independent of passband edge $\omega_p$. In the proposed lowpass filter model, the various regions of the filter are modeled using trigonometric functions of frequency as follows.

In the passband region, the frequency response is given by

$$H_{pm}(\omega) = 1 + \frac{p}{2} \cos k_p \omega, \quad 0 \leq \omega \leq \omega_p$$  \hspace{1cm} (2.96)

where $H_{pm}(0) = 1 + \frac{p}{2}$ and $H_{pm}(\omega_p) = 1$

where $\omega$ is the frequency variable, $H_{pm}(\omega)$ is the pseudo-magnitude of the filter response, $\delta_p$ is the passband ripple, $k_p$ is a filter design parameter and $\omega_p$ is the passband edge. In the linear transition region $[\omega_p, \omega_z]$ the frequency response is given by

$$H_{pm}(\omega) = 1 - \left[ \frac{\omega - \omega_p}{(\omega - \omega_p)} \right], \quad \omega_p \leq \omega \leq \omega_z$$  \hspace{1cm} (2.97)
Fig. 2.11. Illustration of proposed Class IV lowpass filter model with equiripple magnitude response and linear transition region.
Also, \( H_{pm}(\omega_c) = 1 - \frac{\delta}{2} \), \( H_{pm}(\omega_s) = \frac{\delta}{2} \) and \( H_{pm}(\omega_z) = 0 \)

where \( \omega_c \) is the cutoff edge, \( \omega_s \) is the stopband edge and \( \omega_z \) is the frequency at which \( H_{pm}(\omega) \) is zero and \( \delta \) is the stopband attenuation.

In the stopband region the frequency response is given by

\[
H_{pm}(\omega) = -\frac{\delta}{2} \sin k_p (\omega - \omega_z), \quad \omega_z \leq \omega \leq \pi
\]

(2.98)

### 2.8.2 Slope equalization

Slope equalization technique is applied to the equiripple filter model with linear transition region, which further improves the performance of the filter. The filter design parameter \( k_p \) is evaluated by equalizing the slopes of the pseudo-magnitude response function at both the ends of the transition region as described earlier. This allows the proposed function to be continuous thus reducing the ripples due to Gibb’s phenomenon.

Using (2.96), Slope at \( \omega_p \) =

\[
\frac{d}{d\omega} [H_{pm}(\omega)]_{\omega=\omega_p} = \frac{d}{d\omega} \left[ 1 + \frac{\delta_p}{2} \cos k_p \omega \right] = -\frac{\delta_p}{2} k_p \sin k_p \omega_p
\]

(2.99)

At \( \omega = \omega_p, H_{pm}(\omega_p) = 1.0 = 1 + \frac{\delta_p}{2} \cos k_p \omega_p \)

(2.100)

Simplifying, \( \cos k_p \omega_p = 0 \) and \( \sin k_p \omega_p = \pm 1 \)

Since the slope of the magnitude response at \( \omega_p \) is negative we choose the positive sign,
i.e. \( \sin k_p \omega_p = 1 \) \hspace{1cm} (2.101)

Substituting (2.101) in (2.99), we obtain,

\[
\text{Slope at } \omega_p = \frac{-k_p \delta_p}{2}
\]  

(2.102)

Since the transition region \([\omega_p, \omega_z]\) is linear, the slopes of the frequency response at \(\omega_p\), \(\omega_c\), \(\omega_s\), and \(\omega_z\) are identical. Thus transition region slope of the magnitude response is given as

\[
-\frac{\delta_p}{2} = \frac{1}{(\omega_c - \omega_p)} = \frac{1}{(\omega_s - \omega_p)} = \frac{1 - \delta_p}{2}
\]  

(2.103)

Equalizing the slopes at \(\omega_p\) and \(\omega_z\), from (2.102) and (2.103) we obtain,

\[
-\frac{k_p \delta_p}{2} = \frac{\delta_p}{2(\omega_c - \omega_p)}
\]  

(2.104)

Simplifying (2.104), \(\omega_c = \omega_p + \frac{1}{k_p}\) \hspace{1cm} (2.105)

Equalizing the slopes at \(\omega_p\) and \(\omega_z\) from (2.102) and (2.103),

\[
-\frac{k_p \delta_p}{2} = \frac{1 - \delta_p}{2(\omega_s - \omega_p)}
\]  

(2.106)

Simplifying (2.106), \(\omega_s = \omega_p + \frac{2}{k_p \delta_p} - \frac{1}{k_p}\) \hspace{1cm} (2.107)

Equalizing slopes at \(\omega_p\) and \(\omega_z\) from (2.102) and (2.103),

\[
-\frac{k_p \delta_p}{2} = \frac{1}{(\omega_z - \omega_p)}
\]  

(2.108)
Simplifying (2.108),

\[
\omega_z = \omega_p + \frac{2}{k_p \delta_p} \tag{2.109}
\]

From (2.98), Slope at \(\omega_z = \frac{d}{d\omega} \left[ H_{pm} (\omega) \right]_{\omega=\omega_z}

\[
= \frac{d}{d\omega} \left[ \frac{\delta_p}{2} \sin k_p (\omega - \omega_z) \right] = \frac{k_p \delta_p}{2} \cos k_p (\omega - \omega_z) \tag{2.110}
\]

Equalizing the slopes at \(\omega_z\) from (2.102) and (2.110),

\[
\frac{k_p \delta_p}{2} = \frac{k_p \delta_s}{2}
\]

Thus \(\delta_p = \delta_s\) \hspace{1cm} (2.111)

Eq. (2.111) outlines the symmetry of the response.

Using (2.105) and (2.107),

\[
k_p = \frac{2(1-\delta_p)}{\delta_p (\omega_s - \omega_c)} \tag{2.112}
\]

2.8.3 Expressions for Impulse Response Coefficients

Referring to filter design theory of section 2.4, the impulse response coefficients \(h(n)\) for the lowpass filter are obtained by evaluating the integral below,

\[
h(n) = \frac{1}{\pi} \left[ \int_0^\pi H_{pm}(\omega) \cos k\omega \ d\omega \right] \tag{2.113}
\]

\[
h(n) = \frac{1}{\pi} \left[ \int_0^{\omega_p} H_{pm}(\omega) \cos k\omega \ d\omega + \int_{\omega_p}^{\omega_z} H_{pm}(\omega) \cos k\omega \ d\omega + \int_{\omega_z}^{\pi} H_{pm}(\omega) \cos k\omega \ d\omega \right] \tag{2.114}
\]
where $n = 0, 1, 2, \ldots, \frac{N-1}{2}$ for N odd.

$$n = 0, 1, 2, \ldots, \frac{N}{2} - 1 \text{ for } N \text{ even and } k = \left\lfloor \frac{N-1}{2} - n \right\rfloor$$

$$h(n) = \frac{1}{\pi} \left[ \int_{0}^{\frac{\omega_p}{2}} (1 + \frac{\delta_p}{2} \cos k_p \omega) \cos k \omega \, d\omega + \int_{\frac{\omega_p}{2}}^{\frac{\omega_z}{2}} \left( \frac{(\omega - \omega_p)}{(\omega_z - \omega_p)} \right) \cos k \omega \, d\omega \\
+ \int_{\frac{\omega_z}{2}}^{\frac{\omega_p}{2}} \frac{\delta_p}{2} \sin k_p (\omega - \omega_z) \cos k \omega \, d\omega \right]$$

(2.115)

Evaluating (2.115), the expressions obtained for the impulse response coefficients $h(n)$ for the lowpass filter are

$$h(n) = \frac{(\delta_p k_p) \cos (k \omega_p)}{2\pi(k_p^2 - k^2)} \left[ \frac{\cos(k \omega_z) - \cos(k \omega_p)}{k^2 \pi(\omega_z - \omega_p)} \right]$$

$$+ \frac{\delta_p}{2\pi(k_p^2 - k^2)} \left[ k_p \cos k_p (\pi - \omega_z) \cos (k \pi) - k_p \cos (k \omega_z) + k \sin k_p (\pi - \omega_z) \sin (k \pi) \right]$$

(2.116)

Eq. (2.116) is valid for $N$ even where $k$ is a non-integer. For $N$ odd (2.116) is valid except for $k=0$ and $k = k_p$.

For $N$ odd, $k = 0$ we obtain

$$h\left(\frac{N-1}{2}\right) = \frac{1}{\pi} \left[ \frac{\omega_z + \omega_p}{2} + \frac{\delta_p \cos k_p (\pi - \omega_z)}{2k_p} \right]$$

(2.117)

For $N$ odd, $k = k_p$ we obtain

$$h\left(\frac{N-1}{2} - k_p\right) = h\left(\frac{N-1}{2} + k_p\right) = \frac{\delta_p \omega_p}{4\pi} \frac{\cos k_p \omega_z}{k_p^2 \pi(\omega_z - \omega_p)}$$
2.8.4 Filter Synthesis Results

Design Example: Lowpass linear phase sharp transition FIR filters are designed for the desired filter specifications: Passband edge $\omega_p$ is $0.666\pi$, transition bandwidths ($\omega_s - \omega_c$) are $0.005\pi$, $0.01\pi$ and $0.02\pi$, maximum passband ripple $\delta_p$ is $\pm 0.1\text{dB}$ ($0.2\text{dB}$) and minimum stopband attenuation $\delta_s$ is $40\text{dB}$ using the proposed Class IV filter design approach.

The filter is designed using MATLAB with program MLP-5. The filter specifications obtained by measurement of the magnitude response of the filter using Signal Processing toolbox are: passband edge $\omega_p = 0.6667\pi$, cutoff edge $\omega_c = 0.6668\pi$, stopband edge $\omega_s = 0.6767\pi$, transition bandwidth ($\omega_s - \omega_c$) = $0.01\pi$, passband ripple $\delta_p = 0.129\text{dB}$ and stopband attenuation $\delta_s = 40.3\text{dB}$ for a filter order of 701. It is observed that the desired filter specifications are obtained. The magnitude, impulse and phase response of the proposed lowpass filter obtained is shown in Fig.2.12. Also Tables 2.5, 2.6, 2.7 and Figs. 2.13, 2.14 and 2.15 depicts the performance of the filter. No appreciable change in filter performance is observed for narrow and wide passband as seen in Table 2.5 and 2.7. Class IV filter gives the best filter performance compared to Class I, Class II and Class III filter models. From Table 2.6 it is observed that filter order varies almost linearly with a transition bandwidth of the filter for a given passband ripple and stopband attenuation which holds good also with all proposed filter designs. For conventional FIR sharp transition filters the peak passband ripple due to Gibb’s phenomenon is about 18%. In proposed Class IV filter the passband ripple is 1.49% for the filter order 701 and decreases for higher filter order. Filter synthesis and design steps are given in Appendix A.6.1.4.
### Table 2.5

Variation of passband ripple and stopband attenuation with filter order for transition bandwidth of 0.01\( \pi \) and passband edge of 0.666\( \pi \) with slope equalization for proposed Class IV FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passband Ripple in dB</td>
<td>0.33</td>
<td>0.21</td>
<td>0.2</td>
<td>0.197</td>
<td>0.129</td>
<td>0.088</td>
</tr>
<tr>
<td>Stopband Attenuation in dB</td>
<td>32.8</td>
<td>36.46</td>
<td>36.53</td>
<td>36.77</td>
<td>40.3</td>
<td>42.85</td>
</tr>
</tbody>
</table>

### Table 2.6

Variation of transition width with filter order for passband ripple of 0.2dB and stopband attenuation of 40dB for passband edge of 0.666\( \pi \) with slope equalization for proposed Class IV FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>351</th>
<th>701</th>
<th>1401</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transition width</td>
<td>0.022( \pi )</td>
<td>0.010( \pi )</td>
<td>0.0052( \pi )</td>
</tr>
</tbody>
</table>

### Table 2.7

Variation of passband ripple and stopband attenuation with filter order for transition width of 0.01\( \pi \) and passband edge of 0.333\( \pi \) with slope equalization for proposed Class IV FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passband Ripple in dB</td>
<td>0.324</td>
<td>0.21</td>
<td>0.2</td>
<td>0.1965</td>
<td>0.129</td>
<td>0.084</td>
</tr>
<tr>
<td>Stopband Attenuation in dB</td>
<td>32.65</td>
<td>35.54</td>
<td>36.04</td>
<td>36.04</td>
<td>40.2</td>
<td>42.05</td>
</tr>
</tbody>
</table>
Fig. 2.12. (a) Magnitude response of the proposed Class IV lowpass filter (b) magnified view of the passband (c) Impulse response sequence (d) Phase response.
Fig. 2.13. Variation of passband ripple with filter order for constant transition width of $0.011\pi$ and passband width of $0.666\pi$ for various proposed filters.

Fig. 2.14. Variation of stopband attenuation with filter order for constant transition width of $0.011\pi$ and passband width of $0.666\pi$ for various proposed filters.

Fig. 2.15. Plot of Filter order Vs Transition width for Class IV filter model with slope equalization for a given passband ripple of $0.1\text{dB}$, stopband attenuation of $40\text{dB}$ and passband of $0.666\pi$. 
2.9 Class V FIR Lowpass Filter with Equiripple Response, Linear Transition and Variable Density Ripple Cycles

2.9.1 Filter model and design

In this section, the formulation of linear phase, sharp transition lowpass FIR filter model with equiripple response and its design is presented. The transition region is linear and the filter design parameter $k_p$ is varied over the passband and stopbands in this design. The filter model magnitude response $H_{pm}(\omega)$ is as shown in Fig. 2.16. In the proposed model, the various regions of the filter are modeled using trigonometric functions of frequency as follows.

The passband region, consists of two sub-regions whose frequency responses are given by

$$H_{pm}(\omega) = 1 + \frac{\delta}{2} \cos k_p \omega, \quad 0 \leq \omega \leq \omega_p$$  \hspace{1cm} (2.119)

$$H_{pm}(\omega) = 1 + \frac{\delta}{2} \cos k_p (\omega - \omega_{p0}), \quad \omega_{p0} \leq \omega \leq \omega_p$$  \hspace{1cm} (2.120)

where $H_{pm}(0) = 1 + \frac{\delta}{2}$ and $H_{pm}(\omega_p) = 1$

where, as usual $\omega$ is the frequency variable, $H_{pm}(\omega)$ is the pseudo-magnitude of the filter response, $\delta_p$ is passband ripple, $k_p$ and $k_{p0}$ are filter design parameters, $\omega_p$ is the passband edge and $\omega_{p0}$ is an intermediate frequency in the passband such that $0 < \omega_{p0} < \omega_p$.

In the linear transition region, the frequency response is given by

$$H_{pm}(\omega) = 1 - \left[ \frac{\omega - \omega_p}{\omega_z - \omega_p} \right], \quad \omega_p \leq \omega \leq \omega_z$$  \hspace{1cm} (2.121)
Fig. 2.16. Illustration of proposed Class V lowpass filter model with equiripple magnitude response, linear transition region and variable $k_p$. 
where $\omega_z$ is the frequency at which $H_{pm}(\omega)$ is zero, i.e., $H_{pm}(\omega_z) = 0$. Also $H_{pm}(\omega_c) = \frac{\delta}{2}$.

where $\omega_c$ is the cutoff frequency.

The stopband region, consists of two sub-regions in which the frequency responses are given by

$$H_{pm}(\omega) = -\frac{\delta}{2} \sin k_p(\omega - \omega_z), \quad \omega_z \leq \omega \leq \omega_{s0} \quad (2.122)$$

$$H_{pm}(\omega) = -\frac{\delta}{2} \cos k_p(\omega - \omega_{s0}), \quad \omega_{s0} \leq \omega \leq \pi \quad (2.123)$$

where $\delta_s$ is the stopband attenuation and $\omega_{s0}$ is an intermediate frequency in the stopband such that $\omega_s < \omega_{s0} < \pi$.

Also $H_{pm}(\omega_s) = -\frac{\delta}{2}$ and $H_{pm}(\omega_{s0}) = -\frac{\delta}{2}$ where $\omega_s$ is the stopband edge frequency.

For the Class V lowpass filter design,

The stopband region $[\omega_z, \omega_{s0}]$ of the filter model possesses $\left(m + \frac{1}{4}\right)$ number of ripple cycles and is characterized by filter design parameter $k_p$. Therefore,

$$k_p(\omega_{s0} - \omega_z) = 2\pi \left(m + \frac{1}{4}\right) \quad (2.124)$$

where $m$ is an integer.

From (2.124) we get, $\sin k_p(\omega_{s0} - \omega_z) = 1 \quad (2.125)$

The passband region $[\omega_{p0}, \omega_p]$ of the filter model possess $\left(m + \frac{1}{4}\right)$ number of ripple cycles and is characterized by filter design parameter $k_p$. Therefore,
\[ k_p (\omega_p - \omega_{p0}) = 2\pi \left( m + \frac{1}{4} \right) \quad (2.126) \]

From (2.126), \[ \sin k_p (\omega_p - \omega_{p0}) = 1 \quad (2.127) \]

From (2.126), \[ \omega_{p0} = \omega_p - \left( m + \frac{1}{4} \right) \frac{2\pi}{k_p} \quad (2.128) \]

The passband region \([0, \omega_{p0}]\) of the filter model possesses 'n' number of ripple cycles (n is an integer) and is characterized by filter design parameter \(k_p\). Therefore, \(k_p \omega_{p0} = 2\pi n\).

The filter design parameter \(k_{p0} = \frac{2\pi n}{\omega_{p0}}\) \quad (2.129)

### 2.9.2 Slope Equalization

Slope equalization technique is applied to the filter model which further improves the performance of the filter. The filter design parameters are evaluated by equalizing the slopes of the pseudo-magnitude response function at both the ends of the transition region. Use of variable filter design parameter \(k_p\) further reduces the effects due to Gibb's phenomenon and hence reduces passband ripple and improves stopband attenuation.

Using (2.120), slope at \(\omega_p\) \[ \frac{d}{d\omega} \left[ H_{pm} (\omega) \right]_{\omega=\omega_p} \]

\[ = \frac{d}{d\omega} \left[ \frac{\delta_p}{2 + \frac{\delta_p}{2} \cos k_p (\omega - \omega_{p0})} \right] \]

\[ = -\frac{\delta_p}{2} k_p \sin k_p (\omega_p - \omega_{p0}) \quad (2.130) \]

Substituting (2.127) in (2.130),

70
Slope at $\omega_p = -\frac{k_p \delta_p}{2}$  \hspace{1cm} (2.131)

Since the transition region $[\omega_p, \omega_z]$ is linear, the slopes of the frequency response at $\omega_p$, $\omega_c$, $\omega_s$, and $\omega_z$ are identical. Thus transition region slope of the magnitude response $H_{pm}(\omega)$ can be variously given as

$$\frac{\delta_p}{2} = \frac{1}{(\omega_c - \omega_p)} = \frac{1}{(\omega_z - \omega_p)} = \frac{1}{(\omega_s - \omega_p)}$$  \hspace{1cm} (2.132)

Equalizing the slopes at $\omega_p$ and $\omega_z$ using (2.131) and (2.132) we obtain,

$$\frac{k_p \delta_p}{2} = \frac{1}{(\omega_z - \omega_p)}$$  \hspace{1cm} (2.133)

Using (2.133) we have, $\omega_z = \omega_p + \frac{2}{k_p \delta_p}$  \hspace{1cm} (2.134)

Using (2.122) and simplifying, slope at $\omega_z = \frac{d}{d\omega} \left[ -\frac{\delta}{2} \sin k_p (\omega - \omega_z) \right]$

$$= -\frac{k_p \delta_s}{2}$$  \hspace{1cm} (2.135)

Equalization of slopes at $\omega_z$ using (2.131) and (2.135),

$$-\frac{k_p \delta_p}{2} = -\frac{k_p \delta_s}{2}$$  \hspace{1cm} (2.136)

Simplifying (2.136), we obtain, $\delta_s = \delta_p$  \hspace{1cm} (2.137)

Eq. (2.137) outlines the symmetry of the response. Equating the slopes at $\omega_p$ and $\omega_z$ using (2.131) and (2.132) we obtain,
\[
\frac{-k_p \delta_p}{2} = \frac{\delta_p}{2 \left( \omega_c - \omega_p \right)}
\]

\[
\omega_c = \omega_p + \frac{1}{k_p}
\]  
(2.138)

\[
\omega = \omega_s, H_{pm}(\omega_s) = \frac{\delta_p}{2} = 1 - \frac{\left( \omega_s - \omega_p \right)}{\left( \omega_s - \omega_c \right)} = \frac{\delta_p}{2}
\]  
(2.139)

Substituting (2.134) in (2.139) and simplifying,

\[
\omega_s = \omega_z - \frac{1}{k_p} = \omega_p + \frac{2}{k_p \delta_p} - \frac{1}{k_p}
\]  
(2.140)

Using (2.138) and (2.140), we obtain the filter design parameter

\[
k_p = \frac{\delta_p}{\left( \omega_s - \omega_c \right)}
\]  
(2.141)

2.9.3 Expressions for Impulse Response Coefficients

Referring to filter design theory of section 2.4, the impulse response coefficients \( h(n) \) for the lowpass filter are obtained by evaluating the integral below.

\[
h(n) = \frac{1}{\pi} \int_0^\pi H_{pm}(\omega) \cos k\omega \, d\omega
\]  
(2.142)

\[
h(n) = \frac{1}{\pi} \left[ \int_0^{\omega_p} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_p} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_p} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_p} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_p} H_{pm}(\omega) \cos k\omega \, d\omega + \int_{\omega_p}^{\omega_p} H_{pm}(\omega) \cos k\omega \, d\omega \right]
\]  
(2.143)
where \( n = 0,1,2, \ldots, \frac{N-1}{2} \) for \( N \) odd

\[
n = 0,1,2, \ldots, \frac{N}{2} - 1 \text{ for } N \text{ even and } k = \left( \frac{N-1}{2} - n \right)
\]

\[
h(n) = \frac{1}{\pi} \int_0^{\omega_p} \left[ \frac{\delta}{2} \cos k_p \omega \right] \cos k \omega \ d\omega + \int_{\omega_p}^{\omega} \left[ \frac{\delta}{2} \cos k_p (\omega - \omega_p) \right] \cos k \omega \ d\omega
\]

\[
+ \int_{\omega_p}^{\omega} \left[ \frac{1}{\omega_p} - \frac{\delta}{2} \cos k_p (\omega - \omega_p) \right] \cos k \omega \ d\omega
\]

\[
+ \int_{\omega_p}^{\omega} \left[ \frac{\delta}{2} \cos k_p (\omega - \omega_p) \right] \cos k \omega \ d\omega
\]

\[ (2.144) \]

Evaluating (2.144) the expressions obtained for the impulse response coefficients \( h(n) \) for the lowpass filter are

\[
h(n) = -\frac{[\delta_p \sin(k \omega_p)]}{2\pi(k_p^2 - k^2)} + \frac{\delta_p [k_p \cos(k \omega_p) + k \sin(k \omega_p)]}{2\pi(k_p^2 - k^2)} - \frac{\cos(k \omega - \omega_p)}{k^2 \pi(k - \omega)}
\]

\[
+ \frac{\delta_p [k \sin(k \omega_p) - k_p \cos(k \omega_p)]}{2\pi(k_p^2 - k^2)} - \frac{\delta_p [k \sin(k \omega_p) + k_p \cos(k \omega_p)]}{2\pi(k_p^2 - k^2)}
\]

\[
+ \frac{\delta_p}{2\pi(k_p^2 - k^2)} [k \cos k \omega_p (\pi - \omega_p) \sin(k \pi)]
\]

\[ (2.145) \]

Eq. (2.145) is valid for \( N \) even where \( k \) is a non-integer. For \( N \) odd (2.145) is valid except for \( k = 0 \), \( k = k_p \) and \( k = k_{p0} \).

For \( N \) odd, \( k = 0 \) we obtain,
\[ h \left( \frac{N-1}{2} + k_p \right) = h \left( \frac{N-1}{2} - k_p \right) = -\frac{\delta_p}{2\pi k_p} \frac{\sin \left( k_p \omega z \right) - \cos \left( k_p \omega_0 \right)}{2k_p} \]

\[ + \frac{\delta_p}{4\pi} \left[ \frac{\sin \left( \frac{2k_p \omega - k_p \omega_0}{2} \right) - \sin \left( \frac{k_p \omega_0}{2} \right)}{2k_p} + \left( \omega - \omega_0 \right) \cos \left( k_p \omega_0 \right) \right] \]

\[ - \frac{\delta_p}{2\pi k_p} \frac{\sin \left( k_p \omega z \right) + k_p \sin k_p \left( \pi - \omega_0 \right) \cos \left( k_p \pi \right) - k_p \cos k_p \left( \pi - \omega_0 \right) \sin \left( k_p \pi \right)}{2k_p} \]

\[ + \frac{\delta_p}{4\pi} \left[ \frac{\cos \left( \frac{2k_p \omega - k_p \omega_0}{2} \right) - \cos \left( \frac{k_p \omega_0}{2} \right)}{2k_p} + \left( \omega - \omega_0 \right) \sin \left( k_p \omega z \right) \right] \] (2.147)

For \( N \) odd, \( k = k_p \) we obtain,

\[ h \left( \frac{N-1}{2} + k_p \right) = h \left( \frac{N-1}{2} - k_p \right) = \frac{\delta_p}{4\pi} \frac{\sin \left( k_p \omega_0 \right) + k_p \sin k_p \left( \pi - \omega_0 \right) \cos \left( k_p \pi \right) - k_p \cos k_p \left( \pi - \omega_0 \right) \sin \left( k_p \pi \right)}{2k_p} \]

\[ + \frac{\delta_p}{4\pi} \left[ \frac{\cos \left( \frac{2k_p \omega - k_p \omega_0}{2} \right) - \cos \left( \frac{k_p \omega_0}{2} \right)}{2k_p} + \left( \omega - \omega_0 \right) \sin \left( k_p \omega z \right) \right] \] (2.148)

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2.9.4 Filter Synthesis Results

Design Example: A lowpass linear phase sharp transition FIR filter is designed for the desired filter specifications: Passband edge $\omega_p$ is $0.666\pi$, transition bandwidth $(\omega_s - \omega_c)$ is $0.01\pi$, maximum passband ripple $\delta_p$ is $\pm 0.1\text{dB}$ ($0.2\text{dB}$) and minimum stopband attenuation $\delta_s$ is $40\text{dB}$ using the proposed Class V filter design approach.

The filter is designed using MATLAB with program MLP-6. The filter specifications obtained by measurement of the magnitude response of the filter using Signal Processing toolbox are passband edge $\omega_p = 0.6667\pi$, cutoff edge $\omega_c = 0.6668\pi$, stopband edge $\omega_s = 0.6767\pi$, transition bandwidth $(\omega_s - \omega_c) = 0.01\pi$, passband ripple $\delta_p = 0.13\text{dB}$ and stopband attenuation $\delta_s = 40.19\text{dB}$ are obtained for a filter order of 701. It is observed that the desired filter specifications are obtained. The magnitude and impulse response of the proposed lowpass filter obtained is shown in Fig.2.17. Table 2.8 depicts the performance of the filter. Class V also gives the best filter performance for a given filter order compared to all filter models developed i.e. Class I, Class II and Class III filters with no appreciable change in filter performance compared to Class IV. But peak passband ripple is reduced compared to Class IV filter indicating the further reduction of Gibb's phenomenon with this filter design. For conventional FIR sharp transition filters the peak passband ripple due to Gibb's phenomenon is about $18\%$. In proposed Class V filter the passband ripple is $1.47\%$ for the filter order 701 and decreases for higher filter order. Filter synthesis and design steps are given in Appendix A6.1.5.
Table 2.8

Variation of passband ripple and stopband attenuation with filter order for transition bandwidth of 0.01π and passband edge of 0.666π with slope equalization for proposed Class V FIR filter.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passband Ripple in dB</td>
<td>0.337</td>
<td>0.192</td>
<td>0.19</td>
<td>0.189</td>
<td>0.13</td>
<td>0.085</td>
</tr>
<tr>
<td>Stopband Attenuation in dB</td>
<td>31.96</td>
<td>35.2</td>
<td>35.71</td>
<td>36.46</td>
<td>40.19</td>
<td>43.01</td>
</tr>
</tbody>
</table>
Fig. 2.17. (a) Magnitude response of the proposed Class V lowpass filter (b) Linear plot (c) magnified view of the passband (d) Impulse response sequence.
2.10 Comparison of Various Proposed FIR Filters with Conventional FIR filters

For comparison of proposed FIR filter design with filters designed using conventional techniques, consider the design example of the proposed FIR filter. Using our proposed approach, the desired filter specifications are obtained for a low filter order of 701 without any optimization technique being used. In window method, the cutoff is at 6 dB. Such a large passband ripple is undesirable and does not meet the desired passband ripple specifications. In frequency sampling method, it is found that for obtaining the desired filter specifications the minimum FIR filter order required is 1025 which is large compared to proposed approach. In Remez approach, which employs optimization techniques, to obtain the desired filter specifications the filter order required is 401. But in this case closed form expressions for impulse response coefficients cannot be obtained because optimization techniques are used. Our proposed filter can be also used as an initial filter for optimization which may then match the filter order obtained in Remez approach. Also, if the filter is suitably truncated it will reduce the filter complexity appreciably with minimum deterioration in passband ripple and stopband attenuation. As seen in Table 2.9 and Fig. 2.19, peak passband ripple due to Gibb's phenomenon is drastically reduced for the proposed lowpass filters with class V filter having the least peak passband ripple of 1.47% for a filter order of 701 and decreases for higher filter order. In conventional filter design peak passband ripple is fixed at about 18% and does not decrease with filter order. In the proposed approach as seen in Table 2.9 peak passband ripple reduces with increase in filter order.
Table 2.9

Peak passband ripple for conventional FIR filter and proposed FIR filters for various filter orders to illustrate reduction in Gibb’s phenomenon.

<table>
<thead>
<tr>
<th>Filter Order</th>
<th>% Peak Passband Ripple</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Conventional Filter</td>
</tr>
<tr>
<td>301</td>
<td>18</td>
</tr>
<tr>
<td>401</td>
<td>18</td>
</tr>
<tr>
<td>501</td>
<td>18</td>
</tr>
<tr>
<td>601</td>
<td>18</td>
</tr>
<tr>
<td>701</td>
<td>18</td>
</tr>
<tr>
<td>801</td>
<td>18</td>
</tr>
<tr>
<td>901</td>
<td>18</td>
</tr>
<tr>
<td>1001</td>
<td>18</td>
</tr>
</tbody>
</table>

* Without slope equalization
** With slope equalization
Fig. 2.18. Plot of Filter order Vs peak passband ripple for conventional FIR filter and various proposed FIR filters for a given passband ripple of 0.2dB, stopband attenuation of 40dB, transition bandwidth of 0.01π and passband edge of 0.666π.

Fig. 2.19. Bar chart showing peak passband ripple for conventional FIR filter and various proposed FIR filters for a transition width of 0.01π, and filter order 701.
2.11 Conclusions

Various proposed sharp transition, linear phase, lowpass FIR filters are designed. Various regions of the filter are approximated with trigonometric functions of frequency. The design possesses closed form expressions for impulse response coefficients of the filter and its transfer function is evolved in frequency and time domain.

A novel technique is devised to reduce Gibb's phenomenon. Equations are derived for slopes of the frequency response of the filter, at the edges of the transition region and the slopes are matched. The filter design parameters of the model are evaluated by equalizing the slopes of the pseudo-magnitude response function at both the ends of the transition region. It is proved in the proposed approach that equalizing the slopes at the edges of the transition region makes the proposed function of frequency continuous between a pair of adjoining regions defined by the model equations and hence reduces the effects due to Gibb's phenomenon thereby reducing ripples at the edges of the transition region of the filter. Comparison is made with filters designed with the same set of specifications without slope equalization and it was found that the ripple at the transition edges has reduced with slope equalization. This reduces passband ripple and improves stopband attenuation of the filter as observed in Class III filter without and with slope equalization.

This proposed design approach is without optimizations and hence computation is reduced unlike filter designs based on optimization techniques. In the proposed filters, if the impulse response sequence is suitably truncated subjected to a finite word length there is a appreciable reduction in filter order with marginal deterioration in filter performance i.e. increase in passband ripple and decrease in stopband attenuation. The proposed filters performance is better compared to window and frequency sampling techniques and simpler than optimum filter design without
the need for optimization and complex computational procedures. Peak passband ripple due to Gibb’s phenomenon is drastically reduced for the proposed lowpass filters and decreases for higher filter order. In conventional filter design peak passband ripple is about 18% and does not decrease with filter order.

The proposed design approach is found to compare favorably with frequency response masking techniques [7] in terms of sharp transition, least passband ripple and good stopband attenuation with least filter order. The proposed filter design approach is a direct one without subfilters and filter optimizations and closed form expressions for impulse response coefficients are obtained unlike FRM approach where four subfilters are to be synthesized and are to be optimized. With the proposed approach sharp transition filters for any narrow transition bandwidth can be synthesized. The actual filter length (including zero and nonzero coefficients) and delays in FRM approach is higher than our approach to meet a given filter specification. Lesser group delay which is very much desired particularly in speech processing applications.