ABSTRACT

STUDY OF VARIOUS ASPECTS IN THE THEORY OF UNIVALENT AND MULTIVALENT FUNCTIONS

The theory of univalent and multivalent functions is an old branch of mathematics, particularly complex analysis attracting a large number of mathematicians and research workers because of its only shear beauty of its geometrical aspects and abundant avenues for research work. The study of univalent and multivalent functions is one of the leading branch of the geometric function theory. An univalent function is an analytic or meromorphic function $f$ in a domain of extended complex plane such that $f(z_1) = f(z_2)$ for $z_1 = z_2$, where $z_1$ and $z_2$ are members of the domain.

This study consists of the family of functions like starlike, convex, close-to-convex etc. defined on a domain of the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and also punctured unit disk $U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$.

In Chapter 0, we give the list of all the definitions, lemmas, theorems and basic results which we require during the course of our research work.

In the Chapter 1, we have studied the class $A_{\lambda,\mu,\nu}(n,\beta)$ of the analytic and univalent functions with negative coefficients in the unit disk $U$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k , \ (a_k \geq 0, n \in \mathbb{N})$$

defined by making use of the generalized Ruscheweyh derivatives involving a
general fractional derivative operator satisfying:

$$\text{Re}\left\{ \frac{z(J_1^{\lambda,\mu} f(z))'}{(1 - \gamma)J_1^{\lambda,\mu} f(z) + \gamma z^2 (J_1^{\lambda,\mu} f(z))''} \right\} > \beta$$

for $z \in U, 0 \leq \gamma < 1, n \in \mathbb{N}, 0 \leq \beta < 1, \lambda > -1$, where $J_1^{\lambda,\mu}$ is a generalized Ruscheweyh derivatives defined as

$$J_1^{\lambda,\mu} f(z) = \frac{\Gamma(\mu + 2) - \lambda + \nu + 2}{\Gamma(\nu + 2)\Gamma(\mu + 1)} z^{J_0,z}^{\lambda,\mu}(z^{|\lambda-1| f(z))}
= z - \sum_{k=n+1}^{\infty} a_k C_1^{\lambda,\mu}(k) z^k,$$

where

$$C_1^{\lambda,\mu}(k) = \frac{\Gamma(k + \mu)\Gamma(\nu + 2 + \mu - \lambda)\Gamma(k + \nu + 1)}{\Gamma(k)\Gamma(k + \nu + 1 + \mu - \lambda)\Gamma(\nu + 2)\Gamma(1 + \mu)}.$$ 

A necessary and sufficient condition for a function to be in the class $A_\gamma^{\lambda,\mu,n}(n, \beta)$ is obtained. We define the generalized fractional derivative operator

$$J_0^{\lambda,\mu} f(z) = \begin{cases} \frac{1}{(1 - \lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z - \xi)^{-\lambda} f(\xi) \right\}, & 0 \leq \lambda < 1 \\ \frac{d^n}{dz^n} J_0^{\lambda-n,\mu,n} f(z), & n \leq \lambda < n + 1, n \in \mathbb{N} \end{cases}$$

where the function $f(z)$ is analytic in a simply connected region of the $z$-plane containing the origin with the order

$$f(z) = O(|z|^\epsilon); \quad z \to 0,$$

for $\epsilon > \max\{0, \mu - \nu - 1\} - 1$.

In continuation of this study, we have also studied radii of starlikeness, convexity and close-to-convexity, extreme points, convex combination and integral operators.
We can introduce the subclass $A^{\lambda,\mu,\nu}_{\gamma,\epsilon}(1,\beta)$ consisting of functions with negative and fixed finitely many coefficients. Also we have obtained some interesting properties of $A^{\lambda,\mu,\nu}_{\gamma,\epsilon}(1,\beta)$.

We have also introduced a class $k$-uniformly convex functions and related class of $k$-starlike functions with negative coefficients based on integral operator $Q_\delta^\beta$ of the function $f(z)$. We denote $k$-uniformly class by $k^{-}\text{UCV}^{\lambda, \gamma, \beta, \delta}(1, \beta)$ the subclass of $T$ satisfying the condition:

$$
\Re \left\{ \frac{z(Q_\delta^\beta f(z))' + \lambda z^2(Q_\delta^\beta f(z))''}{(1 - \lambda)Q_\delta^\beta f(z) + \lambda (Q_\delta^\beta f(z))'} \right\} \\
\geq k \left| \frac{z(Q_\delta^\beta f(z))' + \lambda z^2(Q_\delta^\beta f(z))''}{(1 - \lambda)Q_\delta^\beta f(z) + \lambda (Q_\delta^\beta f(z))'} - 1 \right| + \gamma.
$$

For $0 \leq \lambda \leq 1, 0 \leq \gamma < 1, k \geq 0, \beta \geq 0$ and $\delta > -1$, where $T$ denotes the class of the analytic and univalent functions in the open unit disk $U$ of the form:

$$
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in \mathbb{N}),
$$

and $Q_\delta^\beta$ is the generalized Jung-Kim-Srivastava integral operator defined by

$$
Q_\delta^\beta f(z) = \frac{\Gamma(\beta + \delta + 1)}{z\Gamma(\beta)\Gamma(\delta + 1)} \int_0^z t^{\beta-1}(1 - \frac{t}{z})^{\delta-1} f(t) dt, \beta \geq 0, \delta > -1
$$

$$
= z - \sum_{n=2}^{\infty} \frac{\Gamma(\beta + \delta + 1)\Gamma(\beta + n)}{\Gamma(\beta + 1)\Gamma(\delta + n)} a_n z^n.
$$

For $\beta = 0$ we get $Q_\delta^0 f(z) = f(z)$. Here we obtain neighbourhood result. Partial sums $h_m(z)$ of the functions $f(z)$ in this class are considered. Also we have studied coefficient bounds and radius of close-to-convexity, starlikeness and convexity of this class.

We have introduced the class $WAG(\beta, \alpha, A, B, \gamma, \sigma)$ of univalent meromorphic functions as defined by
WAG(β, α, A, B, γ, σ) = \{ f \in A^*U : \\
\left| \frac{[(B - A)\gamma + A][z^2(I^\sigma f(z))' + \beta z(I^\sigma f(z)) + (1 - \beta)]}{[z^2(I^\sigma f(z))' + \beta z(I^\sigma f(z))][(B - A)\gamma + A + (1 - \beta)][(B - A)\gamma + (A + 1)]} \right| < \alpha,
\}

for some 0 ≤ β < 1, 0 < α ≤ 1, −1 ≤ A < B ≤ 1, \frac{A}{A-B} < γ ≤ 1 and σ > 0}, where \( A^*U \) denotes the class of univalent meromorphic functions in the punctured unit disc \( U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) of the form:

\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n , \quad n \in \mathbb{N} = \{1, 2, \ldots\} \]

and \( I^\sigma \) is the Jung-Kim-Srivastava integral operator defined by

\[ I^\sigma f(z) = \frac{1}{z^2 \Gamma(\sigma)} \int_0^z (\log \frac{z}{t})^{\sigma-1} tf(t)dt = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{n+2} \right)^\sigma a_n z^n. \]

Here, we have searched for coefficient inequalities, distortion bounds, extreme points and Quasi-Hadamard product.

The Chapter 2 is fully devoted for the study of \( p \)-valent analytic functions with negative coefficients in the unit disk \( U \) of the form:

\[ f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k , \quad (a_k \geq 0, \quad n, p \in \mathbb{N}) \]

defined by making use of the generalized Ruscheweyh derivatives involving a general fractional derivative operator, satisfying:

\[ \Re \left\{ \frac{z(J_p^{\lambda,\mu} f(z))' + \gamma z^2(J_p^{\lambda,\mu} f(z))''}{\gamma z(J_p^{\lambda,\mu} f(z))' + (1 - \gamma)(J_p^{\lambda,\mu} f(z))} \right\} > \beta, \]

for 0 ≤ β < p, λ > −1, p ∈ \( \mathbb{N} \), 0 ≤ γ ≤ 1, where \( J_p^{\lambda,\mu} f(z) \) is a generalized Ruscheweyh derivatives defined by

\[ J_p^{\lambda,\mu} f(z) = \frac{\Gamma(\mu - \lambda + \nu + 2)}{\Gamma(\nu + 2)\Gamma(\mu + 1)} z^p J_{0,z}^{\lambda,\mu}(z^{\mu-p} f(z)) \]
\[ f(z) = z^p - \sum_{n=m+p}^{\infty} a_n z^n, \quad (a_n \geq 0, m, p \in \mathbb{N}) \]

where
\[ C_p^{\lambda, \mu}(k) = \frac{\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}. \]

Here, we obtain several nice characterizations like coefficient bounds, growth and distortion bounds and \((k, \theta)\)-neighbourhood. In continuation of this generalized Ruscheweyh derivatives, we have obtained the region of univalency in particular test of starlikeness, convexity. Quasi-Hadamard product is also taken into consideration for our research.

We have also studied application of differential operator of a subclass of \(k\)-uniformly \(p\)-valent starlike and convex functions with negative coefficients defined in the form:
\[ f(z) = z^p - \sum_{n=m+p}^{\infty} a_n z^n, \quad (a_n \geq 0, m, p \in \mathbb{N}) \]
satisfying the condition:
\[ \text{Re} \left\{ \frac{z(Q_3^p f(z))^{(1+q)}}{(1-\lambda)(Q_3^p f(z))^{(q)}} + \lambda z^2(Q_3^p f(z))^{(2+q)}} \right\} \geq k \left| \frac{z(Q_3^p f(z))^{(1+q)}}{(1-\lambda)(Q_3^p f(z))^{(q)}} + \lambda z^2(Q_3^p f(z))^{(2+q)}} \right| - 1 + \gamma. \]

Here, we have studied some interesting properties of the subclass of \(k\)-uniformly starlike and convex functions by using higher order derivatives of Taylor series expansions of some \(p\)-valent functions with negative coefficients defined by integral operator. We have also obtained the necessary and sufficient condition for \(f(z)\) to be in the class \(k - UCV^m_p(\lambda, \gamma, \beta, \delta, q)\). Also we have obtained various geometric properties like convex combination, integral representation, Quasi-Hadamard product and extreme points.
We have also considered meromorphic $p$-valent functions by defining the functions in the form: $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n, (a_n \geq 0, p \in \mathbb{N})$ in the punctured unit disc $U^*$. We define the subclass $\delta^+(p, \beta, \alpha, \gamma, \lambda)$ and obtain the routine geometric properties.

Here, we have also defined the condition:

$$
\text{Re} \left\{ -\frac{z(I_p^\sigma(f \ast g)(z))'}{I_p^\sigma(f \ast g)(z)} \right\} > p\alpha, \quad z \in U
$$

for $0 \leq \alpha < 1, \sigma > 0$, where

$$
I_p^\sigma f(z) = \frac{1}{z^{p+1}} \Gamma(\sigma) \int_0^z \log(\frac{\tilde{z}}{t})^{\sigma-1} t^p f(t) dt, \quad p \in \mathbb{N}.
$$

We have studied some properties like coefficient inequalities, growth inequalities as well as closure theorems. Also we have applied some integral operators on the class $A^+_p(\alpha, \sigma)$.

In the Chapter 3, we have investigated the application of linear operator on meromorphically univalent functions involving differential subordination. By making use of Briot-Bouquet differential subordination, we have studied several inclusion relationships and other interesting properties and characteristics of the subclasses of meromorphically univalent functions in the punctured unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$, which are defined here by a linear operator. Convolution properties also we have studied.

We have also continued the study of research. Here, we have thought of application of subordination and Ruscheweyh derivative for $p$-valent functions with negative coefficients. We have defined the class $A_p(A, B, \alpha, \lambda)$ of functions $f \in D(\delta, \beta, \lambda, p)$ of the form:

$$
f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad (a_k \geq 0, \quad p \in \mathbb{N}),
$$
which are analytic and $p$-valent in the unit disk $U$ and satisfying:

$$
\frac{z(D^{\lambda+p-1}f(z))'}{D^{\lambda+p-1}f(z)} < \frac{p + (pB + (A - B)(p - \alpha))z}{1 + Bz}, \quad z \in U,
$$

where $-1 \leq B < A \leq 1, \quad 0 \leq \alpha < p, \lambda > -p$, and $D(\delta, \beta, \lambda, p)$ denote the class of functions $f$ satisfying the condition:

$$
\text{Re} \left\{ \frac{z(D^{\lambda+p-1}f(z))'}{D^{\lambda+p-1}f(z)} \right\} > \delta \left| \frac{z(D^{\lambda+p-1}f(z))'}{D^{\lambda+p-1}f(z)} - p \right| + \beta
$$

for $0 \leq \beta < p, \delta \geq 0, \lambda > -p, p \in \mathbb{N}$ and $z \in U$, where $D^{\lambda+p-1}f(z)$ is the Ruscheweyh derivative of $f$ of order $\lambda + p - 1$.

We have investigated the properties of the class $A_p(A, B, \alpha, \lambda)$ of $p$-valent functions with negative coefficients defined by Ruscheweyh derivative and subordination. We obtain coefficient bounds, convolution properties for the class $A_p(A, B, \alpha, \lambda)$ under the assumption $-1 \leq B < 0$. Also, we have obtained the class preserving operator of the form:

$$
G_c(z) = \frac{c + p}{z^c} \int_0^z s^{c-1} f(s) ds, \quad c > -p \quad \text{for the class } A_p(A, B, \alpha, \lambda).
$$

In continuation of the study, we have studied the application of differential subordination for certain subclass of meromorphically $p$-valent functions with positive coefficients in the punctured unit disk $U^*$ of the form:

$$
f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k, \quad \text{for any } m \geq p, p \in \mathbb{N}, a_k \geq 0
$$

defined by linear operator satisfying the following subordination condition:

$$
-\frac{z^{p+1}(L^n f(z))'}{p} < \frac{1 + Az}{1 + Bz}, \quad n \in \mathbb{N}_0, \quad z \in U,
$$

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where we define the linear operator $L^n$ by

$$L^0 f(z) = f(z),$$

$$L^1 f(z) = z^{-p} + \sum_{k=m}^{\infty} (p+k+1)a_k z^k = \frac{(z^{p+1} f(z))'}{z^p}$$

and in general

$$L^n f(z) = L(L^{n-1} f(z)) = z^{-p} + \sum_{k=m}^{\infty} (p+k+1)^n a_k z^k = \frac{(z^{p+1} L^{n-1} f(z))'}{z^p}, \quad (n \in \mathbb{N}).$$

We obtain the coefficient bounds, $\delta$-neighbourhood and partial sums, integral representation, convex combination, arithmetic mean and convolution properties by using the above linear operator.

In Chapter 4, we have discussed the application of hypergeometric functions using fractional derivative operator, convolution operator and Dziok-Srivastava linear operator and properties of univalent and multivalent functions with negative coefficients by using fractional calculus techniques.

We have investigated the properties for multivalent functions using hypergeometric functions and fractional derivatives. Here we have defined $T_p$ the class of multivalent analytic functions $f(z)$ having Taylor series form:

$$f(z) = mz^p + \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} - \text{2F1}(a, b; c; z), \quad |z| < 1,$$

where $\text{2F1}(a, b; c; z)$ is Gaussian hypergeometric function defined by

$$\text{2F1}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1,$$

where $(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}$, $c > b > 0$, $c > a + b$.

$m > 0$ and $t_{n-p+1} = \frac{(a, n-p+1)(b, n-p+1)}{(c, n-p+1)(n-p+1)!}$.
We have obtained coefficient bounds, radii of starlikeness and convexity, distortion bounds, integral representation, extreme points, convex combination and arithmetic mean. Also we have applied some integral operators on the class $T^\delta_p(\alpha, \beta)$ if $f(z)$ satisfies the inequality:

$$
\Re\left\{ \frac{z(Q^\delta_z f(z))'}{Q^\delta_z f(z)} \right\} - \alpha > \beta \left| \frac{z(Q^\delta_z f(z))'}{Q^\delta_z f(z)} - p \right|
$$

For $0 \leq \alpha < p, \beta \geq 0, 0 \leq \delta < 1, p \in \mathbb{N} = \{1, 2, \cdots\}$ and $z \in U$, where $Q^\delta_z f(z) = mz^p - \sum_{n=p+1}^{\infty} \frac{\Gamma(2p-\delta)\Gamma(n+p)}{\Gamma(2p)\Gamma(n+p-\delta)} g_n z^n$ and

$$
g_n = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)}, \quad n \geq p+1.
$$

We have also introduced the class $A(a, b, c, \sigma, A, B, \beta)$ of the analytic and univalent functions with negative coefficients in the unit disk $U$ of the form:

$$
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (n \in \mathbb{N}, a_n \geq 0),
$$

defined by convolution operator in terms of hypergeometric function $H_{a,b,c}(f)(z)$ satisfying:

$$
\left| \frac{\sigma z(H_{a,b,c}(f)(z))' - \sigma(H_{a,b,c}(f)(z))}{(\sigma B + (A - B)\beta)(H_{a,b,c}(f)(z)) - B\sigma z(H_{a,b,c}(f)(z))'} \right| < 1,
$$

where $-1 \leq B < A \leq 1, 0 < \beta < 1, \sigma > 0$ and

$$
H_{a,b,c}(f)(z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n,
$$

where

$$
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad c > b > 0, \quad \text{and} \quad c > a + b.
$$

We study for this class some geometric properties. We also investigate some interesting properties of integral operator introduced here due to Bernardi:

$$
L_k(f(z)) = \frac{1 + k}{z^k} \int_{0}^{z} (f(t)) t^{k-1} dt, \quad (k > -1).
$$

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We have also defined a new class of $p$-valent functions defined by Dziok-Srivastava linear operator

$$DS_p^{m,k}(f)(z) = z^p - \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_m)_n a_{n+p} z^{n+p}}{n!}.$$  

We have defined the class

$$W_p(m,k,A,\alpha,\beta) = \left\{ f(z) \in W_p : \left| \frac{z(DS_p^{m,k}(f)(z))'}{DS_p^{m,k}(f)(z)} - p \right| < \beta, \right\}$$

$$(-1 \leq A < 1, 0 < \beta \leq 1, 0 \leq \alpha < p, z \in U),$$

where $W_p$ is the class of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad (a_{n+p} \geq 0, p \in \mathbb{N}),$$

which are analytic and $p$-valent with negative coefficients in the unit disk $U$.

Here, we have obtained coefficient estimates, distortion bounds and closure theorem. Also we show the class $W_p(m,k,A,\alpha,\beta)$ is closed under convolution.

We have also applied the fractional calculus techniques for the subclass $WA(n,p,\beta,\gamma,\lambda)$. By using the definitions of fractional derivative and integration, we have obtained some theorems leading to distortion theorems. We have also studied neighbourhood properties.

We have established that $WA(n,p,\beta,\gamma,\lambda)$ satisfies the inequality:

$$\text{Re} \left\{ \frac{D^{\lambda+p-1} f(z)}{(1-\gamma)z^2(D^{\lambda+p-1} f(z)))'' + \gamma z(D^{\lambda+p-1} f(z))'} + D^{\lambda+p-1} f(z) \right\} > \beta, \quad z \in U, \quad 0 \leq \gamma < 1/2, \quad 0 \leq \beta < \frac{1}{(1-\gamma)(p-1)+\gamma p+1}, \quad \lambda > -p, \quad p \in \mathbb{N}.$$  

Lastly, in the Chapter 4, we have defined a class $G(\alpha,\beta,b)$, $(\alpha \geq 0, -1 \leq \beta \leq 0, b \in \mathbb{C})$ which satisfies the condition:

$$\text{Re} \left\{ \frac{\beta f(z)}{z} + (1-\beta)f'(z) + \alpha zf''(z) \right\} > 1 - |b|$$

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for some $\alpha \geq 0, -1 \leq \beta \leq 0$ and $b \in \mathbb{C}$.

A necessary and sufficient condition for functions to be in the class $G(\alpha, \beta, b)$ is obtained. Also for this class, we get some interesting geometric properties.

Furthermore, we have given an application involving fractional calculus for functions in the class $G(\alpha, \beta, b)$.

The last Chapter 5 has been fully dealt with the study of harmonic univalent functions, Sălăgean-type multivalent harmonic functions, harmonic univalent functions defined by Ruscheweyh derivatives and also involving a generalized Ruscheweyh type operator. Varieties of different geometrical properties have been investigated, in this Chapter.

We note that while getting the geometrical results pertaining to different classes and conditions, we have tried our level best for obtaining the sharp results.