CHAPTER 4

FUZZY GENERALIZED SEMI GENERALIZED CLOSED SETS

4.1 INTRODUCTION

The fuzzy concept has invaded almost all branches of mathematics ever since the introduction of fuzzy set by Zadeh (1965). The theory of fuzzy topological space was introduced and developed by Chang (1968) and since then various notions in classical topology have been extended to fuzzy topological space. The extensions of functions in fuzzy setting can very interestingly and effectively be carried out by the concept of quasi-coincidence and quasi-neighborhoods introduced by Pu and Liu (1980).

The concept of generalized semi generalized closed set in topological space was studied by Lellis Thivagar et al (2011). The purpose of this chapter is to study the structures of generalized semi generalized closed set and its characterizations in fuzzy setting. Fuzzy $g_{sg}$-closure and fuzzy $g_{sg}$-interior with its properties are discussed. As an application this set fuzzy $T_{g_{sg}}$-space, $F_{g_{sg}}$-continuity and $F_{g_{sg}}$-irresolute mappings are introduced.

Throughout this chapter, $(X, \tau)$ represents the fuzzy topological space with the topology $\tau$. The members of the fuzzy topology are called fuzzy open sets and their complements are known as fuzzy closed sets.
Some basic definitions required for this chapter are already given in chapter 1. Here are a few more definitions and results which are to be used in this chapter.

**Lemma 4.1.1 (Murugasen & Thangavelu 2008)** Let \( \lambda \) be a fuzzy set in a fuzzy topological space \((X, \tau)\). Then

\[
(i) \quad spCl(\lambda) \leq sCl(\lambda) \leq \alpha Cl(\lambda) \leq Cl(\lambda)
\]

\[
(ii) \quad spCl(\lambda) \leq pCl(\lambda) \leq \alpha Cl(\lambda) \leq Cl(\lambda)
\]

**Definition 4.1.1** A fuzzy topological space \((X, \tau)\) is called a **fuzzy \textit{T}_\omega\text{-space}** if every \(F_\omega\)-closed set in it is fuzzy closed.

**Definition 4.1.2** A fuzzy topological space \((X, \tau)\) is called a **fuzzy \textit{T}_b\text{-space}** if every \(F_{g_2}\)-closed set in it is fuzzy closed.

### 4.2 Fuzzy \textit{gsg}-Closed Sets and Fuzzy \textit{gsg}-Open Sets

This section introduce the concept of fuzzy generalized semi generalized closed set in fuzzy topological space and discusses the comparative study of fuzzy \textit{gsg}-closed sets with other types of fuzzy closed sets that already exists in literature, by giving counter examples whenever necessary.

**Definition 4.2.1** A fuzzy set \( \lambda \) of \((X, \tau)\) is called a **fuzzy generalized semi generalized closed** set (in short, \(F_{gsg}\)-closed) if \(Cl(\lambda) \leq \mu \) whenever \( \lambda \leq \mu \) and \( \mu \) is \(F_{g_2}\)-open in \((X, \tau)\).

**Proposition 4.2.1** In a fuzzy topological space \((X, \tau)\), every fuzzy closed set is \(F_{gsg}\)-closed.
Proof: Let $\lambda$ be fuzzy closed set and $\mu$ be any $F_{sg}$-open set such that $\lambda \leq \mu$. Since $\lambda$ is fuzzy closed, $Cl(\lambda) = \lambda \leq \mu$. Hence $\lambda$ is $F_{gsg}$-closed.

Proposition 4.2.2 In a fuzzy topological space $(X, \tau)$, every $F_{gsg}$-closed set is $F_{g}$-closed.

Proof: Let $\lambda$ be any $F_{gsg}$-closed set and $\mu$ be any fuzzy open set such that $\lambda \leq \mu$. Since every fuzzy open set is $F_{sg}$-open and $\lambda$ is $F_{gsg}$-closed, $Cl(\lambda) \leq \mu$. Hence $\lambda$ is $F_{g}$-closed.

Proposition 4.2.3 In a fuzzy topological space $(X, \tau)$, every $F_{gsg}$-closed set is $F_{o}$-closed.

Proof: Let $\lambda$ be any $F_{gsg}$-closed set and $\mu$ be any $F_{s}$-open set such that $\lambda \leq \mu$. Since every fuzzy semi open set is $F_{sg}$-open and $\lambda$ is $F_{gsg}$-closed, $Cl(\lambda) \leq \mu$. Hence $\lambda$ is $F_{o}$-closed.

Proposition 4.2.4 In a fuzzy topological space $(X, \tau)$, every $F_{gsg}$-closed set is $F_{g\alpha}$-closed.

Proof: Let $\lambda$ be any $F_{gsg}$-closed set and $\mu$ be any fuzzy open set such that $\lambda \leq \mu$. Since every fuzzy open set is $F_{sg}$-open and $\lambda$ is $F_{gsg}$-closed, $\alpha Cl(\lambda) \leq Cl(\lambda) \leq H$. Hence $\lambda$ is $F_{g\alpha}$-closed.

Proposition 4.2.5 In a fuzzy topological space $(X, \tau)$, every $F_{gsg}$-closed set is $F_{\alpha g}$-closed.

Proof: Let $\lambda$ be any $F_{gsg}$-closed set and $\mu$ be any $F_{\alpha}$-open set such that $\lambda \leq \mu$. Since every $F_{\alpha}$-open set is fuzzy semi open set which is $F_{sg}$-open and $\lambda$ is $F_{gsg}$-closed, $\alpha Cl(\lambda) \leq Cl(\lambda) \leq \mu$. Hence $\lambda$ is $F_{\alpha g}$-closed.
Proposition 4.2.6 In fuzzy topological space $(X, \tau)$, every $F_{gsg}$-closed set is $F_{sg}$-closed and $F_{sp}$-closed.

Proof: Let $\lambda$ be any $F_{gsg}$-closed set and $\mu$ be any $F_{s}$-open set such that $\lambda \leq \mu$. Since every fuzzy semi open set is $F_{sg}$-open and $\lambda$ is $F_{gsg}$-closed, $sCl(\lambda) \leq Cl(\lambda) \leq \mu$. Hence $\lambda$ is $F_{sg}$-closed. Since every $F_{sg}$-closed set is $F_{sp}$-closed, $\lambda$ is $F_{sp}$-closed.

Proposition 4.2.7 In a fuzzy topological space $(X, \tau)$, every $F_{gsg}$-closed set is $F_{gs}$-closed, $F_{gsp}$-closed and $F_{gp}$-closed.

Proof: Let $\lambda$ be any $F_{gsg}$-closed set and $\mu$ be any fuzzy open set such that $\lambda \leq \mu$. Since every fuzzy open set is $F_{sg}$-open and $\lambda$ is $F_{gsg}$-closed, $sCl(\lambda) \leq Cl(\lambda) \leq \mu$. Hence $\lambda$ is $F_{gs}$-closed.

Similarly, $spCl(\lambda) \leq Cl(\lambda) \leq \mu$ and $pCl(\lambda) \leq Cl(\lambda) \leq \mu$. Thus $\lambda$ is $F_{gsp}$-closed and $F_{gp}$-closed. Hence the proposition.

The pictorial representation of the above discussion is expressed in Figure 4.1.

![Figure 4.1](image-url)

**Figure 4.1** Comparison of $F_{gsg}$-closed set with existing fuzzy closed sets in fuzzy topological spaces
The reverse implication in the above propositions is not true as shown in the following examples.

**Example 4.2.1** Let $X = \{x, y, z\}$ and the fuzzy sets $\lambda_1, \lambda_2$ and $\lambda_3$ from $X$ to $[0,1]$ be defined as

\[
\begin{align*}
\lambda_1(x) &= 0.0, \lambda_1(y) = 0.0, \lambda_1(z) = 0.4; \\
\lambda_2(x) &= 0.9, \lambda_2(y) = 0.6, \lambda_2(z) = 0.0; \\
\lambda_3(x) &= 1.0, \lambda_3(y) = 0.7, \lambda_3(z) = 1.0.
\end{align*}
\]

Let $\tau = \{0, \lambda_1, \lambda_2, \lambda_1 \lor \lambda_2, 1\}$. Clearly $(X, \tau)$ is a fuzzy topological space. Then the set $\lambda_3$ is $F_{gss}$-closed but not fuzzy closed in $(X, \tau)$.

**Example 4.2.2** Let $X = \{x, y, z\}$ and the fuzzy sets $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_5$ from $X$ to $[0,1]$ be defined as

\[
\begin{align*}
\lambda_1(x) &= 0.7, \lambda_1(y) = 0.3, \lambda_1(z) = 1.0; \\
\lambda_2(x) &= 0.7, \lambda_2(y) = 0.0, \lambda_2(z) = 0.0; \\
\lambda_3(x) &= 0.9, \lambda_3(y) = 0.2, \lambda_3(z) = 0.1; \\
\lambda_4(x) &= 0.2, \lambda_4(y) = 0.7, \lambda_4(z) = 0.0; \\
\lambda_5(x) &= 0.2, \lambda_5(y) = 0.7, \lambda_5(z) = 0.2.
\end{align*}
\]

Let $\tau = \{0, \lambda_1, \lambda_2, 1\}$. Clearly $(X, \tau)$ is a fuzzy topological space.
Then

- The fuzzy set $\lambda_3$ is $F_g$-closed and hence $F_{ga}$-closed, $F_{gp}$-closed, $F_{gs}$-closed and $F_{gsp}$-closed, but not $F_{gsg}$-closed in $(X, \tau)$.
- The fuzzy set $\lambda_4$ is $F_\omega$-closed but not $F_{gsg}$-closed in $(X, \tau)$.
- The fuzzy set $\lambda_5$ is $F_{dg}$-closed and hence $F_{sg}$-closed and $F_{sp'}$-closed but not $F_{gsg}$-closed in $(X, \tau)$.

**Definition 4.2.2** A fuzzy set $\lambda$ of a fuzzy topological space $(X, \tau)$ is called $F_{gsg}$-open set if and only if $1 - \lambda$ is $F_{gsg}$-closed.

**Proposition 4.2.8** In a fuzzy topological space $(X, \tau)$, every fuzzy open set is $F_{gsg}$-open.

**Proof:** Let $\lambda$ be any fuzzy open set. Let $\mu$ be any $F_{sg}$-open set such that $1 - \lambda \leq \mu$. Since $\lambda$ is fuzzy open, $Cl(1 - \lambda) = 1 - Int(\lambda) = 1 - \lambda \leq \mu$.

This gives $1 - \lambda$ is $F_{gsg}$-closed and hence $\lambda$ is $F_{gsg}$-open.

**Proposition 4.2.9** In a fuzzy topological space $(X, \tau)$,

i. Every $F_{gsg}$-open set is $F_g$-open and $F_\omega$-open.

ii. Every $F_{gsg}$-open set is $F_{gs}$-open, $F_{sg}$-open, $F_{sp}$-open, $F_{gsp}$-open, $F_{g\alpha}$-open and $F_{\alpha g}$-open.

**Proof:** Obvious.
4.3 CHARACTERIZATION OF $F_{gssg}$-CLOSED SETS AND $F_{gssg}$-OPEN SETS

In this section, some characterizations of $F_{gssg}$-closed and $F_{gssg}$-open sets in fuzzy topological space are studied.

**Definition 4.3.1** A fuzzy set $\lambda$ in a fuzzy topological space $(X, \tau)$ is called $F_{gssg}$-nhd of a fuzzy point $x_\kappa$ if there exists a $F_{gssg}$-open set $\mu$ such that $x_\kappa \in \mu \leq \lambda$.

**Definition 4.3.2** A $F_{gssg}$-nhd $\lambda$ in a fuzzy topological space $(X, \tau)$ is said to be $F_{gssg}$-open-nhd (resp. $F_{gssg}$-closed-nhd) if and only if $\lambda$ is $F_{gssg}$-open (resp. $F_{gssg}$-closed).

**Definition 4.3.3** A fuzzy set $\lambda$ in a fuzzy topological space $(X, \tau)$ is called fuzzy $gssg$-$q$-nhd of a fuzzy point $x_\kappa$ (resp. fuzzy set $\mu$), if there exists a $F_{gssg}$-open set $\delta$ in $(X, \tau)$ such that $x_\kappa q\delta \leq \lambda$ (resp. $\mu q\delta \leq \lambda$).

**Theorem 4.3.1** If $\lambda$ and $\mu$ are $F_{gssg}$-closed sets in a fuzzy topological space $(X, \tau)$ then $\lambda \lor \mu$ is $F_{gssg}$-closed.

**Proof**: Let $\lambda$ and $\mu$ be two fuzzy $F_{gssg}$-closed sets in $(X, \tau)$ and let $\delta$ be any $F_{sg}$-open set such that $\lambda \leq \delta$ and $\mu \leq \delta$. Therefore $Cl(\lambda) \leq \delta$ and $Cl(\mu) \leq \delta$. Since $\lambda \leq \delta$ and $\mu \leq \delta$, $\lambda \lor \mu \leq \delta$. Now $Cl(\lambda \lor \mu) = Cl(\lambda) \lor Cl(\mu) \leq \delta$. Hence $\lambda \lor \mu$ is $F_{gssg}$-closed.

**Theorem 4.3.2** If $\lambda$ and $\mu$ are $F_{gssg}$-open sets in a fuzzy topological space $(X, \tau)$ then $\lambda \land \mu$ is $F_{gssg}$-open.
Proof: Let $\lambda$ and $\mu$ be two fuzzy $F_{gsg}$-open sets in $(X, \tau)$. Then $1 - \lambda$ and $1 - \mu$ are $F_{gsg}$-closed. By theorem 4.3.1, $(1 - \lambda) \lor (1 - \mu)$ is $F_{gsg}$-closed. Since $(1 - \lambda) \lor (1 - \mu) = 1 - (\lambda \land \mu)$, $1 - (\lambda \land \mu)$ is $F_{gsg}$-closed. Hence $\lambda \land \mu$ is $F_{gsg}$-open.

**Theorem 4.3.3** If a fuzzy set $\lambda$ is $F_{gsg}$-closed in a fuzzy topological space $(X, \tau)$ and $Cl(\lambda) \land (1 - Cl(\lambda)) = 0$ then $Cl(\lambda) - \lambda$ does not contain any non-zero $F_{sg}$-closed set in $(X, \tau)$.

Proof: Let $\lambda$ be $F_{gsg}$-closed in $(X, \tau)$ and $Cl(\lambda) \land (1 - Cl(\lambda)) = 0$. The result is proved by contradiction. Let $\mu$ be any $F_{sg}$-closed set in $(X, \tau)$ such that $\mu \leq Cl(\lambda) - \lambda$ and $\mu \neq 0$. This gives $\mu \leq Cl(\lambda)$ and $\mu \leq 1 - \lambda$. Therefore $\lambda \leq 1 - \mu$, which is $F_{sg}$-open. Since $\lambda$ is $F_{gsg}$-closed, $Cl(\lambda) \leq 1 - \mu$. This implies $\mu \leq 1 - Cl(\lambda)$. Therefore $\mu \leq Cl(\lambda) \land 1 - Cl(\lambda) = 0$. That is, $\mu = 0$, which is a contradiction. Hence $Cl(\lambda) - \lambda$ does not contain any non-zero $F_{sg}$-closed set in $(X, \tau)$.

**Theorem 4.3.4** If a fuzzy set $\lambda$ is $F_{gsg}$-closed in a fuzzy topological space $(X, \tau)$ and $Cl(\lambda) \land (1 - Cl(\lambda)) = 0$ then $Cl(\lambda) - \lambda$ does not contain any non-zero fuzzy closed set in $(X, \tau)$.

Proof: It follows from the theorem 4.3.3 and the fact that every fuzzy closed set is $F_{sg}$-closed.

**Theorem 4.3.5** If $\lambda$ is $F_{sg}$-open and $F_{gsg}$-closed in a fuzzy topological space $(X, \tau)$ then $\lambda$ is fuzzy closed in $(X, \tau)$.

Proof: Since $\lambda \leq \lambda$ and $\lambda$ is $F_{sg}$-open and $F_{gsg}$-closed, $Cl(\lambda) \leq \lambda$. But $\lambda \leq Cl(\lambda)$, then $\lambda = Cl(\lambda)$. Hence $\lambda$ is fuzzy closed.
Theorem 4.3.6 A fuzzy $\lambda$ of a fuzzy topological space $(X, \tau)$ is $F_{gs_\delta}$-closed if and only if $\lambda M \Rightarrow Cl(\lambda M)$, for every $F_{gs_\delta}$-closed set $\mu$ of $(X, \tau)$.

Proof: Let $\mu$ be $F_{sg}$-closed set and $\lambda M \mu$. Then $\lambda \leq 1- \mu$. Since $\lambda$ is $F_{gs_\delta}$-closed and $1- \mu$ is $F_{sg}$-open, $Cl(\lambda) \leq 1- \mu$. Hence $Cl(\lambda M) \mu$.

Conversely, let $\delta$ be $F_{sg}$-open set such that $\lambda \leq \delta$. By hypothesis $\lambda M (1- \delta) \Rightarrow Cl(\lambda M) (1- \delta)$, as $1- \delta$ is $F_{sg}$-closed. Then $Cl(\lambda) \leq \delta$. Hence $\lambda$ is $F_{gs_\delta}$-closed.

Theorem 4.3.7 If $\lambda$ be a $F_{gs_\delta}$-closed set in $(X, \tau)$ and $x_\kappa$ be a fuzzy point of $X$ such that $x_\kappa q Cl(\lambda)$ then $Cl(x_\kappa) q \lambda$.

Proof: Suppose $Cl(x_\kappa) q \lambda$ then $\lambda \leq 1- Cl(x_\kappa)$. Since $1- Cl(x_\kappa)$ is $F_{sg}$-open and $\lambda$ is $F_{gs_\delta}$-closed, $Cl(\lambda) \leq 1- Cl(x_\kappa) = 1- x_\kappa$. This gives $x_\kappa q Cl(\lambda)$, a contradiction. Hence $Cl(x_\kappa) q \lambda$.

Theorem 4.3.8 If $\lambda$ is $F_{gs_\delta}$-closed set in a fuzzy topological space $(X, \tau)$ and $\lambda \leq \mu \leq Cl(\lambda)$ then $\mu$ is $F_{gs_\delta}$-closed in $(X, \tau)$.

Proof: Let $\delta$ be $F_{sg}$-open set in a fuzzy topological space $(X, \tau)$ such that $\mu \leq \delta$. Since $\lambda \leq \mu$, $\lambda \leq \delta$. Since $\lambda$ is $F_{gs_\delta}$-closed set, $Cl(\lambda) \leq \delta$. But $\mu \leq Cl(\lambda)$ implies $Cl(\mu) \leq Cl(Cl(\lambda)) = Cl(\lambda) \leq \delta$. Hence $\mu$ is $F_{gs_\delta}$-closed.

Theorem 4.3.9 If $\lambda$ is $F_{gs_\delta}$-open set in a fuzzy topological space $(X, \tau)$ and $Int(\lambda) \leq \mu \leq \lambda$, then $\mu$ is $F_{gs_\delta}$-open in $(X, \tau)$.
Proof: Let $\lambda$ is $F_{gsg}$-open set in $(X, \tau)$ and $\text{Int}(\lambda) \leq \mu \leq \lambda$. Then $1- \lambda$ is $F_{gsg}$-closed and $1- \lambda \leq 1- \mu \leq \text{Cl}(1- \lambda)$. Then, by the theorem 4.3.8, $1- \mu$ is $F_{gsg}$-closed. Hence $\mu$ is $F_{gsg}$-open.

Theorem 4.3.10. A fuzzy set $\lambda$ is $F_{gsg}$-open in a fuzzy topological space $(X, \tau)$ if and only if $\mu \leq \text{Int}(\lambda)$ where $\mu$ is $F_{gsg}$-closed and $\mu \leq \lambda$ in $(X, \tau)$.

Proof: Let $\mu \leq \text{Int}(\lambda)$ where $\mu$ is $F_{gsg}$-closed and $\mu \leq \lambda$. Then $1- \lambda \leq 1- \mu$ and $1- \mu$ is $F_{sg}$-open.

Now $\text{Cl}(1- \lambda) = 1- \text{Int}(\lambda) \leq 1- \mu$, by hypothesis. Then $1- \lambda$ is $F_{gsg}$-closed. Hence $\lambda$ is $F_{gsg}$-open.

Conversely, let $\lambda$ is $F_{gsg}$-open and $\mu$ is $F_{sg}$-closed and $\mu \leq \lambda$. Then $1- \lambda \leq 1- \mu$. Since $1- \lambda$ is $F_{gsg}$-closed and $1- \mu$ is $F_{sg}$-open, $\text{Cl}(1- \lambda) \leq 1- \mu$. Then $\mu \leq \text{Int}(\lambda)$.

4.4 $F_{gsg}$-CLOSURE AND $F_{gsg}$-INTERIOR

In this section, $F_{gsg}$-Closure and $F_{gsg}$-Interior of a fuzzy set in fuzzy topological space are introduced and some properties of these fuzzy topological structures are studied in fuzzy topological space.

Definition 4.4.1 Let $\lambda$ be any fuzzy set in $(X, \tau)$ then $F_{gsg}$-Closure and $F_{gsg}$-Interior are defined as,

\[
g_{gsg}\text{-Cl}(\lambda) = \land \{\mu : \mu \text{ is } F_{gsg}\text{-closed and } \mu \geq \lambda\},
\]

\[
g_{gsg}\text{-Int}(\lambda) = \lor \{\mu : \mu \text{ is } F_{gsg}\text{-open and } \mu \leq \lambda\}.
\]
It is evident that

i. \( gsg-Cl(\lambda) = \lambda \) if and only if \( \lambda \) is \( F_{gsg} \)-closed.

ii. \( gsg-Int(\lambda) = \lambda \) if and only if \( \lambda \) is \( F_{gsg} \)-open.

iii. \( gsg-Cl(\lambda) \) is the smallest fuzzy set containing \( \lambda \).

iv. \( gsg-Int(\lambda) \) is the largest fuzzy set contained in \( \lambda \).

**Theorem 4.4.1** Let \( x_\kappa \) and \( \lambda \) be a fuzzy point and fuzzy set respectively in a fuzzy topological space \((X, \tau)\). Then \( x_\kappa \in gsg-Cl(\lambda) \) if and only if every fuzzy \( gsg-q \)-nhd of \( x_\kappa \) is \( q \)-coincident with \( \lambda \).

**Proof:** The result is proved by the method of contradiction. Let \( x_\kappa \in gsg-Cl(\lambda) \). Suppose there exists a fuzzy \( gsg-q \)-nhd \( \mu \) of \( x_\kappa \) such that \( \mu \bar{\lambda} \). Since \( \mu \) is \( gsg-q \)-nhd of \( x_\kappa \) there exists \( F_{gsg} \)-open set \( \delta \) in \((X, \tau)\) such that \( x_\kappa \cdot q \delta \leq \mu \) which gives that \( \delta \bar{\lambda} \lambda \) and hence \( \lambda \leq 1-\delta \). Then \( gsg-Cl(\lambda) \leq 1-\delta \), as \( 1-\delta \) is \( F_{gsg} \)-closed. Since \( x_\kappa \notin 1-\delta \), \( x_\kappa \notin gsg-Cl(\lambda) \), a contradiction. Hence every fuzzy \( gsg-q \)-nhd of \( x_\kappa \) is \( q \)-coincident with \( \lambda \).

Conversely, suppose \( x_\kappa \notin gsg-Cl(\lambda) \). Then There exists a \( F_{gsg} \)-closed set \( \mu \) such that \( \lambda \leq \mu \) and \( x_\kappa \notin \mu \). Then \( x_\kappa \cdot q(1-\mu) \) and \( \lambda \bar{\mu} \), a contradiction. Hence \( x_\kappa \in gsg-Cl(\lambda) \).

**Proposition 4.4.1** Let \( \lambda \) be any fuzzy set in a fuzzy topological space \((X, \tau)\).

Then

i. \( gsg-Int(1-\lambda) = 1-(gsg-Cl(\lambda)) \).

ii. \( gsg-Cl(1-\lambda) = 1-(gsg-Int(\lambda)) \).
Proof: (i) By definition,

\[ gsg-Cl(\lambda) = \bigwedge \{ \mu : \mu \text{ is } F_{gsg}-\text{closed and } \mu \geq \lambda \} \]

\[ 1 - gsg-Cl(\lambda) = 1 - \bigwedge \{ \mu : \mu \text{ is } F_{gsg}-\text{closed and } \mu \geq \lambda \} \]

\[ = \bigvee \{ 1 - \mu : 1 - \mu \text{ is } F_{gsg}-\text{open and } 1 - \mu \leq 1 - \lambda \} \]

\[ = \bigvee \{ \delta : \delta \text{ is } F_{gsg}-\text{open and } \delta \leq 1 - \lambda \}, \]

where \( \delta = 1 - \mu \)

\[ = gsg-\text{Int}(1 - \lambda) \]

(ii) The proof is similar to that of (i)

Proposition 4.4.2 If \( \lambda \) and \( \mu \) are fuzzy sets in a fuzzy topological space \((X, \tau)\)

Then the following are true.

(i) \( gsg-Cl(0) = 0, gsg-Cl(1) = 1. \)

(ii) \( gsg-Cl(\lambda) \) is \( F_{gsg}-\text{closed in } (X, \tau). \)

(iii) \( gsg-Cl(\lambda) \leq gsg-Cl(\mu) \) when \( \lambda \leq \mu. \)

(iv) \( \delta q \lambda \) if and only if \( \delta q gsg-Cl(\lambda), \)

when \( \delta \) is \( F_{gsg}-\text{open set in } (X, \tau). \)

(v) \( gsg-Cl(\lambda) = gsg-Cl(gsg-Cl(\lambda)). \)

(vi) \( gsg-Cl(\lambda \land \mu) \leq gsg-Cl(\lambda) \land gsg-Cl(\mu). \)

(vii) \( gsg-Cl(\lambda \lor \mu) = gsg-Cl(\lambda) \lor gsg-Cl(\mu). \)
Proof:

(i) and (ii) are obvious.

(iii) Let \( x_\kappa \notin gsg-Cl(\mu) \). Then by the theorem 4.4.1, there exist fuzzy \( gsg-q \)-nhd \( \delta \) of \( x_\kappa \) such that \( \delta \overline{q}\mu \). Since \( \delta \) is fuzzy \( gsg-q \)-nhd of \( x_\kappa \) there exists fuzzy open set \( \lambda_1 \) such that \( x_\kappa \overline{q}\lambda_1 \leq \delta \). This gives \( \lambda_1 \overline{q}\mu \). Since \( \lambda \leq \mu \), then \( \lambda_1 \overline{q}\lambda \). Then by the theorem 4.4.1, \( x_\kappa \notin gsg-Cl(\lambda) \). Hence \( gsg-Cl(\lambda) \leq gsg-Cl(\mu) \).

(iv) Let \( \mu \) be any \( F_{gsg} \)-open set in \( (X, \tau) \). Suppose \( \mu \overline{q}\lambda \), then \( \lambda \leq 1 - \mu \). Since \( 1 - \mu \) is \( F_{gsg} \)-closed. By (iii), \( gsg-Cl(\lambda) \leq gsg-Cl(1 - \mu) = 1 - \mu \). Thus \( \mu \overline{q}gsg-Cl(\lambda) \).

Conversely, Let \( \mu \overline{q}gsg-Cl(\lambda) \). Then \( gsg-Cl(\lambda) \leq 1 - \mu \). Since \( \lambda \leq gsg-Cl(\lambda) \), \( \lambda \leq 1 - \mu \). Thus \( \mu \overline{q}\lambda \). Hence \( \mu \overline{q}\lambda \) if and only if \( \mu \overline{q}gsg-Cl(\lambda) \).

(v) Since \( gsg-Cl(\lambda) \leq gsg-Cl(gsg-Cl(\lambda)) \), it is enough to prove \( gsg-Cl(gsg-Cl(\lambda)) \leq gsg-Cl(\lambda) \). Let \( x_\kappa \notin gsg-Cl(\lambda) \). Then by the theorem 4.4.1, there exist fuzzy \( gsg-q \)-nhd \( \delta \) of \( x_\kappa \) such that \( \delta \overline{q}\lambda \) and so there is a \( F_{gsg} \)-open set \( \mu \) in \( (X, \tau) \) such that \( x_\kappa \overline{q}\mu \leq \delta \) and \( \mu \overline{q}\lambda \). By (iv), \( \mu \overline{q}gsg-Cl(\lambda) \). Then by theorem 4.4.1, \( x_\kappa \notin gsg-Cl(gsg-Cl(\lambda)) \). Hence \( gsg-Cl(\lambda) = gsg-Cl(gsg-Cl(\lambda)) \).

(vi) Since \( \lambda \wedge \mu \leq \lambda \) and \( \lambda \wedge \mu \leq \mu \), \( gsg-Cl(\lambda \wedge \mu) \leq gsg-Cl(\lambda) \) and \( gsg-Cl(\lambda \wedge \mu) \leq gsg-Cl(\mu) \). This implies \( gsg-Cl(\lambda \wedge \mu) \leq gsg-Cl(\lambda) Cl(\lambda) \wedge gsg-Cl(\mu) \).
(vii.) Since \( \lambda \leq \lambda \lor \mu \) and \( \mu \leq \lambda \lor \mu \), \( \text{gsg-Cl}(\lambda) \leq \text{gsg-Cl}(\lambda \lor \mu) \) and \( \text{gsg-Cl}(\mu) \leq \text{gsg-Cl}(\lambda \lor \mu) \). Then \( \text{gsg-Cl}(\lambda) \lor \text{gsg-Cl}(\mu) \leq \text{gsg-Cl}(\lambda \lor \mu) \). Let \( x_k \in \text{gsg-Cl}(\lambda \lor \mu) \). Then by theorem 4.4.1, there exist fuzzy \( \text{gsg-q-nhd} \) \( \delta \) of \( x_k \) such that \( \delta q(\lambda \lor \mu) \). By lemma 1.4.2.1, either \( \delta q\lambda \) or \( \delta q\mu \). Then by theorem 4.4.1, \( x_k \in \text{gsg-Cl}(\lambda) \) or \( x_k \in \text{gsg-Cl}(\mu) \). That is, \( x_k \in \text{gsg-Cl}(\lambda) \lor \text{gsg-Cl}(\mu) \). Hence \( \text{gsg-Cl}(\lambda \lor \mu) = \text{gsg-Cl}(\lambda) \lor \text{gsg-Cl}(\mu) \).

**Proposition 4.4.3** If \( \lambda \) and \( \mu \) are Fuzzy sets in \((X, \tau)\), Then the following are true.

(i) \( \text{gsg-Int}(0) = 0, \text{gsg-Int}(1) = 1 \).

(ii) \( \text{gsg-Int}(\lambda) \) is \( F_{\text{gsg}} \)-open in \((X, \tau)\).

(iii) \( \text{gsg-Int}(\lambda) \leq \text{gsg-Int}(\mu) \) when \( \lambda \leq \mu \).

(iv) \( \text{gsg-Int}(\lambda) = \text{gsg-Int}(\text{gsg-Int}(\lambda)) \).

(v) \( \text{gsg-Int}(\lambda \lor \mu) \geq \text{gsg-Int}(\lambda) \lor \text{gsg-Int}(\mu) \).

(vi) \( \text{gsg-Int}(\lambda \land \mu) = \text{gsg-Int}(\lambda) \land \text{gsg-Int}(\mu) \).

Proof: The proof is similar to that of proposition 4.4.2.

### 4.5 FUZZY \( T_{\text{gsg}} \)-SPACE

In this section, fuzzy \( T_{\text{gsg}} \)-space was introduced and this space was compared with already existing spaces in fuzzy topological space.
**Definition 4.5.1** A fuzzy topological space \((X, \tau)\) is called a **fuzzy** \(T_{g_{gs}}\)-space if every \(F_{g_{gs}}\)-closed set in it is fuzzy closed.

**Proposition 4.5.1** Every fuzzy \(T_{1/2}\)-space is fuzzy \(T_{g_{gs}}\)-space.

**Proof:** Let \((X, \tau)\) be a fuzzy \(T_{1/2}\)-space and let \(\lambda\) be \(F_{g_{gs}}\)-closed set in \((X, \tau)\). Then \(\lambda\) is \(F_g\)-closed, by proposition 4.2.2. Since \((X, \tau)\) is \(T_{1/2}\)-space, \(\lambda\) is fuzzy closed in \((X, \tau)\). Hence \((X, \tau)\) is fuzzy \(T_{g_{gs}}\)-space.

**Proposition 4.5.2** Every fuzzy \(T_{\omega}\)-space is a \(T_{gs_{gs}}\)-space.

**Proof:** Let \((X, \tau)\) be a fuzzy \(T_{\omega}\)-space and let \(\lambda\) be \(F_{g_{gs}}\)-closed set in \((X, \tau)\). Then \(\lambda\) is \(F_{\omega}\)-closed, by proposition 4.2.3. Since \((X, \tau)\) is \(T_{\omega}\)-space, \(\lambda\) is fuzzy closed in \((X, \tau)\). Hence \((X, \tau)\) is fuzzy \(T_{gs_{gs}}\)-space.

**Proposition 4.5.3** Every fuzzy \(T_{b}\)-space is a \(T_{gs_{gs}}\)-space.

**Proof:** Let \((X, \tau)\) be a fuzzy \(T_{b}\)-space and let \(\lambda\) be \(F_{g_{gs}}\)-closed set in \((X, \tau)\). Then \(\lambda\) is \(F_{gs}\)-closed, by proposition 4.2.7. Since \((X, \tau)\) is \(T_{b}\)-space, \(\lambda\) is fuzzy closed in \((X, \tau)\). Hence \((X, \tau)\) is fuzzy \(T_{gs_{gs}}\)-space.

The following example shows that the converses of the above propositions are not true.

**Example 4.5.1** Let \(X = \{x, y, z\}\) and the fuzzy sets \(\lambda_1\) and \(\lambda_2\) from \(X\) to \([0,1]\) be defined as

\[
\lambda_1(x) = 0.7, \lambda_1(y) = 0.3, \lambda_1(z) = 1.0; \\
\lambda_2(x) = 0.7, \lambda_2(y) = 0.0, \lambda_2(z) = 0.0.
\]
Let $\tau = \{0, \lambda_1, \lambda_2, 1\}$. Then the fuzzy topological space $(X, \tau)$ is $T_{gsg}$-space but not $T_{1/2}$-space, not $T_\omega$-space and not $T_b$-space.

4.6 $F_{gsg}$-CONTINUOUS AND $F_{gsg}$-IRRESOLUTE MAPPING

In this section, $F_{gsg}$ the continuous and $F_{gsg}$ irresolute mappings are defined by using $gsg$-closed set and some basic topological structures of these mappings are studied.

**Definition 4.6.1** Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called fuzzy $gsg$-continuous (in short, $F_{gsg}$-continuous) if $f^{-1}(\lambda)$ is $F_{gsg}$-closed in $(X, \tau)$ for every fuzzy closed set $\lambda$ of $(Y, \sigma)$.

**Definition 4.6.2** Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called fuzzy $gsg$-irresolute (in short, $F_{gsg}$-irresolute) if $f^{-1}(\lambda)$ is $F_{gsg}$-closed in $(X, \tau)$ for every $F_{gsg}$-closed set $\lambda$ of $(Y, \sigma)$.

**Theorem 4.6.1** Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces and $f : (X, \tau) \rightarrow (Y, \sigma)$ be $F_{gsg}$-continuous. Then $f$ is $F_{gsg}$-continuous.

**Proof:** Let $\lambda$ be a fuzzy closed set in $(Y, \sigma)$. Since $f$ is $F_{gsg}$-continuous, $f^{-1}(\lambda)$ is $F_{gsg}$-closed in $(X, \tau)$. By proposition 4.2.6, $f^{-1}(\lambda)$ is $F_{gsg}$-closed in $(X, \tau)$, Hence $f$ is $F_{gsg}$-continuous.

The converse of above theorem is not true as shown in the following example.

**Example 4.6.1** Let $X = \{x_1, y_1, z_1\}$, $Y = \{x_2, y_2, z_2\}$. Fuzzy sets $\lambda$ and $\mu$ are defined as
\[
\lambda(x_1) = 0.0, \lambda(y_1) = 0.3, \lambda(z_1) = 0.2;
\]
\[
\mu(x_2) = 0.7, \mu(y_2) = 0.6, \mu(z_2) = 0.4.
\]

Let \( \tau = \{0, \lambda, 1\} \) and \( \sigma = \{0, \mu, 1\} \). Then the mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(x_1) = x_2, f(y_1) = y_2, f(z_1) = z_2 \) is \( F_{sg} \)-continuous but not \( F_{gsg} \)-continuous.

**Theorem 4.6.2** Let \( (X, \tau) \) and \( (Y, \sigma) \) be two fuzzy topological spaces and \( f : (X, \tau) \rightarrow (Y, \sigma) \) be \( F_{gsg} \)-irresolute. Then \( f \) is \( F_{gsg} \)-continuous.

**Proof:** Let \( \lambda \) be a fuzzy closed set in \( (Y, \sigma) \). By proposition 4.2.1, \( \lambda \) is \( F_{gsg} \)-closed in \( (Y, \sigma) \). Since \( f \) is \( F_{gsg} \)-irresolute, \( f^{-1}(\lambda) \) is \( F_{gsg} \)-closed in \( (X, \tau) \). Hence \( f \) is \( F_{gsg} \)-continuous.

The converse of above theorem is not true as shown in the following example.

**Example 4.6.2** Let \( X = \{x_1, y_1\}, Y = \{x_2, y_2\} \). Fuzzy sets \( \lambda \) is defined as

\[
\lambda(x_1) = 0.4, \lambda(y_1) = 0.6.
\]

Let \( \tau = \{0, \lambda, 1\} \) and \( \sigma = \{0, 1\} \). Then the mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(x_1) = x_2, f(y_1) = y_2 \) is \( F_{gsg} \)-continuous but not \( F_{gsg} \)-irresolute.

**Theorem 4.6.3** Let \( (X, \tau) \) and \( (Y, \sigma) \) be two fuzzy topological spaces and \( f : (X, \tau) \rightarrow (Y, \sigma) \) be \( F_{gsg} \)-continuous. Then \( f \) is \( F_{gsp} \)-continuous.

**Proof:** Let \( \lambda \) be a fuzzy closed set in \( (Y, \sigma) \). Since \( f \) is \( gsg \)-continuous, \( f^{-1}(\lambda) \) is \( F_{gsg} \)-closed in \( (X, \tau) \). By proposition 4.2.7, \( f^{-1}(\lambda) \) is \( F_{gsp} \)-closed in \( (X, \tau) \). Hence \( f \) is \( F_{gsp} \)-continuous.
The converse of above theorem is not true as shown in the following example.

**Example 4.6.3** Let \( X = \{x_1, y_1\}, Y = \{x_2, y_2\} \). Fuzzy sets \( \lambda \) and \( \mu \) are defined as

\[
\lambda(x_1) = 0.3, \lambda(y_1) = 0.7;
\mu(x_2) = 0.3, \mu(y_2) = 0.4.
\]

Let \( \tau = \{0, \lambda, 1\} \) and \( \sigma = \{0, \mu, 1\} \). Define the mapping \( f : (X, \tau) \to (Y, \sigma) \) by \( f(x_1) = x_2, f(y_1) = y_2 \). Then \( f \) is \( F_{gsp} \)-continuous but not \( F_{gsg} \)-continuous.

**Theorem 4.6.4** Let \((X, \tau)\) and \((Y, \sigma)\) be two fuzzy topological spaces and \( f : (X, \tau) \to (Y, \sigma) \) be \( F_{gsg} \)-continuous if and only if inverse image of each fuzzy open set of \((Y, \sigma)\) is \( F_{gsg} \)-open in \((X, \tau)\).

**Proof:** Let \( f \) be \( F_{gsg} \)-continuous. If \( \lambda \) is any fuzzy open set in \((Y, \sigma)\) then \( f^{-1}(1 - \lambda) = 1 - f^{-1}(\lambda) \) is \( F_{gsg} \)-closed in \((X, \tau)\). Hence \( f^{-1}(\lambda) \) is \( F_{gsg} \)-open in \((X, \tau)\).

Conversely, Let \( \lambda \) be a fuzzy closed set in \((Y, \sigma)\). By hypothesis, \( f^{-1}(1 - \lambda) \) is \( F_{gsg} \)-open in \((X, \tau)\). This gives \( f^{-1}(\lambda) \) is \( F_{gsg} \)-closed. Hence \( f \) is \( F_{gsg} \)-continuous.

**Theorem 4.6.5** Let \((X, \tau)\) and \((Y, \sigma)\) be two fuzzy topological spaces. If \( f : (X, \tau) \to (Y, \sigma) \) is \( F_{gsg} \)-continuous then for each fuzzy point \( x_\kappa \) of \( X \) and \( \lambda \in \sigma \) such that \( f(x_\kappa) \in \lambda \), there exists a \( F_{gsg} \)-open set \( \mu \) of \( X \) such that \( x_\kappa \in \mu \) and \( f(\mu) \leq \lambda \).
**Proof:** Let \( x_\kappa \) be a fuzzy point of \( X \) and \( \lambda \in \sigma \) such that \( f(x_\kappa) \in \lambda \). Take \( \mu = f^{-1}(\lambda) \). Since \( 1 - \lambda \) is fuzzy closed in \( (Y, \sigma) \) and \( f \) is \( F_{gsg} \)-continuous, \( f^{-1}(1 - \lambda) = 1 - f^{-1}(\lambda) \) is \( F_{gsg} \)-closed in \( (X, \tau) \). This gives \( \mu = f^{-1}(\lambda) \) is \( F_{gsg} \)-open in \( (X, \tau) \) and \( x_\kappa \in \mu \) and \( f(\mu) = f(f^{-1}(\lambda)) \leq \lambda \).

**Theorem 4.6.6** Let \( (X, \tau) \) and \( (Y, \sigma) \) be two fuzzy topological spaces. If \( f : (X, \tau) \to (Y, \sigma) \) is \( F_{gsg} \)-continuous then for each fuzzy point \( x_\kappa \) of \( (X, \tau) \) and \( \lambda \in \sigma \) such that \( f(x_\kappa)q\lambda \), there exists a \( F_{gsg} \)-open set \( \mu \) of \( (X, \tau) \) such that \( x_\kappa q\mu \) and \( f(\mu) \leq \lambda \).

**Proof:** Let \( x_\kappa \) be a fuzzy point of \( (X, \tau) \) and \( \lambda \in \sigma \) such that \( f(x_\kappa)q\lambda \). Take \( \mu = f^{-1}(\lambda) \). By the theorem 4.6.5, \( \mu \) is \( F_{gsg} \)-open in \( (X, \tau) \) and \( x_\kappa q\mu \) and \( f(\mu) = f(f^{-1}(\lambda)) \leq \lambda \).

**Theorem 4.6.7** Let \( (X, \tau) \), \( (Y, \sigma) \) and \( (Z, \eta) \) be fuzzy topological spaces. If \( f : (X, \tau) \to (Y, \sigma) \) is \( F_{gsg} \)-continuous and \( g : (Y, \sigma) \to (Z, \eta) \) is \( F_g \)-continuous and \( (Y, \sigma) \) is a fuzzy \( T_{1/2} \)-space. Then \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( F_{gsg} \)-continuous.

**Proof:** Let \( \lambda \) be a fuzzy closed set in \( (Z, \eta) \). Since \( g \) is \( F_g \)-continuous and \( (Y, \sigma) \) is a fuzzy \( T_{1/2} \)-space, \( g^{-1}(\lambda) \) is fuzzy closed in \( (Y, \sigma) \). Since \( f \) is \( F_{gsg} \)-continuous, \( (g \circ f)^{-1}(\lambda) = f^{-1}(g^{-1}(\lambda)) \) is \( F_{gsg} \)-closed in \( (X, \tau) \). Hence \( g \circ f \) is \( F_{gsg} \)-continuous.

### 4.7 CONCLUSION

In this chapter, the fuzzy structures of generalized semi generalized closed sets and its characterizations are studied in fuzzy topological spaces. Fuzzy \( gsg \)-closure and fuzzy \( gsg \)-interior are introduced and its properties are discussed. As an application this set fuzzy \( T'_{gsg} \)-space, \( F_{gsg} \)-continuity and \( F_{gsg} \)-irresolute mappings are studied.