CHAPTER 3

GENERALIZED SEMI GENERALIZED CLOSED SETS IN BITOPOLITICAL SPACES

3.1 INTRODUCTION

A triple $(X, \tau_1, \tau_2)$, where $X$ is a nonempty set and $\tau_1$ and $\tau_2$ are topologies on $X$ is called a bitopological space and Kelly (1963) initiated the systematic study of such spaces. In this chapter, generalized semi generalized closed sets ($(i,j)$-gsg-closed sets) in bitopological spaces are analyzed and basic properties of these sets are studied. The notion of $(i,j)$-$T_{gsg}$ space and $(i,j)$-$gsg$ continuous mapping in bitopological space are introduced and some of their properties are investigated. Throughout this chapter, $i, j, k = 1, 2$ where $i \neq j \neq k$.

Here are a few more definitions and results which are to be used in this chapter.

**Definition 3.1.1** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(i,j)$-gsp-closed if $\tau_j$-$spCl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_i$-open in $(X, \tau_1, \tau_2)$.

**Definition 3.1.2** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(i,j)$-gp-closed if $\tau_j$-$pCl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_i$-open in $(X, \tau_1, \tau_2)$.
Definition 3.1.3 Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A mapping \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is called \((i, j)\)-gsp-\(\sigma_k\)-continuous if the inverse image of every \(\sigma_k\)-closed in \((Y, \sigma_1, \sigma_2)\) is \((i, j)\)-gsp-closed in \((X, \tau_1, \tau_2)\).

Definition 3.1.4 Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. A mapping \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is called \((i, j)\)-gp-\(\sigma_k\)-continuous if the inverse image of every \(\sigma_k\)-closed in \((Y, \sigma_1, \sigma_2)\) is \((i, j)\)-gp-closed in \((X, \tau_1, \tau_2)\).

3.2 \(gsg\)-CLOSED SETS IN BITOPOLOGICAL SPACE

In this section, the concept of \((i, j)\)-gsg-closed sets in bitopological spaces are introduced and this set is compared with already existing closed sets in bitopological spaces. The counter examples are given whenever necessary.

Definition 3.2.1 A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be a \((i, j)\)-generalized semi generalized closed set (in short, \((i, j)\)-gsg-closed) if \(\tau_j-\text{Cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_l\)-s\(g\)-open in \(X\).

Proposition 3.2.1 Every \(\tau_j\)-closed set is \((i, j)\)-gsg-closed in a bitopological space \((X, \tau_1, \tau_2)\).

Proof: Let \(A\) be any \(\tau_j\)-closed set and \(U\) be any \(\tau_l\)-s\(g\)-open set containing \(A\), in a bitopological space \((X, \tau_1, \tau_2)\). Then \(\tau_j-\text{Cl}(A) = A \subseteq U\). Hence \(A\) is \((i, j)\)-gsg-closed in a bitopological space \((X, \tau_1, \tau_2)\).

Proposition 3.2.2 Every \((i, j)\)-gsg-closed set is \((i, j)\)-g-closed in a bitopological space \((X, \tau_1, \tau_2)\).
**Proof**: Let $A$ be any $(i,j)$-$gsg$-closed set and $U$ be any $\tau_i$-open set containing $A$ in a bitopological space $(X, \tau_1, \tau_2)$. Since every $\tau_i$-open set is $\tau_i$-$sg$-open set and $A$ is $(i,j)$-$gsg$-closed set, then $\tau_j$-$Cl(A) \subseteq U$. Hence $A$ is $(i,j)$-$g$-closed in a bitopological space $(X, \tau_1, \tau_2)$.

**Proposition 3.2.3** Every $(i,j)$-$gsg$-closed set is $(i,j)$-$\omega$-closed in a bitopological space $(X, \tau_1, \tau_2)$.

**Proof**: Let $A$ be any $(i,j)$-$gsg$-closed set and $U$ be any $\tau_i$-semi open set containing $A$ in a bitopological space $(X, \tau_1, \tau_2)$. Since every $\tau_i$-semi open set is $\tau_i$-$sg$-open set and $A$ is $(i,j)$-$gsg$-closed set, then $\tau_j$-$Cl(A) \subseteq U$. Hence $A$ is $(i,j)$-$\omega$-closed in a bitopological space $(X, \tau_1, \tau_2)$.

**Proposition 3.2.4** Every $(i,j)$-$gsg$-closed set is $(i,j)$-$sg$-closed in a bitopological space $(X, \tau_1, \tau_2)$.

**Proof**: Let $A$ be any $(i,j)$-$gsg$-closed set and $U$ be any $\tau_i$-semi open set containing $A$ in a bitopological space $(X, \tau_1, \tau_2)$. Since every $\tau_i$-semi open set is $\tau_i$-$sg$-open set and $A$ is $(i,j)$-$gsg$-closed set, $\tau_j$-$SgCl(A) \subseteq \tau_j$-$Cl(A) \subseteq U$. Hence $A$ is $(i,j)$-$sg$-closed in a bitopological space $(X, \tau_1, \tau_2)$.

**Proposition 3.2.5** Every $(i,j)$-$gsg$-closed set is $(i,j)$-$gs$-closed, $(i,j)$-$gps$-closed and $(i,j)$-$gp$-closed in a bitopological space $(X, \tau_1, \tau_2)$.

**Proof**: Let $A$ be any $(i,j)$-$gsg$-closed set and $U$ be any $\tau_i$-open set containing $A$ in a bitopological space $(X, \tau_1, \tau_2)$. Then $\tau_j$-$SgCl(A) \subseteq \tau_j$-$Cl(A) \subseteq U$. Hence $A$ is $(i,j)$-$g$s$-closed in a bitopological space $(X, \tau_1, \tau_2)$. Similarly, $\tau_j$-$SpCl(A) \subseteq \tau_j$-$Cl(A)$ and $\tau_j$-$pCl(A) \subseteq \tau_j$-$Cl(A)$ gives $A$ is $(i,j)$-$gps$-closed and $(i,j)$-$gp$-closed in a bitopological space $(X, \tau_1, \tau_2)$.
The pictorial representation of the above discussion is expressed in Figure 3.1.

![Figure 3.1](image)

**Figure 3.1** Comparison of \((i,j)\)-gsg-closed set with other closed sets in bitopological space

The following examples show that the converses of the above implications are not true in general.

**Example 3.2.1**

Let \(X = \{x, y, z\}\), \(\tau_1 = \{\varnothing, \{x\}, X\}\), \(\tau_2 = \{\varnothing, \{x, y\}, X\}\). Clearly \((X, \tau_1, \tau_2)\) is a bitopological space. Then \(\{y, z\}\) is \((1,2)\)-gsg-closed but not \(\tau_2\)-closed.

**Example 3.2.2**

Let \(X = \{x, y, z\}\), \(\tau_1 = \{\varnothing, \{x\}, X\}\), \(\tau_2 = \{\varnothing, \{x\}, \{y\}, \{x, y\}, X\}\). Clearly \((X, \tau_1, \tau_2)\) is a bitopological space. Then \(\{x, y\}\) is \((1,2)\)-g-closed but not \((1,2)\)-gsg-closed.
Example 3.2.3

Let $X = \{x, y, z\}$, $\tau_1 = \{\varnothing, \{x, y\}, X\}$, $\tau_2 = \{\varnothing, \{x\}, \{z\}, \{x, z\}, X\}$. Clearly $(X, \tau_1, \tau_2)$ is a bitopological space. Then $\{x\}$ is $(1,2)$-$\omega$-closed but not $(1,2)$-$gs\,g$-closed.

Example 3.2.4

Let $X = \{x, y, z\}$, $\tau_1 = \{\varnothing, \{x\}, X\}$, $\tau_2 = \{\varnothing, \{x\}, \{y, z\}, X\}$. Clearly $(X, \tau_1, \tau_2)$ is a bitopological space. Then $\{x, y\}$ is $(1,2)$-$gs$-closed but not $(1,2)$-$gs\,g$-closed.

Example 3.2.5

Let $X = \{x, y, z, w\}$, $\tau_1 = \{\varnothing, \{x, y, z\}, X\}$, $\tau_2 = \{\varnothing, \{x, w\}, \{x, y, w\}, X\}$. Clearly $(X, \tau_1, \tau_2)$ is a bitopological space. Then $\{y\}$ is $(1,2)$-$sg$-closed but not $(1,2)$-$gs\,g$-closed.

Example 3.2.6

Let $X = \{x, y, z\}$, $\tau_1 = \{\varnothing, \{x, \{y\}, \{x, y\}, X\}$, $\tau_2 = \{\varnothing, \{x\}, \{y\}, \{x, y\}, X\}$. Clearly $(X, \tau_1, \tau_2)$ is a bitopological space. Then $\{x\}$ is $(1,2)$-$gs\,p$-closed but not $(1,2)$-$gs\,g$-closed.

Example 3.2.7

Let $X = \{x, y, z\}$, $\tau_1 = \{\varnothing, \{z\}, X\}$, $\tau_2 = \{\varnothing, \{x\}, \{x, y\}, X\}$. Clearly $(X, \tau_1, \tau_2)$ is a bitopological space. Then $\{x\}$ is $(1,2)$-$gp$-closed but not $(1,2)$-$gs\,g$-closed.
Definition 3.2.2 A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be \textit{(i, j)-generalized semi generalized open set} (in short, \textit{(i, j)-gsg-open}) if $X - A$ is \textit{(i, j)-gsg-closed} in $(X, \tau_1, \tau_2)$.

Theorem 3.2.1 In a bitopological space $(X, \tau_1, \tau_2)$,

i. Every $\tau_i$-open set is \textit{(i, j)-gsg-open}.

ii. Every \textit{(i, j)-gsg-open} set is \textit{(i, j)-g-open} and \textit{(i, j)-o-open}.

iii. Every \textit{(i, j)-gsg-open} set is \textit{(i, j)-sg-open} and \textit{(i, j)-gs-open}.

iv. Every \textit{(i, j)-gsg-open} set is \textit{(i, j)-gps-open} and \textit{(i, j)-gp-open}.

Proof: It is easy to prove.

Theorem 3.2.2 If $A$ and $B$ are \textit{(i, j)-gsg-closed} in a bitopological space $(X, \tau_1, \tau_2)$ then $A \cup B$ is \textit{(i, j)-gsg-closed}.

Proof: Let $U$ be any $\tau_i$-sg-open set containing $A$ and $B$ in a bitopological space $(X, \tau_1, \tau_2)$. Then $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since $A$ and $B$ are \textit{(i, j)-gsg-closed}, $\tau_j$-$\text{Cl}(A) \subseteq U$ and $\tau_j$-$\text{Cl}(B) \subseteq U$. That is, $\tau_j$-$\text{Cl}(A \cup B) = \tau_j$-$\text{Cl}(A) \cup \tau_j$-$\text{Cl}(B) \subseteq U$, so $\tau_j$-$\text{Cl}(A \cup B) \subseteq U$. Hence $A \cup B$ is \textit{(i, j)-gsg-closed}.

Theorem 3.2.3 If a set $A$ is \textit{(i, j)-gsg-closed} in a bitopological space $(X, \tau_1, \tau_2)$ then $\tau_j$-$\text{Cl}(A) - A$ contains no nonempty $\tau_i$-closed set.

Proof: Let $A$ be any \textit{(i, j)-gsg-closed} and $F$ be a $\tau_i$-closed set in a bitopological space $(X, \tau_1, \tau_2)$ such that $F \subseteq \tau_j$-$\text{Cl}(A) - A$. Since $A$ is
(i,j)-gs-g-closed, \( \tau_j-Cl(A) \subseteq X - F \). Then \( F \subseteq \tau_j-Cl(A) \cap (X - \tau_j-Cl(A)) = \emptyset \). Hence \( F \) is empty.

The converse of the above theorem is not true as shown in the following example.

**Example 3.2.8**

Let \( X = \{x, y, z\} \), \( \tau_1 = \{\varnothing, \{z\}, X\} \), \( \tau_2 = \{\varnothing, \{x\}, \{x, y\}, X\} \). Clearly \((X, \tau_1, \tau_2)\) is a bitopological space. If \( A = \{x\} \) then \( \tau_2-Cl(A) - A = \{y, z\} \) does not contain nonempty \( \tau_1 \)-closed set. But \( \{x\} \) is not \((1,2)\)-gs-g-closed.

**Theorem 3.2.4** A set \( A \) is \((i,j)\)-gs-g-closed in a bitopological space \((X, \tau_1, \tau_2)\) if and only if \( \tau_j-Cl(A) - A \) contains no nonempty \((i,j)\)-gs-g-closed set in a bitopological space \((X, \tau_1, \tau_2)\).

**Proof**: Let \( A \) be any \((i,j)\)-gs-g-closed in a bitopological space \((X, \tau_1, \tau_2)\) and \( D \) be a \((i,j)\)-gs-g-closed set in a bitopological space \((X, \tau_1, \tau_2)\) such that \( D \subseteq \tau_j-Cl(A) - A \). Since \( A \) is \((i,j)\)-gs-g-closed, \( \tau_j-Cl(A) \subseteq X - D \). Then \( D \subseteq \tau_j-Cl(A) \cap (X - \tau_j-Cl(A)) = \emptyset \). Thus \( D \) is empty.

Conversely, suppose that \( \tau_j-Cl(A) - A \) contains no nonempty \((i,j)\)-gs-g-closed set. Let \( A \subseteq G \) and \( G \) is \((i,j)\)-gs-g-open. If \( \tau_j-Cl(A) \not\subseteq G \) then \( \tau_j-Cl(A) \cap X - G \) is nonempty. Since \( \tau_j-Cl(A) \) is closed set and \( X - G \) is \((i,j)\)-gs-g-closed, \( \tau_j-Cl(A) \cap X - G \) is nonempty \((i,j)\)-gs-g-closed set of \( \tau_j-Cl(A) - A \) which is a contradiction. Therefore \( \tau_j-Cl(A) \subseteq G \). Hence \( A \) is \((i,j)\)-gs-g-closed in a bitopological space \((X, \tau_1, \tau_2)\).

**Theorem 3.2.5** If a set \( A \) is \((i,j)\)-gs-g-closed in a bitopological space \((X, \tau_1, \tau_2)\) then \( \tau_i-Cl(\{x\}) \cap A \neq \emptyset \) holds for each \( x \in \tau_j-Cl(A) \).
Proof : If \( \tau_i^{-}\text{Cl}(\{x\}) \cap A = \emptyset \) for some \( x \in \tau_j^{-}\text{Cl}(A) \), then \( A \subseteq X - (\tau_i^{-}\text{Cl}(\{x\})) \). Since \( A \) is \((i,j)\)-\text{gsg}-closed in a bitopological space \((X, \tau_1, \tau_2)\), \( \tau_j^{-}\text{Cl}(A) \subseteq (X - \tau_i^{-}\text{Cl}(\{x\})) \). This shows that \( x \notin \tau_j^{-}\text{Cl}(A) \). This contradicts the assumption. Hence \( \tau_i^{-}\text{Cl}(\{x\}) \cap A \neq \emptyset \) holds for each \( x \in \tau_j^{-}\text{Cl}(A) \).

The converse of the above theorem is not true as shown in the following example.

Example 3.2.9 Let \( X = \{x, y, z\} \), \( \tau_1 = \{\emptyset, \{x\}, X\} \), \( \tau_2 = \{\emptyset, \{x\}, \{y, z\}, X\} \). Clearly \((X, \tau_1, \tau_2)\) is a bitopological space. A subset \( A = \{x, y\} \) is not \((1,2)\)-\text{gsg}-closed set, but \( \tau_1^{-}\text{Cl}(\{x\}) \cap A \neq \emptyset \), for each \( x \in \tau_2^{-}\text{Cl}(A) \).

Theorem 3.2.6 If \( A \) is a \((i,j)\)-\text{gsg}-closed set of a bitopological space \((X, \tau_1, \tau_2)\) such that \( A \subseteq B \subseteq \tau_j^{-}\text{Cl}(A) \), then \( B \) is an \((i,j)\)-\text{gsg}-closed set of a bitopological space \((X, \tau_1, \tau_2)\).

Proof : Let \( U \) be \( \tau_i^{-}\text{sg}\)-open set such that \( B \subseteq U \) in a bitopological space \((X, \tau_1, \tau_2)\). Since \( A \) is \((i,j)\)-\text{gsg}-closed and \( A \subseteq U \), \( \tau_j^{-}\text{Cl}(A) \subseteq U \). Now \( B \subseteq \tau_j^{-}\text{Cl}(A) \) which gives, \( \tau_j^{-}\text{Cl}(B) \subseteq \tau_j^{-}\text{Cl}\{\tau_j^{-}\text{Cl}(A)\} = \tau_j^{-}\text{Cl}(A) \subseteq U \). Thus \( \tau_j^{-}\text{Cl}(B) \subseteq U \). Hence \( B \) is \((i,j)\)-\text{gsg}-closed set of a bitopological space \((X, \tau_1, \tau_2)\).

Theorem 3.2.7 In a bitopological space \((X, \tau_1, \tau_2)\), \( \text{SGO}(X, \tau_i) \subseteq \{F \subseteq X : X - F \in \tau_j\} \) if and only if every subset of \((X, \tau_1, \tau_2)\) is an \((i,j)\)-\text{gsg}-closed set in \((X, \tau_1, \tau_2)\).

Proof : Suppose that \( \text{SGO}(X, \tau_i) \subseteq \{F \subseteq X : X - F \in \tau_j\} \). Let \( A \) be a subset of \( X \) and \( U \) be \( \tau_i^{-}\text{sg}\)-open set in a bitopological space \((X, \tau_1, \tau_2)\) such that \( A \subseteq U \). Then \( \tau_j^{-}\text{Cl}(A) \subseteq \tau_j^{-}\text{Cl}(U) = U \). Hence \( A \) is an \((i,j)\)-\text{gsg}-closed set in a bitopological space \((X, \tau_1, \tau_2)\).
Conversely, suppose that every subset of \((X, \tau_1, \tau_2)\) is a \((i, j)\)-gsg-closed set. Let \(U \in SGO(X, \tau_i)\). Since \(U\) is a \((i, j)\)-gsg-closed set, \(\tau_j-\text{Cl}(U) \subseteq U\). Therefore \(U \in \{F \subseteq X: X - F \in \tau_j\}\). Hence \(SGO(X, \tau_i) \subseteq \{F \subseteq X: X - F \in \tau_j\}\).

**Theorem 3.2.8** If \(A\) is \(\tau_i\)-sg-open and \((i, j)\)-gsg-closed in a bitopological space \((X, \tau_1, \tau_2)\) then \(A\) is \(\tau_j\)-closed in \((X, \tau_1, \tau_2)\).

**Proof :** Since \(A\) is \(\tau_i\)-sg-open and \((i, j)\)-gsg-closed in \((X, \tau_1, \tau_2)\), then \(\tau_j-\text{Cl}(A) \subseteq A\). But \(A \subseteq \tau_j-\text{Cl}(A)\), which gives \(A = \tau_j-\text{Cl}(A)\). Hence \(A\) is \(\tau_j\)-closed in \((X, \tau_1, \tau_2)\).

**Theorem 3.2.9** For each point \(x\) of \((X, \tau_1, \tau_2)\), either a singleton set \(\{x\}\) is \(\tau_i\)-sg-closed or \(X - \{x\}\) is \((i, j)\)-gsg-closed in \((X, \tau_1, \tau_2)\).

**Proof :** If set \(\{x\}\) is not \(\tau_i\)-sg-closed in \((X, \tau_1, \tau_2)\) then \(X - \{x\}\) is not \(\tau_i\)-sg-open in \((X, \tau_1, \tau_2)\) and the only \(\tau_i\)-sg-open set containing \(X - \{x\}\) is the space \((X, \tau_1, \tau_2)\) itself. Then \(\tau_j-\text{Cl}(X - \{x\}) \subseteq (X, \tau_1, \tau_2)\) and so \(X - \{x\}\) is \((i, j)\)-gsg-closed in \((X, \tau_1, \tau_2)\).

### 3.3 APPLICATION OF \((i, j)\)-gsg-CLOSED SETS

In this section as an application of \((i, j)\)-gsg-closed sets, \((i, j)\)-\(T_{gsg}\) -space in bitopological spaces are introduced and investigated some of its properties.

**Definition 3.3.1** A bitopological space \((X, \tau_1, \tau_2)\) is called a \((i, j)\)-\(T_{gsg}\) -space if every \((i, j)\)-gsg-closed set in it is \(\tau_j\)-closed.

**Proposition 3.3.1** Every \((i, j)\)-\(T_{1/2}\) -space is a \((i, j)\)-\(T_{gsg}\) -space.
Proof: Let \((X, \tau_1, \tau_2)\) be a \((i,j)\)-\(T_{1/2}\)-space and let \(A\) be a \((i,j)\)-\(gsg\)-closed set in \((X, \tau_1, \tau_2)\). By proposition 3.2.2, \(A\) is a \((i,j)\)-\(g\)-closed in \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_{1/2}\)-space, \(A\) is \(\tau_j\)-closed in \((X, \tau_1, \tau_2)\). Hence \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_{gsg}\)-space.

**Proposition 3.3.2** Every \((i,j)\)-\(T_{\omega}\)-space is a \((i,j)\)-\(T_{gsg}\)-space.

**Proof:** Let \((X, \tau_1, \tau_2)\) be a \((i,j)\)-\(T_{\omega}\)-space and let \(A\) be a \((i,j)\)-\(gsg\)-closed set in \((X, \tau_1, \tau_2)\). By proposition 3.2.3, \(A\) is a \((i,j)\)-\(\omega\)-closed in \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_{\omega}\)-space, \(A\) is \(\tau_j\)-closed in \((X, \tau_1, \tau_2)\). Hence \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_{gsg}\)-space.

**Proposition 3.3.3** Every \((i,j)\)-\(T_b\)-space is a \((i,j)\)-\(T_{gsg}\)-space.

**Proof:** Let \((X, \tau_1, \tau_2)\) be a \((i,j)\)-\(T_b\)-space and let \(A\) be a \((i,j)\)-\(gsg\)-closed set in \((X, \tau_1, \tau_2)\). By proposition 3.2.5, \(A\) is a \((i,j)\)-\(g\)-closed set in \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_b\)-space, \(A\) is \(\tau_j\)-closed in \((X, \tau_1, \tau_2)\). Hence \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_{gsg}\)-space.

**Example 3.3.1** In Example 3.2.1, the space \((X, \tau_1, \tau_2)\) is a \((1,2)\)-\(T_{gsg}\)-space but not a \((1,2)\)-\(T_b\)-space and \((1,2)\)-\(T_{1/2}\)-space.

**Theorem 3.3.1** A bitopological space \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_{gsg}\)-space if and only if every singleton set \(\{x\}\) of \((X, \tau_1, \tau_2)\) is either \(\tau_i\)-\(Sg\)-closed or \(\tau_j\)-open.

**Proof:** Suppose that \(\{x\}\) is not \(\tau_i\)-\(Sg\)-closed. Then \(X - \{x\}\) is \((i,j)\)-\(gsg\)-closed by theorem 3.2.9. Since \((X, \tau_1, \tau_2)\) is a \((i,j)\)-\(T_{gsg}\)-space, \(X - \{x\}\) is \(\tau_j\)-closed. Hence \(\{x\}\) is \(\tau_j\)-open in \((X, \tau_1, \tau_2)\).
Conversely, Let $A$ be a $(i, j)\cdot gsg$-closed set of $(X, \tau_1, \tau_2)$. Clearly $A \subseteq \tau_j-Cl(A)$. Let $x \in \tau_j-Cl(A)$. Then by hypothesis $\{x\}$ is either $\tau_i-gsg$-closed or $\tau_j$-open.

**Case 1:** Suppose $\{x\}$ is $\tau_i-gsg$-closed. If $x \notin A$, then $\{x\} \subseteq \tau_j-Cl(A) - A$, this is a contradiction to the theorem 3.2.4. Therefore $x \in A$. So $\tau_j-Cl(A) \subseteq A$.

**Case 2:** Suppose $\{x\}$ is $\tau_j$-open. Since $x \in \tau_j-Cl(A)$, $\{x\} \cap A \neq \varnothing$. Therefore $x \in A$ So $\tau_j-Cl(A) \subseteq A$.

Hence in both the cases $A = \tau_j-Cl(A)$. That is, $A$ is $\tau_j$-closed.

Thus $(X, \tau_1, \tau_2)$ is a $(i, j)$- $T_{gsg}$-space.

### 3.4 $(i, j)$-gsg-CONTINUOUS MAPPING

In this section, the concept of $(i, j)$-gsg-continuous mapping in bitopological spaces are introduced and some of its properties are established.

**Definition 3.4.1** Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ is two bitopological spaces. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be $\textbf{(i, j)-gsg-}\sigma_k$-continuous if the inverse image of every $\sigma_k$-closed in $(Y, \sigma_1, \sigma_2)$ is $(i, j)$-gsg-closed in $(X, \tau_1, \tau_2)$.

**Theorem 3.4.1** Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ is two bitopological spaces. If a mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(i, j)$-gsg-\sigma_k-continuous then $f$ is $(i, j)$-sg-\sigma_k-continuous.

**Proof**: Let $A$ be any $\sigma_k$-closed in $(Y, \sigma_1, \sigma_2)$. Since $f$ is $(i, j)$-gsg-\sigma_k-continuous, $f^{-1}(A)$ is $(i, j)$-gsg-closed in $(X, \tau_1, \tau_2)$. Then by proposition (3.2.4), $f^{-1}(A)$ is $(i, j)$-sg-closed in $(X, \tau_1, \tau_2)$. 
Hence $f$ is $(i,j)$-$s\,g\,\sigma_k$-continuous.

**Theorem 3.4.2** Let $(X,\tau_1,\tau_2)$ and $(Y,\sigma_1,\sigma_2)$ be two bitopological spaces. If a mapping $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is $(i,j)$-$g\,s\,g\,-\sigma_k$-continuous then $f$ is $(i,j)$-$g\,s\,p\,-\sigma_k$-continuous.

**Proof**: Let $A$ be any $\sigma_k$-closed in $(Y,\sigma_1,\sigma_2)$. Since $f$ is $(i,j)$-$g\,s\,g\,-\sigma_k$-continuous, $f^{-1}(A)$ is $(i,j)$-$g\,s\,g$-closed in $(X,\tau_1,\tau_2)$. Then by proposition 3.2.5, $f^{-1}(A)$ is $(i,j)$-$g\,s\,p$-closed in $(X,\tau_1,\tau_2)$. Hence $f$ is $(i,j)$-$g\,s\,p\,-\sigma_k$-continuous.

**Theorem 3.4.3** Let $(X,\tau_1,\tau_2)$ and $(Y,\sigma_1,\sigma_2)$ be two bitopological spaces. If a mapping $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is $(i,j)$-$g\,s\,g\,-\sigma_k$-continuous then $f$ is $(i,j)$-$g\,p\,-\sigma_k$-continuous, $(i,j)$-$g\,s\,-\sigma_k$-continuous and $(i,j)$-$\omega\,-\sigma_k$-continuous.

**Proof**: The proof is similar to that of theorem 3.4.2.

**Theorem 3.4.4** Let $(X,\tau_1,\tau_2)$ and $(Y,\sigma_1,\sigma_2)$ be two bitopological spaces. If a mapping $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is $(i,j)$-$g\,s\,g\,-\sigma_k$-continuous if and only if inverse image of each $\sigma_k$-open set of $(Y,\sigma_1,\sigma_2)$ is $(i,j)$-$g\,s\,g$-open in $(X,\tau_1,\tau_2)$.

**Proof**: Let $f$ be $(i,j)$-$g\,s\,g\,-\sigma_k$-continuous. If $A$ is any $\sigma_k$-open set of $(Y,\sigma_1,\sigma_2)$ then $X - A$ is $\sigma_k$-closed in $(Y,\sigma_1,\sigma_2)$. Since $f$ is $(i,j)$-$g\,s\,g\,-\sigma_k$-continuous, $f^{-1}(X - A) = X - f^{-1}(A)$ is $(i,j)$-$g\,s\,g$-closed in $(X,\tau_1,\tau_2)$. Hence $f^{-1}(A)$ is $(i,j)$-$g\,s\,g$-open in $(X,\tau_1,\tau_2)$.

Conversely, let $A$ be any $\sigma_k$-closed in $(Y,\sigma_1,\sigma_2)$. By hypothesis, $f^{-1}(X - A)$ is $(i,j)$-$g\,s\,g$-open in $(X,\tau_1,\tau_2)$. Then $f^{-1}(A)$ is $(i,j)$-$g\,s\,g$-closed in $(X,\tau_1,\tau_2)$. Hence $f$ is $(i,j)$-$g\,s\,g\,-\sigma_k$-continuous.
**Theorem 3.4.5** Let \((X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)\) and \((Z, \eta_1, \eta_2)\) be bitopological spaces. If \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is \((i, j)\)-gs\(g\)-\(\sigma_k\)-continuous and \(g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)\) is \((i, j)\)-\(g\)-\(\sigma_k\)-continuous and \((Y, \sigma_1, \sigma_2)\) is \((i, j)\)-\(T_{1/2}\)-space. Then \(g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)\) is \((i, j)\)-gs\(g\)-\(\sigma_k\)-continuous.

**Proof:** Let \(A\) be any \(\eta_k\)-closed in \((Z, \eta_1, \eta_2)\). Since \(g\) is \((i, j)\)-\(g\)-\(\sigma_k\)-continuous and \((Y, \sigma_1, \sigma_2)\) is \((i, j)\)-\(T_{1/2}\)-space, \(g^{-1}(A)\) is \(\sigma_j\)-closed in \((Y, \sigma_1, \sigma_2)\). Since \(f\) is \((i, j)\)-gs\(g\)-\(\sigma_k\)-continuous, \(f^{-1}(g^{-1}(A))\) is \((i, j)\)-gs\(g\)-closed in \((X, \tau_1, \tau_2)\). Hence \(g \circ f\) is \((i, j)\)-gs\(g\)-\(\sigma_k\)-continuous.

**Theorem 3.4.6** Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two bitopological spaces. Let \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be a map.

i. If \((X, \tau_1, \tau_2)\) is an \((i, j)\)-\(T_{1/2}\)-space then \(f\) is \((i, j)\)-\(g\)-\(\sigma_k\)-continuous if and only if it is \((i, j)\)-gs\(g\)-\(\sigma_k\)-continuous.

ii. If \((X, \tau_1, \tau_2)\) is an \((i, j)\)-\(T_{gs\!g}\)-space then \(f\) is \(\tau_j\)-\(\sigma_k\)-continuous if and only if it is \((i, j)\)-gs\(g\)-\(\sigma_k\)-continuous.

**Proof:**

i. Let \(A\) be any \(\sigma_k\)-closed in \((Y, \sigma_1, \sigma_2)\). Since \(f\) is \((i, j)\)-\(g\)-\(\sigma_k\)-continuous, \(f^{-1}(A)\) is \((i, j)\)-\(g\)-closed in \((X, \tau_1, \tau_2)\). But \((X, \tau_1, \tau_2)\) is an \((i, j)\)-\(T_{1/2}\)-space implies \(f^{-1}(A)\) is \(\tau_j\)-closed. By proposition 3.2.1, \(f^{-1}(A)\) is \((i, j)\)-gs\(g\)-\(\sigma_k\)-closed in \((X, \tau_1, \tau_2)\). Hence \(f\) is \((i, j)\)-gs\(g\)-\(\sigma_k\)-continuous.
Conversely, suppose that \( f \) is \((i,j)\)-gs\(g\)-\(\sigma_k\)-continuous. Let \( A \) be any \( \sigma_k \)-closed in \((Y,\sigma_1,\sigma_2)\). Then \( f^{-1}(A) \) is \((i,j)\)-gs\(g\)-closed in \((X,\tau_1,\tau_2)\). By proposition 3.2.2, \( f^{-1}(A) \) is \((i,j)\)-gs\(g\)-closed in \((X,\tau_1,\tau_2)\). Hence \( f \) is \((i,j)\)-gs\(g\)-\(\sigma_k\)-continuous.

ii. Let \( A \) be any \( \sigma_k \)-closed in \((Y,\sigma_1,\sigma_2)\). Since \( f \) is \( \tau_j \)-\(\sigma_k\)-continuous, \( f^{-1}(A) \) is \( \tau_j \)-closed in \((X,\tau_1,\tau_2)\) By proposition 3.2.1, \( f^{-1}(A) \) is \((i,j)\)-gs\(g\)-closed in \((X,\tau_1,\tau_2)\). Hence \( f \) is \((i,j)\)-gs\(g\)-\(\sigma_k\)-continuous.

Conversely, suppose that \( f \) is \((i,j)\)-gs\(g\)-\(\sigma_k\)-continuous. Let \( A \) be any \( \sigma_k \)-closed in \((Y,\sigma_1,\sigma_2)\). Then \( f^{-1}(A) \) is \((i,j)\)-gs\(g\)-closed in \((X,\tau_1,\tau_2)\). But \((X,\tau_1,\tau_2)\) is an \((i,j)\)-\(T_{gs\!g}\)-space implies \( f^{-1}(A) \) is \( \tau_j \)-closed in \((X,\tau_1,\tau_2)\). Hence \( f \) is \( \tau_j \)-\(\sigma_k\)-continuous.

3.5 CONCLUSION

In this chapter, generalized semi generalized closed sets ((\(i,j\)-gs\(g\)-closed sets) in bitopological space is introduced and basic properties of these sets are analyzed. As an application, the notion of \((i,j)\)-\(T_{gs\!g}\)-space is investigated. Further, \((i,j)\)-gs\(g\) continuous mapping in bitopological space is introduced and some of their properties are investigated.