CHAPTER 2

A NEW CLASS OF MINIMAL AND MAXIMAL SETS VIA $gsg$-CLOSED SET

2.1 INTRODUCTION

In this chapter, the concept of minimal $gsg$-open sets, minimal $gsg$-closed sets, maximal $gsg$-open sets, maximal $gsg$-closed sets, $gsg$-semi-maximal open sets and $gsg$-semi-minimal closed sets are introduced. And also, minimal-$gsg$-continuous and maximal-$gsg$-continuous mappings are introduced. Several properties and characterization of these concepts are established.

Throughout this chapter, $(X, \tau)$ represents the topological space with the topology $\tau$.

**Definition 2.1.1 (Nakaoka & Oda 2001)** A proper nonempty open subset $A$ of a topological space $(X, \tau)$ is said to be **minimal open** if any open set which is contained in $A$ is $\varnothing$ or $A$.

**Definition 2.1.2 (Nakaoka & Oda 2001)** A proper nonempty open subset $A$ of topological space $(X, \tau)$ is said to be **maximal open** if any open set which contains $A$ is $X$ or $A$.

**Definition 2.1.3 (Nakaoka & Oda 2006)** A proper nonempty closed subset $A$ of topological space $(X, \tau)$ is said to be **minimal closed** if any closed set which is contained in $A$ is $\varnothing$ or $A$. A proper nonempty closed subset $A$ of
topological space \((X, \tau)\) is said to be **maximal closed** if any closed set which contains \(A\) is \(X\) or \(A\).

**Note 2.1.1** The following duality principle holds in Nakaoka & Oda (2006) for subset \(A\) of a topological space \((X, \tau)\):

i. \(A\) is minimal closed if and only if \(X - A\) is maximal open.

ii. \(A\) is maximal closed if and only if \(X - A\) is minimal open.

**Definition 2.1.4** (Nakaoka & Oda 2006) A topological space \((X, \tau)\) is said to be **locally finite space** if each of its elements are contained in a finite open set.

### 2.2 MINIMAL \(gsg\)-OPEN SETS AND MAXIMAL \(gsg\)-CLOSED SETS

In this section, minimal \(gsg\)-open sets and maximal \(gsg\)-closed sets are introduced and some basic properties are studied.

**Definition 2.2.1** A proper nonempty \(gsg\)-open subset \(A\) of a topological space \((X, \tau)\) is said to be a **minimal \(gsg\)-open** if any \(gsg\)-open set contained in \(A\) is \(\varnothing\) or \(A\).

**Definition 2.2.2** A proper nonempty \(gsg\)-closed subset \(A\) of a topological space \((X, \tau)\) is said to be a **maximal \(gsg\)-closed** if any \(gsg\)-closed set containing \(A\) is either \(X\) or \(A\).

**Example 2.2.1** Let \(X = \{x, y, z, w\}\), \(\tau = \{\varnothing, \{z\}, \{x, z\}, X\}\). Clearly \((X, \tau)\) is a topological space. The set \(\{z\}\) is both minimal open and minimal \(gsg\)-open. And the set \(\{x\}\) is minimal \(gsg\)-open but not minimal open.
The following example shows that the minimal open and minimal $gssg$-open sets are independent of each other.

**Example 2.2.2** Let $X = \{x, y, z\}, \tau = \{\varnothing, \{x, y\}, X\}$. Clearly $(X, \tau)$ is a topological space. Then $\{x, y\}$ is minimal open set but not minimal $gssg$-open and $\{x\}, \{y\}$ are minimal $gssg$-open but not minimal open.

The following example shows that the maximal closed and maximal $gssg$-closed sets are independent of each other.

**Example 2.2.3** In Example 2.2.2, $\{z\}$ is maximal closed but not maximal $gssg$-closed and $\{y, z\}, \{x, z\}$ are maximal $gssg$-closed but not maximal closed.

**Theorem 2.2.1** Let $(X, \tau)$ be a topological space and $A, B \subseteq (X, \tau)$.

i. Let $A$ be minimal $gssg$-open set and $B$ be a $gssg$-open set. Then $A \cap B = \varnothing$ or $A \subset B$.

ii. Let $A$ and $B$ be minimal $gssg$-open sets. Then $A \cap B = \varnothing$ or $A = B$.

**Proof:**

i. Let $A$ be a minimal $gssg$-open set and $B$ be a $gssg$-open set.

If $A \cap B = \varnothing$, then there is nothing to prove.

If $A \cap B \neq \varnothing$. Then $A \cap B \subset A$. Now $A$ is minimal $gssg$-open set gives $A \cap B = A$. Therefore $A \subset B$. 
ii. Let $A$ and $B$ be minimal $g_{sg}$-open sets. Suppose $A \cap B \neq \emptyset$. Here, $A$ and $B$ are $g_{sg}$-open sets and then by (i), $A \subset B$ and $B \subset A$. Therefore $A = B$.

**Theorem 2.2.2** Let $A$ be a minimal $g_{sg}$-open set of a topological space $(X, \tau)$. If $x \in A$, then $A \subset B$ for some $g_{sg}$-open set $B$ in $(X, \tau)$ containing $x$.

**Proof:** Let $x \in A$, a minimal $g_{sg}$-open set of a topological space $(X, \tau)$. Let $B$ is any $g_{sg}$-open set in $(X, \tau)$ containing $x$. Now $A \cap B \neq \emptyset$. Then by theorem 2.2.1, $A \subset B$.

**Theorem 2.2.3** For an $x \in X$ and a subset $A$ in a topological space $(X, \tau)$, $x \in g_{sg}-\text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $g_{sg}$-open set $U$ containing $x$.

**Proof:** Let $x \in X, A \subset X$ and $x \in g_{sg}-\text{Cl}(A)$. The result is proved by the method of contradiction. Suppose there exists a $g_{sg}$-open set $U$ containing $x$ such that $U \cap A = \emptyset$. Then $A \subset X - U$ and $X - U$ is $g_{sg}$-closed. Then $g_{sg}-\text{Cl}(A) \subset X - U$. This gives $x \notin g_{sg}-\text{Cl}(A)$, which is contradiction. Hence $U \cap A \neq \emptyset$ for every $g_{sg}$-open set $U$ containing $x$.

Conversely, let $U \cap A \neq \emptyset$ for every $g_{sg}$-open set $U$ containing $x$. The result is proved by the method of contradiction. Suppose $x \notin g_{sg}-\text{Cl}(A)$. Then there exists a $g_{sg}$-closed subset $V$ containing $A$ such that $x \notin V$. Then $x \in X - V$ and $X - V$ is $g_{sg}$-open. Also $(X - V) \cap A = \emptyset$, this is a contradiction. Hence $x \in g_{sg}-\text{Cl}(A)$.

**Theorem 2.2.4** Let $A$ be a nonempty $g_{sg}$-open set of a topological space $(X, \tau)$. Then the following are equivalent:
(1) $A$ is a minimal $gsg$-open set.

(2) $A \subset gsg-Cl(U)$, for any nonempty subset $U$ of $A$.

(3) $gsg-Cl(A) = gsg-Cl(U)$, for any nonempty subset $U$ of $A$.

**Proof:** (1) $\Rightarrow$ (2): Let $A$ be minimal $gsg$-open set of a topological space $(X, \tau)$. Let $x \in A$ and $U$ be a nonempty subset of $A$. By theorem 2.2.2, there exists a $gsg$-open set $B$ containing $x$ such that $A \subset B$. Then $U \subset A \subset B$, which implies $U \subset B$. Now $U = U \cap A \subset U \cap B$. Since $U$ is nonempty, $U \cap B \neq \emptyset$. Since $B$ is any $gsg$-open set containing $x$, by the theorem 2.2.3, $x \in gsg-Cl(U)$. That is, $x \in A$ implies $x \in gsg-Cl(U)$. Hence $A \subset gsg-Cl(U)$, for any nonempty subset $U$ of $A$.

(2) $\Rightarrow$ (3): Let $U$ be a nonempty subset of $A$ and $A \subset gsg-Cl(U)$. Then $gsg-Cl(U) \subset gsg-Cl(A)$ and $gsg-Cl(A) \subset gsg-Cl(U)$. Hence $gsg-Cl(A) = gsg-Cl(U)$, for any nonempty subset $U$ of $A$.

(3) $\Rightarrow$ (1): Let $gsg-Cl(A) = gsg-Cl(U)$, for any nonempty subset $U$ of $A$. Suppose $A$ is not a minimal $gsg$-open set. Then there exists a nonempty $gsg$-open set $B$ such that $B \subset A$ and $B \neq A$. Now there exists an element $x \in A$ such that $x \notin B$, which implies $x \in X - B$. That is, $gsg-Cl(\{x\}) \subset gsg-Cl(X - B) = X - B$, as $X - B$ is $gsg$-closed in $X$. It follows that $gsg-Cl(\{x\}) \neq gsg-Cl(A)$. This is a contradiction to the fact that $gsg-Cl(\{x\}) = gsg-Cl(A)$, for any nonempty subset $\{x\}$ of $A$. Thus $A$ is a minimal $gsg$-open set.

**Theorem 2.2.5** Let $B$ be a nonempty finite $gsg$-open set of a topological space $(X, \tau)$. Then there exists at least one (finite) minimal $gsg$-open set $A$ of such that $A \subset B$. 
Proof: Let $B$ be a nonempty $g_{sg}$-open set. Then the following two cases arise:

1. $B$ is a minimal $g_{sg}$-open.
2. $B$ is not a minimal $g_{sg}$-open.

Case (1): If $B = A$, then the theorem is proved.

Case (2): If $B$ is not a minimal $g_{sg}$-open, then there exists a nonempty (finite) $g_{sg}$-open set $B_1$ such that $B_1 \subset B$. If $B_1$ is minimal $g_{sg}$-open, take $A = B_1$. If $B_1$ is not a minimal $g_{sg}$-open set, then there exists a nonempty (finite) $g_{sg}$-open set $B_2$ such that $B_2 \subset B_1 \subset B$. Continuing this process and have a sequence of $g_{sg}$-open sets, $\ldots \subset B_n \subset \ldots B_2 \subset B_1 \subset B$. Since $B$ is finite, this process will end at finite number of steps. That is, for any natural number $k$, there is a minimal $g_{sg}$-open set $B_k$ such that $B_k = A$. Hence there exists at least one (finite) minimal $g_{sg}$-open set $A$ of such that $A \subset B$.

Corollary 2.2.1 Let $X$ be a locally finite space and $B$ be a nonempty $g_{sg}$-open set. Then there exists at least one (finite) minimal $g_{sg}$-open set $A$ such that $A \subset B$.

Proof: Let $X$ be a locally finite space and $B$ be nonempty $g_{sg}$-open set. Let $x \in B$. Since $X$ is finite, there is a finite open set $B_x$ such that $x \in B_x$. Then $B \cap B_x$ is a nonempty finite $g_{sg}$-open set. By theorem 2.2.5, there exists at least one (finite) minimal $g_{sg}$-open set $A$ such that $A \subset B \cap B_x$. That is, $A \subset B$. Hence there exists at least one (finite) minimal $g_{sg}$-open set $A$ such that $A \subset B$.

Theorem 2.2.6 Let $A$ and $A_\alpha$ be minimal $g_{sg}$-open sets for any $\alpha \in \Gamma$. If $A \subset \bigcup_{\alpha \in \Gamma} A_\alpha$ then there exists an element $\alpha \in \Gamma$ such that $A = A_\alpha$. 
Proof : Let $A \subseteq \bigcup_{\alpha \in \Gamma} A_\alpha$. Then $A = (\bigcup_{\alpha \in \Gamma} A_\alpha) \cap A$. This gives $\bigcup_{\alpha \in \Gamma} (A \cap A_\alpha) = A$. Also by theorem 2.2.1, $A \cap A_\alpha = \emptyset$ or $A = A_\alpha$ for any $\alpha \in \Gamma$. Hence there exists an element $\alpha \in \Gamma$ such that $A = A_\alpha$.

Theorem 2.2.7 Let $A$ and $A_\alpha$ be minimal $gsg$-open sets for any $\alpha \in \Gamma$. If $A \neq A_\alpha$ for any $\alpha \in \Gamma$, then $(\bigcup_{\alpha \in \Gamma} A_\alpha) \cap A = \emptyset$.

Proof : Suppose $(\bigcup_{\alpha \in \Gamma} A_\alpha) \cap A \neq \emptyset$. That is, $\bigcup_{\alpha \in \Gamma} (A_\alpha \cap A) \neq \emptyset$. Then there exists an element $\alpha \in \Gamma$ such that $A \cap A_\alpha \neq \emptyset$. By theorem 2.2.1, $A = A_\alpha$, which is a contradiction to the fact that $A \neq A_\alpha$ for any $\alpha \in \Gamma$. Hence $(\bigcup_{\alpha \in \Gamma} A_\alpha) \cap A = \emptyset$.

Theorem 2.2.8 A proper nonempty subset $A$ of a topological space $(X, \tau)$ is maximal $gsg$-closed if and only if $X - A$ is minimal $gsg$-open.

Proof : Let $A$ be a maximal $gsg$-closed set of a topological space $(X, \tau)$. Suppose $X - A$ is not a minimal $gsg$-open set. Then there exists a nonempty $gsg$-open set $B$ such that $B \subset X - A$. That is, $A \subset X - B$ and $X - B$ is a $gsg$-closed set. This is a contradiction to the fact that $A$ is a maximal $gsg$-closed set.

Conversely, Let $X - A$ is a minimal $gsg$-open set. Suppose $A$ is not a maximal $gsg$-closed set. Then there exists a $gsg$-closed set $B \neq A$ such that $A \subset B \neq X$. That is, $\emptyset \neq X - B \subset X - A$ and $X - B$ is a $gsg$-open set. This contradicts the fact that $X - A$ is a minimal $gsg$-open set. Hence $A$ is a maximal $gsg$-closed set.

Theorem 2.2.9 Let $(X, \tau)$ be a topological space and $A, B \subseteq X$.

i. Let $A$ be maximal $gsg$-closed set and $B$ be a $gsg$-closed set. Then $A \cup B = X$ or $B \subseteq A$. 
ii. Let $A$ and $B$ be maximal $gsg$-closed sets. Then $A \cup B = X$ or $A = B$.

**Proof:**

i. Let $A$ be a maximal $gsg$-closed set and $B$ be a $gsg$-closed set. If $A \cup B = X$, then there is nothing to prove.

If $A \cup B \neq X$. Then $A \subseteq A \cup B$ and $A \cup B$ is $gsg$-closed.

Therefore $A \cup B = A$, which implies $B \subseteq A$.

ii. Let $A$ and $B$ be maximal $gsg$-closed sets. If $A \cup B \neq X$, then $A \subseteq B$ and $B \subseteq A$ by (i). Therefore $A = B$.

**Theorem 2.2.10** Let $A$ be a maximal $gsg$-closed set of a topological space $(X, \tau)$. If $x \in A$ then for any $gsg$-closed set $B$ containing $x$, $A \cup B = X$ or $B \subseteq A$.

**Proof:** Let $A$ be a maximal $gsg$-closed set and $x \in A$. Suppose there exists $gsg$-closed set $B$ containing $x$ such that $A \cup B \neq X$. Then $A \subseteq A \cup B$ and $A \cup B$ is a $gsg$-closed, as the finite union of $gsg$-closed set is a $gsg$-closed set. Since $A$ is a maximal $gsg$-closed, $A \cup B = A$. Hence $B \subseteq A$.

**Theorem 2.2.11** Let $A_\alpha, A_\beta, A_\nu$ be maximal $gsg$-closed sets of a topological space $(X, \tau)$ such that $A_\alpha \neq A_\beta$. If $A_\alpha \cap A_\beta \subset A_\nu$, then either $A_\alpha = A_\nu$ or $A_\beta = A_\nu$.

**Proof:** Given that $A_\alpha \cap A_\beta \subset A_\nu$. If $A_\alpha = A_\nu$, then there is nothing to prove. But if $A_\alpha \neq A_\nu$, then it is need to prove $A_\beta = A_\nu$. Now
\[ A_\beta \cap A_v = A_\beta \cap (A_v \cap X) \]
\[ = A_\beta \cap (A_v \cap (A_\alpha \cup A_\beta)), \text{ by theorem 2.2.9.} \]
\[ = A_\beta \cap ((A_v \cap A_\alpha) \cup (A_v \cap A_\beta)) \]
\[ = (A_\beta \cap A_v \cap A_\alpha) \cup (A_\beta \cap A_v \cap A_\beta) \]
\[ = (A_\alpha \cap A_\beta) \cup (A_v \cap A_\beta), \text{ (since } A_\alpha \cap A_\beta \subset A_v\text{) } \]
\[ = (A_\alpha \cup A_v) \cap A_\beta \]
\[ = X \cap A_\beta, \text{ by theorem 2.2.9.} \]
\[ = A_\beta. \]

That is, \((A_\beta \cap A_v) = A_\beta\) which implies \(A_\beta \subset A_v\). Since \(A_\beta\) and \(A_v\) are maximal \(gsg\)-closed sets, \(A_\beta = A_v\). Therefore \(A_\beta = A_v\).

**Theorem 2.2.12** Let \(A_\alpha, A_\beta\) and \(A_v\) be maximal \(gsg\)-closed sets in a topological space \((X, \tau)\) which are different from each other. Then \((A_\alpha \cap A_\beta) \subsetneq (A_\alpha \cap A_v)\).

**Proof:** Let \((A_\alpha \cap A_\beta) \subset (A_\alpha \cap A_v)\). Then \((A_\alpha \cap A_\beta) \cup (A_v \cap A_\beta) \subset (A_\alpha \cap A_v) \cup (A_v \cap A_\beta)\), which implies \((A_\alpha \cup A_v) \cap A_\beta \subset A_v \cap (A_\alpha \cup A_\beta)\). Since by theorem 2.2.9, \(A_\alpha \cup A_v = X\) and \(A_\alpha \cup A_\beta = X\) which implies \(X \cap A_\beta \subset A_v \cap X\). This gives \(A_\beta \subset A_v\). From the definition of maximal \(gsg\)-closed set, it follows that \(A_\beta = A_v\), which contradicts that \(A_\alpha, A_\beta\) and \(A_v\) are different from each other. Hence \((A_\alpha \cap A_\beta) \not\subset (A_\alpha \cap A_v)\).

**Theorem 2.2.13** Let \(A\) be a maximal \(gsg\)-closed set in a topological space \((X, \tau)\) and \(x \in A\). Then \(A = \bigcup \{B : B\text{ is a }gsg\text{-closed set containing }x\text{ such that }A \cup B \neq X\}\).
Proof: By theorem 2.2.10 and the fact that $A$ is a $g_{sg}$-closed set containing $x$, it is clear that, $A \subset \cup \{B : B$ is a $g_{sg}$-closed set containing $x \text{ such that } A \cup B \neq X\} \subset A$. Hence the result.

Theorem 2.2.14 Let $A$ be a proper nonempty cofinite $g_{sg}$-closed set in a topological space $(X, \tau)$. Then there exists (cofinite) maximal $g_{sg}$-closed set $B$ such that $A \subset B$.

Proof: If $A$ is maximal $g_{sg}$-closed, take $B = A$. If $A$ is not a maximal $g_{sg}$-closed set, then there exists (cofinite) $g_{sg}$-closed set $A_1$ such that $A \subset A_1 \neq X$. If $A_1$ is a maximal $g_{sg}$-closed set, take $B = A_1$. If $A_1$ is not a maximal $g_{sg}$-closed set, then there exists (cofinite) $g_{sg}$-closed set $A_2$ such that $A \subset A_1 \subset A_2 \neq X$. Continuing this process, there is a sequence of $g_{sg}$-closed sets such that $A \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots \neq X$. Since $A$ is a cofinite set, this process will end in a finite number of steps. Then finally there will be a maximal $g_{sg}$-closed set $B = B_n$ for some natural number $n$. Hence there exists (cofinite) maximal $g_{sg}$-closed set $B$ such that $A \subset B$.

Theorem 2.2.15 Let $A$ be a maximal $g_{sg}$-closed set in a topological space $(X, \tau)$. If $x \in X - A$ then $X - A \subset B$ for any $g_{sg}$-closed set $B$ containing $x$.

Proof: Let $A$ be a maximal $g_{sg}$-closed set and $x \in X - A$. Let $B$ be $g_{sg}$-closed set containing $x$. Then by theorem 2.2.9, either $A \cup B = X$ or $B \subset A$. But $B \not\subset A$, for any $g_{sg}$-closed set $B$ containing $x$. Therefore $X - A \subset B$. 

2.3 MINIMAL $gsg$-CLOSED SETS AND MAXIMAL $gsg$-OPEN SETS

In this section, the concepts of minimal $gsg$-closed sets and maximal $gsg$-open sets are introduced and the properties of minimal $gsg$-closed sets and maximal $gsg$-open sets are analyzed.

**Definition 2.3.1** A proper nonempty $gsg$-closed subset $A$ of $X$ is said to be a **minimal $gsg$-closed** if any $gsg$-closed set contained in $A$ is $\varnothing$ or $A$.

**Definition 2.3.2** A proper nonempty $gsg$-open subset $A$ of $X$ is said to be a **maximal $gsg$-open** if any $gsg$-open set containing $A$ is either $X$ or $A$.

**Example 2.3.1** Let $X = \{x, y, z, w\}$, $\tau = \{\varnothing, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}, X\}$. Then $\{w\}$ is both minimal closed and minimal $gsg$-closed and $\{x, y, z\}$ is both maximal open and maximal $gsg$-open.

**Theorem 2.3.1** Let $(X, \tau)$ be a topological space and $A, B \subseteq (X, \tau)$.

i. Let $A$ be minimal $gsg$-closed set and $B$ be a $gsg$-closed set. Then $A \cap B = \varnothing$ or $A \subset B$.

ii. Let $A$ and $B$ be minimal $gsg$-closed sets. Then $A \cap B = \varnothing$ or $A = B$.

**Proof:**

i. Let $A$ be a minimal $gsg$-closed set and $B$ be a $gsg$-closed set. If $A \cap B = \varnothing$, then there is nothing to prove. If $A \cap B \neq \varnothing$. Then $A \cap B \subset A$. Since $A$ is minimal $gsg$-closed set, $A \cap B = A$. Therefore $A \subset B$. 
ii. Let $A$ and $B$ be minimal $gsg$-closed sets. If $A \cap B \neq \varnothing$, then $A \subseteq B$ and $B \subseteq A$ by (i). Therefore $A = B$.

**Theorem 2.3.2** Let $A$ be a minimal $gsg$-closed set in a topological space $(X, \tau)$. If $x \in A$, then $A \subset B$ for some $gsg$-closed set $B$ containing $x$.

**Proof:** Let $x \in A$, a minimal $gsg$-closed set of a topological space $(X, \tau)$. Let $B$ be any $gsg$-closed set in $(X, \tau)$ containing $x$. Now $A \cap B \neq \varnothing$. Then by theorem 2.3.1, $A \subset B$.

**Corollary 2.3.1** Let $A$ be a minimal $gsg$-closed set in a topological space $(X, \tau)$. If $x \in A$, then $A = \cap \{B : B \text{ is } gsg\text{-closed set containing } x\}$ for any element $x$ in $A$.

**Theorem 2.3.3** For an $x \in X$ and a subset $A$ in $X$, $x \in gsg\text{-Int}(A)$ if and only if $U \cap A \neq \varnothing$ for every $gsg$-closed set $U$ containing $x$.

**Proof:** Let $x \in X, A \subset X$ and $x \in gsg\text{-Int}(A)$. The result is proved by the method of contradiction. Suppose there exists a $gsg$-closed set $U$ containing $x$ such that $U \cap A = \varnothing$. Then $A \subset X - U$ and $X - U$ is $gsg$-open. Then $gsg\text{-Int}(A) \subset X - U$. This gives $x \notin gsg\text{-Int}(A)$, which is contradiction. Hence $U \cap A \neq \varnothing$ for every $gsg$-closed set $U$ containing $x$.

Conversely, let $U \cap A \neq \varnothing$ for every $gsg$-closed set $U$ containing $x$. Again the result is proved by the method of contradiction. Suppose $x \notin gsg\text{-Int}(A)$. Then there exists a $gsg$-open subset $V$ containing $A$ such that $x \notin V$. Then $x \in X - V$ and $X - V$ $gsg$-closed. Also $(X - V) \cap A = \varnothing$, which is a contradiction. Hence $x \in gsg\text{-Int}(A)$.
Theorem 2.3.4 Let \( A \) be a nonempty \( gsg \)-closed set in a topological space \((X, \tau)\). Then the following are equivalent:

1. \( A \) is a minimal \( gsg \)-closed set.
2. \( A \subset gsg\text{-}Int(U) \), for any nonempty subset \( U \) of \( A \).
3. \( gsg\text{-}Int(A) = gsg\text{-}Int(U) \), for any nonempty subset \( U \) of \( A \).

Proof: (1) \( \Rightarrow \) (2): Let \( A \) be minimal \( gsg \)-closed set. Let \( x \in A \) and \( U \) be a nonempty subset of \( A \). By theorem 2.3.2, there exists a \( gsg \)-closed set \( B \) containing \( x \) such that \( A \subset B \). Then \( U \subset A \subset B \) which implies \( U \subset B \). Now \( U = U \cap A \subset U \cap B \). Since \( U \) is nonempty, \( U \cap B \neq \emptyset \). Since \( B \) is any \( gsg \)-closed set containing \( x \), by the theorem 2.2.3, \( x \in gsg\text{-}Int(U) \). That is, \( x \in A \) implies \( x \in gsg\text{-}Int(U) \). Hence \( A \subset gsg\text{-}Int(U) \), for any nonempty subset \( U \) of \( A \).

(2) \( \Rightarrow \) (3): Let \( U \) be a nonempty subset of \( A \) and \( A \subset gsg\text{-}Int(U) \). Then \( gsg\text{-}Int(U) \subset gsg\text{-}Int(A) \) and \( gsg\text{-}Int(A) \subset gsg\text{-}Int(U) \). Hence \( gsg\text{-}Int(A) = gsg\text{-}Int(U) \), for any nonempty subset \( U \) of \( A \).

(3) \( \Rightarrow \) (1): Let \( gsg\text{-}Int(A) = gsg\text{-}Int(U) \), for any nonempty subset \( U \) of \( A \). Suppose \( A \) is not a minimal \( gsg \)-closed set. Then there exists a nonempty \( gsg \)-closed set \( B \) such that \( B \subset A \) and \( B \neq A \). Now there exists an element \( x \in A \) such that \( x \notin B \), which implies \( x \in X - B \). That is, \( gsg\text{-}Int(\{x\}) \subset gsg\text{-}Int(X - B) = X - B \), as \( X - B \) is \( gsg \)-open in \( X \).

It follows that \( gsg\text{-}Int(\{x\}) \neq gsg\text{-}Int(A) \). This is a contradiction to the fact that \( gsg\text{-}Int(\{x\}) = gsg\text{-}Int(A) \), for any nonempty subset \( \{x\} \) of \( A \). Thus \( A \) is a minimal \( gsg \)-closed set.
Theorem 2.3.5 Let $B$ be a nonempty finite $gsg$-closed set in a topological space $(X, \tau)$. Then there exists at least one (finite) minimal $gsg$-closed set $A$ of $(X, \tau)$ such that $A \subset B$.

Proof: Let $B$ be a nonempty $gsg$-closed set. Then the following two cases arise:

1) $B$ is a minimal $gsg$-closed.

2) $B$ is not a minimal $gsg$-closed.

Case (1): If $B = A$, then the theorem is proved.

Case (2): If $B$ is not a minimal $gsg$-closed, then there exists a nonempty (finite) $gsg$-closed set $B_1$ such that $B_1 \subset B$. If $B_1$ is minimal $gsg$-closed, take $A = B_1$. If $B_1$ is not a minimal $gsg$-closed set, then there exists a nonempty (finite) $gsg$-closed set $B_2$ such that $B_2 \subset B_1 \subset B$. Continue this process and have a sequence of $gsg$-closed sets $\ldots \subset B_i \subset \ldots \subset B_2 \subset B_1 \subset B$. Since $B$ is finite, this process will end at finite number of steps. That is, for any natural number $k$, there is a minimal $gsg$-closed set $B_k$ such that $B_k = A$. Hence there exists at least one (finite) minimal $gsg$-closed set $A$ of $(X, \tau)$ such that $A \subset B$.

Corollary 2.3.2 Let $X$ be a locally finite space and $B$ be a nonempty $gsg$-closed set. Then there exists at least one (finite) minimal $gsg$-closed set $A$ such that $A \subset B$.

Proof: Let $X$ be a locally finite space and $B$ be nonempty $gsg$-closed set. Let $x \in B$. Since $X$ is finite, there is a finite closed set $B_x$ such that $x \in B_x$. Then $B \cap B_x$ is a nonempty finite $gsg$-closed set. By theorem 2.3.5, there exists at least one (finite) minimal $gsg$-closed set $A$ such that $A \subset B \cap B_x$. 
That is, \( A \subset B \). Hence there exists at least one (finite) minimal \( g_{sg} \)-closed set \( A \) such that \( A \subset B \).

**Theorem 2.3.6** Let \( A \) and \( A_\alpha \) be minimal \( g_{sg} \)-closed sets for any \( \alpha \in \Gamma \). If \( A \subset \bigcup_{\alpha \in \Gamma} A_\alpha \), then there exists an element \( \alpha \in \Gamma \) such that \( A = A_\alpha \).

**Proof:** Let \( A \subset \bigcup_{\alpha \in \Gamma} A_\alpha \). Then \( A \cap (\bigcup_{\alpha \in \Gamma} A_\alpha) = A \). That is, \( \bigcup_{\alpha \in \Gamma} (A \cap A_\alpha) = A \). Also by theorem 2.3.1, \( A \cap A_\alpha = \varnothing \) or \( A = A_\alpha \) for any \( \alpha \in \Gamma \). Hence there exists an element \( \alpha \in \Gamma \) such that \( A = A_\alpha \).

**Theorem 2.3.7** Let \( A \) and \( A_\alpha \) be minimal \( g_{sg} \)-closed sets for any \( \alpha \in \Gamma \). If \( A \neq A_\alpha \) for any \( \alpha \in \Gamma \), then \( (\bigcup_{\alpha \in \Gamma} A_\alpha) \cap A = \varnothing \).

**Proof:** Suppose \( (\bigcup_{\alpha \in \Gamma} A_\alpha) \cap A \neq \varnothing \). That is, \( \bigcup_{\alpha \in \Gamma} (A_\alpha \cap A) \neq \varnothing \). Then there exists an element \( \alpha \in \Gamma \) such that \( A \cap A_\alpha \neq \varnothing \). By theorem 2.3.1, \( A = A_\alpha \), which is a contradiction to the fact that \( A \neq A_\alpha \) for any \( \alpha \in \Gamma \). Hence \( (\bigcup_{\alpha \in \Gamma} A_\alpha) \cap A = \varnothing \).

**Theorem 2.3.8** A proper nonempty subset \( A \) of \( X \) is maximal \( g_{sg} \)-open if and only if \( X - A \) is minimal \( g_{sg} \)-closed.

**Proof:** Let \( A \) be a maximal \( g_{sg} \)-open set. Suppose \( X - A \) is not a minimal \( g_{sg} \)-closed set. Then there exists a nonempty \( g_{sg} \)-closed set \( B \) such that \( B \subset X - A \). That is, \( A \subset X - B \) and \( X - B \) is a \( g_{sg} \)-open set. This is a contradiction to the fact that \( A \) is a maximal \( g_{sg} \)-open set.

Conversely, Let \( X - A \) is a minimal \( g_{sg} \)-closed set. Suppose \( A \) is not a maximal \( g_{sg} \)-open set. Then there exists a \( g_{sg} \)-open set \( B \neq A \) such that \( A \subset B \neq X \). That is, \( \varnothing \neq X - B \subset X - A \) and \( X - B \) is a \( g_{sg} \)-closed set. This contradicts the fact that \( X - A \) is a minimal \( g_{sg} \)-closed set. Hence \( A \) is a maximal \( g_{sg} \)-open set.
Theorem 2.3.9 Let $(X, \tau)$ be a topological space and $A, B \subseteq (X, \tau)$.

i. Let $A$ be maximal $gsg$-open set and $B$ be a $gsg$-open set. Then $A \cup B = X$ or $B \subset A$.

ii. Let $A$ and $B$ be maximal $gsg$-open sets. Then $A \cup B = X$ or $A = B$.

Proof:

i. Let $A$ be a maximal $gsg$-open set and $B$ be a $gsg$-open set. If $A \cup B = X$, then there is nothing to prove. If $A \cup B \neq X$, then $A \subset A \cup B$ and $A \cup B$ is $gsg$-open. Then $A \cup B = A$, which implies $B \subset A$.

ii. Let $A$ and $B$ be maximal $gsg$-open sets. If $A \cup B \neq X$, then $A \subset B$ and $B \subset A$, by (i). Therefore $A = B$.

Theorem 2.3.10 Let $A$ be a maximal $gsg$-open set. If $x \in A$ then for any $gsg$-open set $B$ containing $x$, $A \cup B = X$ or $B \subset A$.

Proof: Let $A$ be a maximal $gsg$-open set and $x \in A$. Suppose there exists $gsg$-open set $B$ containing $x$ such that $A \cup B \neq X$. Then $A \subset A \cup B$ and $A \cup B$ is a $gsg$-open. Since $A$ is a maximal $gsg$-open, $A \cup B = A$. Hence $B \subset A$.

Theorem 2.3.11 Let $A_\alpha, A_\beta, A_\gamma$ be maximal $gsg$-open sets such that $A_\alpha \neq A_\beta$. If $A_\alpha \cap A_\beta \subset A_\gamma$, then either $A_\alpha = A_\gamma$ or $A_\beta = A_\gamma$.

Proof: Given that $A_\alpha \cap A_\beta \subset A_\gamma$. If $A_\alpha = A_\gamma$, then there is nothing to prove. But if $A_\alpha \neq A_\gamma$ then it is need to prove $A_\beta = A_\gamma$. Now
\[ A_\beta \cap A_v = A_\beta \cap (A_v \cap X) \]

\[ = A_\beta \cap (A_v \cap (A_\alpha \cup A_\beta)), \text{ by theorem 2.3.9.} \]

\[ = A_\beta \cap (A_v \cap A_\alpha) \cup (A_v \cap A_\beta) \]

\[ = (A_\beta \cap A_v \cap A_\alpha) \cup (A_\beta \cap A_v \cap A_\beta) \]

\[ = (A_\alpha \cap A_\beta) \cup (A_v \cap A_\beta) \text{ (since } A_\alpha \cap A_\beta \subset A_v \text{)} \]

\[ = (A_\alpha \cup A_v) \cap A_\beta \]

\[ = X \cap A_\beta, \text{ by theorem 2.3.9.} \]

\[ = A_\beta. \]

That is, \((A_\beta \cap A_v) = A_\beta\) which implies \(A_\beta \subset A_v\). Since \(A_\beta\) and \(A_v\) are maximal \(gsg\)-open sets, \(A_\beta = A_v\). Therefore \(A_\beta = A_v\).

**Theorem 2.3.12** Let \(A_\alpha, A_\beta\) and \(A_v\) be maximal \(gsg\)-open sets which are different from each other. Then \((A_\alpha \cap A_\beta) \not\subset (A_\alpha \cap A_v)\).

**Proof:** Let \((A_\alpha \cap A_\beta) \subset (A_\alpha \cap A_v)\). Then \((A_\alpha \cap A_\beta) \cup (A_v \cap A_\beta) \subset (A_\alpha \cap A_v) \cup (A_v \cap A_\beta)\), which implies \((A_\alpha \cup A_v) \cap A_\beta \subset A_v \cap (A_\alpha \cup A_\beta)\). Since by theorem 2.3.9, \(A_\alpha \cup A_v = X\) and \(A_\alpha \cup A_\beta = X\) which implies \(X \cap A_\beta \subset A_v \cap X\), which gives \(A_\beta \subset A_v\). From the definition of maximal \(gsg\)-open set it follows that \(A_\beta = A_v\), which is contradicts that \(A_\alpha, A_\beta\) and \(A_v\) are different from each other. Hence \((A_\alpha \cap A_\beta) \not\subset (A_\alpha \cap A_v)\).

**Theorem 2.3.13** Let \(A\) be a maximal \(gsg\)-open set and \(x \in A\). Then \(A = \cup \{B : B \text{ is a } gsg\text{-open set containing } x \text{ such that } A \cup B \neq X\}\).
Proof: By theorem 2.3.10 and the fact that $A$ is a $gsg$-open set containing $x$, it is clear that, $A \subset \bigcup \{B: B \text{ is a } gsg\text{-open set containing } x \text{ such that } A \cup B \neq X\} \subset A$. Hence the result.

**Theorem 2.3.14** Let $A$ be a proper nonempty cofinite $gsg$-open set. Then there exists (cofinite) maximal $gsg$-open set $B$ such that $A \subset B$.

Proof: If $A$ is maximal $gsg$-open, take $B = A$. If $A$ is not a maximal $gsg$-open set, then there exists (cofinite) $gsg$-open set $A_1$ such that $A \subset A_1 \neq X$. If $A_1$ is a maximal $gsg$-open set, take $B = A_1$. If $A_1$ is not a maximal $gsg$-open set, then there exists (cofinite) $gsg$-open set $A_2$ such that $A \subset A_1 \subset A_2 \neq X$. Continuing this process, there will be a sequence of $gsg$-open sets such that $A \subset A_1 \subset A_2 \subset \ldots \subset A_k \subset \ldots \neq X$. Since $A$ is a cofinite set, this process will end in a finite number of steps. Then finally there is a maximal $gsg$-open set $B = B_n$ for some natural number $n$.

**Theorem 2.3.15** Let $A$ be a maximal $gsg$-open set. If $x \in X - A$ then $X - A \subset B$ for any $gsg$-open set $B$ containing $x$.

Proof: Let $A$ be a maximal $gsg$-open set and $x \in X - A$. Let $B$ be $gsg$-open set containing $x$. Then by theorem 2.3.14, either $A \cup B = X$ or $B \subset A$. But $B \not\subset A$, for any $gsg$-open set $B$ containing $x$. Therefore $X - A \subset B$.

### 2.4 $gsg$-SEMI-MAXIMAL OPEN SETS AND $gsg$-SEMI-MINIMAL CLOSED SETS

In this section, the new sets namely $gsg$-semi-maximal open sets and $gsg$-semi-minimal closed sets are introduced and the properties of $gsg$-semi-maximal open sets and $gsg$-semi-minimal closed sets are studied.
Definition 2.4.1 A set $A$ in a topological space $(X, \tau)$ is said to be \textit{gsg-semi-maximal open} if there exists a maximal $gsg$-open set $U$ such that $U \subset A \subset Cl(U)$. A set $A$ of a topological space $(X, \tau)$ is $gsg$-semi-maximal open if and only if $X - A$ is \textit{gsg-semi-minimal closed sets}. That is, the complement of $gsg$-semi-maximal open sets is called as $gsg$-semi-minimal closed sets.

Example 2.4.1 Let $X = \{x, y, z\}, \tau = \{\varnothing, \{x\}, X\}$. Then $(X, \tau)$ is clearly a topological space. Here $\{x, y\}, \{x, z\}$ are $gsg$-semi-maximal open and $\{z\}, \{y\}$ are $gsg$-semi-minimal closed.

Remark 2.4.1 Every maximal $gsg$-open (resp. minimal $gsg$-closed) set is $gsg$-semi-maximal open (resp. $gsg$-semi-minimal closed) set.

Theorem 2.4.1 Let $A$ be a $gsg$-semi-maximal open set of a topological space $(X, \tau)$ and $A \subset B \subset Cl(A)$, then $B$ is a $gsg$-semi-maximal open set of $(X, \tau)$.

Proof: Since $A$ is a $gsg$-semi-maximal open set of $(X, \tau)$, then there exists a maximal $gsg$-open set $U$ of $(X, \tau)$ such that $U \subset A \subset Cl(U)$. Then $U \subset A \subset B \subset Cl(A) \subset Cl(U)$. That is, $U \subset B \subset Cl(U)$. Thus $B$ is a $gsg$-semi-maximal open set of $X$.

Theorem 2.4.2 Let $A$ be a $gsg$-semi-minimal closed set of a topological space $(X, \tau)$ if and only if there exists a minimal $gsg$-closed set $B$ in $(X, \tau)$ such that $Int(B) \subset A \subset B$.

Proof: Suppose $A$ is $gsg$-semi-minimal closed set of $(X, \tau)$. By definition $X - A$ is $gsg$-semimaximal open set of $(X, \tau)$. Then there exists a maximal $gsg$-open set $U$ such that $U \subset X - A \subset Cl(U)$. That is, $Int(X - U) = X - Cl(U) \subset A \subset X - U$. Let $B = X - U$, so that $B$ is a minimal $gsg$-closed set in $X$ such that $Int(B) \subset A \subset B$. 
Conversely, Suppose that there exists a minimal $gsg$-closed set $B$ in $X$ such that $\text{Int}(B) \subset A \subset B$. Hence $X - B \subset X - A \subset X - \text{Int}(B) = \text{Cl}(X - B)$. That is, there exists a maximal $gsg$-open set $U = X - B$ such that $U \subset X - A \subset \text{Cl}(U)$. This implies $X - A$ is $gsg$-semi-maximal open in $(X, \tau)$. Hence $A$ is $gsg$-semi-minimal closed in $(X, \tau)$.

**Theorem 2.4.3** Let $B$ be a $gsg$-semi-minimal closed set of $(X, \tau)$. If $\text{Int}(B) \subset A \subset B$, then $A$ is also $gsg$-semi-minimal closed in $(X, \tau)$.

**Proof:** Let $B$ be a $gsg$-semi-minimal closed set of $(X, \tau)$. Then there exists a minimal $gsg$-closed set $U$ such that $\text{Int}(U) \subset B \subset U$. Since $\text{Int}(B) \subset A \subset B$, $\text{Int}(U) \subset \text{Int}(B) \subset A \subset B \subset U$. That is, $\text{Int}(U) \subset A \subset U$. Therefore $A$ is a $gsg$-semi-minimal closed set in $(X, \tau)$.

**Theorem 2.4.4** Let $Y$ be any subspace of $(X, \tau)$ and $A \subset Y$. If $A$ is a $gsg$-semi-maximal open set of $(X, \tau)$, then $A$ is also a $gsg$-semi-maximal open set of $Y$.

**Proof:** Since $A$ is $gsg$-semi-maximal open set of $(X, \tau)$, there exists a maximal $gsg$-open set $U$ such that $U \subset A \subset \text{Cl}(U)$. Hence $U$ is subset of $Y$. Since $U$ is maximal $gsg$-open in $(X, \tau)$, $Y \cap U = U$ is maximal $gsg$-open in $Y$ and $U = Y \cap U \subset Y \cap A \subset Y \cap \text{Cl}(U)$. That is, $U \subset A \subset \text{Cl}_Y(U)$. Hence $A$ is a $gsg$-semi-maximal open set of $Y$.

**Theorem 2.4.5** Let $A_i$ be a $gsg$-semi-maximal open set of $X_i$, $(i = 1, 2)$, then $A_1 \times A_2$ is a $gsg$-semi-maximal open set of $X_1 \times X_2$.

**Proof:** For $i = 1, 2$ there exists a maximal $gsg$-open set $U_i$ such that $U_i \subset A_i \subset \text{Cl}_{X_i}(U_i)$. Therefore $U_1 \times U_2 \subset A_1 \times A_2 \subset \text{Cl}_{X_1}(U_1) \times \text{Cl}_{X_2}(U_2) = \text{Cl}_{X_1 \times X_2}(U_1 \times U_2)$. Hence $A_1 \times A_2$ is $gsg$-semi-maximal open in $X_1 \times X_2$. 

2.5 CONTINUOUS, IRRESOLUTE MAPPINGS AND T-SPACES

In this section the continuous, irresolute mappings and T-spaces in the maximal $gsg$-closed and minimal $gsg$-closed sets are defined and its various properties are established.

**Definition 2.5.1.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. A map $f : (X, \tau) \to (Y, \sigma)$ is called

i. **minimal $gsg$-continuous** (in short, $\text{min-}gsg$-continuous) if $f^{-1}(A)$ is an $gsg$-closed set in $(X, \tau)$ for every minimal closed set $A$ in $(Y, \sigma)$.

ii. **maximal $gsg$-continuous** (in short, $\text{max-}gsg$-continuous) if $f^{-1}(A)$ is an $gsg$-closed set in $(X, \tau)$ for every maximal closed set $A$ in $(Y, \sigma)$.

iii. **minimal $gsg$-irresolute** (in short, $\text{min-}gsg$-irresolute) if $f^{-1}(A)$ is minimal $gsg$-closed set in $(X, \tau)$ for every minimal closed set $A$ in $(Y, \sigma)$.

iv. **maximal $gsg$-irresolute** (in short, $\text{max-}gsg$-irresolute) if $f^{-1}(A)$ is maximal $gsg$-closed set in $(X, \tau)$ for every maximal closed set $A$ in $(Y, \sigma)$.

v. **minimal-maximal $gsg$-continuous** (in short, $\text{min-max-}gsg$ continuous) $f^{-1}(A)$ is maximal $gsg$-closed set in $(X, \tau)$ for every minimal closed set $A$ in $(Y, \sigma)$.

vi. **maximal-minimal $gsg$-continuous** (in short, $\text{max-min-}gsg$ continuous) $f^{-1}(A)$ is minimal $gsg$-closed set in $(X, \tau)$ for every maximal closed set $A$ in $(Y, \sigma)$.
**Definition 2.5.2** A topological space \((X, \tau)\) is called a

i. **\(T_{\text{min-gsg}}\) space** if every \(gsg\)-closed set in it is minimal closed.

ii. **\(T_{\text{max-gsg}}\) space** if every \(gsg\)-closed set in it is maximal closed.

iii. **\(\text{Min-}\text{T}_{\text{gsg}}\) space** if every minimal \(gsg\)-closed set in it is minimal closed.

iv. **\(\text{Max-}\text{T}_{\text{gsg}}\) space** if every maximal \(gsg\)-closed set in it is maximal closed.

**Theorem 2.5.1** Every \(gsg\)-continuous map is minimal \(gsg\)-continuous.

**Proof:** Let \(f : (X, \tau) \to (Y, \sigma)\) be a \(gsg\)-continuous map. To prove \(f\) is minimal \(gsg\)-continuous, let \(A\) be any minimal closed set in \((Y, \sigma)\). Since every minimal closed set is an closed set, \(A\) is an closed set in \((Y, \sigma)\). Since \(f\) is \(gsg\)-continuous, \(f^{-1}(A)\) is an \(gsg\)-closed set in \((X, \tau)\). Hence \(f\) is a minimal \(gsg\)-continuous.

**Theorem 2.5.2** Every \(gsg\)-continuous map is maximal \(gsg\)-continuous.

**Proof:** The proof is similar to that of theorem 2.5.1.

**Theorem 2.5.3** Let \(f : (X, \tau) \to (Y, \sigma)\) be a minimal \(gsg\)-continuous and \((Y, \sigma)\) be a \(T_{\text{min-gsg}}\) space. Then \(f\) is a \(gsg\)-continuous.

**Proof:** Let \(f\) be a minimal \(gsg\)-continuous. Let \(A\) be a closed set in \((Y, \sigma)\). Since every closed set is \(gsg\)-closed, \(A\) is \(gsg\)-closed. By hypothesis, \(Y\) is \(T_{\text{min-gsg}}\) space, it follows that \(A\) is a minimal closed set in
(X, \tau). Since f is minimal \(gsg\)-continuous, \(f^{-1}(A)\) is an \(gsg\)-closed in \((X, \tau)\). Therefore f is a \(gsg\)-continuous.

**Theorem 2.5.4** Let \(f : (X, \tau) \to (Y, \sigma)\) be a maximal \(gsg\)-continuous and let Y be a \(T_{max-gsg}\) space. Then f is a \(gsg\)-continuous.

**Proof:** The proof is similar to that of theorem 2.5.3.

**Theorem 2.5.5** Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces. A map \(f : (X, \tau) \to (Y, \sigma)\) is minimal \(gsg\)-continuous if and only if the inverse image of each maximal open set in \((Y, \sigma)\) is a \(gsg\)-open set in \((X, \tau)\).

**Proof:** Let \(A\) be maximal open set in \((Y, \sigma)\). Then \(X - A\) is minimal closed set in \((Y, \sigma)\). Since f is minimal \(gsg\)-continuous, \(f^{-1}(X - A) = X - f^{-1}(A)\), is \(gsg\)-closed set in \((X, \tau)\). That is, \(f^{-1}(A)\) is \(gsg\)-open set in \((X, \tau)\). Hence the inverse image of each maximal open set in \((Y, \sigma)\) is a \(gsg\)-open set in \((X, \tau)\).

Conversely, let \(A\) be minimal closed set in \((Y, \sigma)\). Then \(X - A\) is maximal open set in \((Y, \sigma)\). By hypothesis, \(f^{-1}(X - A) = X - f^{-1}(A)\), is \(gsg\)-open set in \((X, \tau)\). That is, \(f^{-1}(A)\) is \(gsg\)-closed set in \((X, \tau)\). Hence f is minimal \(gsg\)-continuous.

**Theorem 2.5.6** Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces. A map \(f : (X, \tau) \to (Y, \sigma)\) is maximal \(gsg\)-continuous if and only if the inverse image of each minimal open set in \((Y, \sigma)\) is a \(gsg\)-open set in \((X, \tau)\).

**Proof:** The proof is similar to that of theorem 2.5.5.
**Theorem 2.5.7** Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces and \(A\) be a nonempty subset of \((X, \tau)\). If \(f : (X, \tau) \to (Y, \sigma)\) is minimal \(g_{sg}\)-continuous then the restriction map \(f_A : A \to (Y, \sigma)\) is a minimal \(g_{sg}\)-continuous.

**Proof**: Let \(f : (X, \tau) \to (Y, \sigma)\) is minimal \(g_{sg}\)-continuous map. Let \(B\) be any minimal closed set in \((Y, \sigma)\). Since \(f\) is minimal \(g_{sg}\)-continuous, \(f^{-1}(B)\) is an \(g_{sg}\)-closed set in \((X, \tau)\). But \(f_A^{-1}(B) = A \cap f^{-1}(B)\) and \(A \cap f^{-1}(B)\) is a \(g_{sg}\)-closed set in \(A\). Therefore \(f_A\) is a minimal \(g_{sg}\)-continuous.

**Theorem 2.5.8** Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces and let \(A\) be a nonempty subset of \((X, \tau)\). If \(f : (X, \tau) \to (Y, \sigma)\) is maximal \(g_{sg}\)-continuous then the restriction map \(f_A : A \to (Y, \sigma)\) is a maximal \(g_{sg}\)-continuous.

**Proof**: The proof is similar to that of theorem 2.5.4.

**Theorem 2.5.9** If \(f : (X, \tau) \to (Y, \sigma)\) is \(g_{sg}\)-continuous map, \((Y, \sigma)\) is a \(T_{g_{sg}}\)-space and \(g : (Y, \sigma) \to (Z, \eta)\) is minimal \(g_{sg}\)-continuous map. Then \(g \circ f : (X, \tau) \to (Z, \eta)\) is a minimal \(g_{sg}\)-continuous.

**Proof**: Let \(A\) be any minimal closed set in \((Z, \eta)\). Since \(g\) is minimal \(g_{sg}\)-continuous, \(g^{-1}(A)\) is an \(g_{sg}\)-closed set in \((Y, \sigma)\). Since \((Y, \sigma)\) is a \(T_{g_{sg}}\)-space, \(g^{-1}(A)\) closed in \((Y, \sigma)\). Again, since \(f\) is \(g_{sg}\)-continuous, \(f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)\) is a \(g_{sg}\)-closed set in \((X, \tau)\). Hence \(g \circ f\) is a minimal \(g_{sg}\)-continuous.

**Theorem 2.5.10** If \(f : (X, \tau) \to (Y, \sigma)\) is \(g_{sg}\)-continuous map, \((Y, \sigma)\) is a \(T_{g_{sg}}\)-space and \(g : (Y, \sigma) \to (Z, \eta)\) is maximal \(g_{sg}\)-continuous maps, then \(g \circ f : (X, \tau) \to (Z, \eta)\) is a maximal \(g_{sg}\)-continuous.
Proof: The proof is similar to that of theorem 2.5.9.

Theorem 2.5.11 Every minimal gsg-irresolve map is minimal gsg-continuous map.

Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) be a minimal gsg-irresolve map. Let \( A \) be any minimal closed set in \((Y, \sigma)\). Since \( f \) is minimal gsg-irresolve, \( f^{-1}(A) \) is a minimal gsg-closed set in \((X, \tau)\). That is, \( f^{-1}(A) \) is a gsg-closed set in \((X, \tau)\). Hence \( f \) is a minimal gsg-continuous.

Theorem 2.5.12 Every maximal gsg-irresolve map is maximal gsg-continuous map.

Proof: The proof is similar to that of theorem 2.5.11.

Theorem 2.5.13 Let \( f : (X, \tau) \to (Y, \sigma) \) be a minimal gsg-irresolve and let \((Y, \sigma)\) be a \( T_{min-gsg} \) space. Then \( f \) is gsg-continuous.

Proof: Let \( A \) be closed set in \((Y, \sigma)\). Since every closed set is gsg-closed set, \( A \) is gsg-closed set in \((Y, \sigma)\). Since \((Y, \sigma)\) is \( T_{min-gsg} \) space, \( A \) is minimal closed in \((Y, \sigma)\). The mapping \( f : (X, \tau) \to (Y, \sigma) \) is minimal gsg-irresolve implies \( f^{-1}(A) \) is minimal gsg-closed set in \((X, \tau)\). That is, \( f^{-1}(A) \) is gsg-closed set in \((X, \tau)\). Hence \( f \) is gsg-continuous.

Theorem 2.5.14 Let \( f : (X, \tau) \to (Y, \sigma) \) be a maximal gsg-irresolve and let \((Y, \sigma)\) be a \( T_{max-gsg} \) space. Then \( f \) is a gsg-continuous.

Proof: The proof is similar to that of theorem 2.5.13.

Theorem 2.5.15 Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces. A map \( f : (X, \tau) \to (Y, \sigma) \) is minimal gsg-irresolve if and only if the inverse image of each maximal open set in \((Y, \sigma)\) is a maximal gsg-open in \((X, \tau)\).
Proof: Let $A$ be maximal open set in $(Y, \sigma)$. Then $X - A$ is minimal closed set in $(Y, \sigma)$. Since $f$ is minimal $gsg$-irresolute, $f^{-1}(X - A) = X - f^{-1}(A)$, is minimal $gsg$-closed set in $(X, \tau)$. That is, $f^{-1}(A)$ is maximal $gsg$-open set in $(X, \tau)$. Hence the inverse image of each maximal open set in $(Y, \sigma)$ is a maximal $gsg$-open set in $(X, \tau)$.

Conversely, let $A$ be minimal closed set in $(Y, \sigma)$. Then $X - A$ is maximal open set in $(Y, \sigma)$. By hypothesis, $f^{-1}(X - A) = X - f^{-1}(A)$, is maximal $gsg$-open set in $(X, \tau)$. That is, $f^{-1}(A)$ is minimal $gsg$-closed set in $(X, \tau)$. Hence $f$ is minimal $gsg$-irresolute.

Theorem 2.5.16 Let $(X, \tau)$ and $(Y, \sigma)$ be the topological spaces. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is maximal $gsg$-irresolute if and only if the inverse image of each minimal open set in $(Y, \sigma)$ is a minimal $gsg$-open in $(X, \tau)$.

Proof: The proof is similar to that of theorem 2.5.15.

Theorem 2.5.17 If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are minimal $gsg$-irresolute maps and $(Y, \sigma)$ is a $Min-T_{gsg}$ space, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a minimal $gsg$-irresolute map.

Proof: Let $A$ be any minimal closed set in $(Z, \eta)$. Since $g$ is minimal $gsg$-irresolute, $g^{-1}(A)$ is a minimal $gsg$-closed set in $(Y, \sigma)$. Since $(Y, \sigma)$ is a $Min-T_{gsg}$ space, $g^{-1}(A)$ is a minimal closed set in $(Y, \sigma)$. Again, since $f$ is minimal $gsg$-irresolute, $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is a minimal $gsg$-closed set in $(X, \tau)$. Therefore $g \circ f$ is a minimal $gsg$-irresolute.

Theorem 2.5.18 If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are maximal $gsg$-irresolute maps and $(Y, \sigma)$ is a $Max-T_{gsg}$ space, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a maximal $gsg$-irresolute map.
**Proof:** The proof is similar to that of theorem 2.5.17.

**Theorem 2.5.19** Every minimal-maximal $gsg$-continuous map is minimal $gsg$-continuous map.

**Proof:** Let $f : (X, \tau) \to (Y, \sigma)$ be a minimal-maximal $gsg$-continuous map. Let $A$ be any minimal closed set in $(Y, \sigma)$. Since $f$ is minimal-maximal continuous, $f^{-1}(A)$ is a maximal $gsg$-closed set in $(X, \tau)$. Since every maximal $gsg$-closed set is an $gsg$-closed set, $f^{-1}(A)$ is $gsg$-closed set in $(X, \tau)$ . Hence $f$ is a minimal $gsg$-continuous.

**Theorem 2.5.20** Every maximal-minimal $gsg$-continuous map is maximal $gsg$-continuous.

**Proof:** The proof is similar to that of theorem 2.5.19.

**Theorem 2.5.21** Let $f : (X, \tau) \to (Y, \sigma)$ be a minimal-maximal $gsg$-continuous and let $(Y, \sigma)$ be a $T_{min-gsg}$ space. Then $f$ is a $gsg$-continuous.

**Proof:** Let $A$ be closed set in $(Y, \sigma)$. Since every closed set is $gsg$-closed set, $A$ is $gsg$-closed set in $(Y, \sigma)$. Since $(Y, \sigma)$ is $T_{min-gsg}$ space, $A$ is minimal closed in $(Y, \sigma)$. The mapping $f$ is minimal-maximal $gsg$-continuous implies $f^{-1}(A)$ is maximal $gsg$-closed set in $(X, \tau)$. That is, $f^{-1}(A)$ is $gsg$-closed set in $(X, \tau)$. Hence $f$ is $gsg$-continuous.

**Theorem 2.5.22** Let $f : (X, \tau) \to (Y, \sigma)$ be a maximal-minimal $gsg$-continuous and let $(Y, \sigma)$ be a $T_{max-gsg}$ space. Then $f$ is a $gsg$-continuous.

**Proof:** The proof is similar to that of theorem 2.5.21.
Theorem 2.5.23 Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces. A map \(f : (X, \tau) \to (Y, \sigma)\) is minimal-maximal \(gsg\)-continuous if and only if the inverse image of each maximal open in \((Y, \sigma)\) is a minimal \(gsg\)-open set in \((X, \tau)\).

**Proof:** Let \(A\) be maximal open set in \((Y, \sigma)\). Then \(X - A\) is minimal closed set in \((Y, \sigma)\). Since \(f\) is minimal-maximal \(gsg\)-continuous, \(f^{-1}(X - A) = X - f^{-1}(A)\), is maximal \(gsg\)-closed set in \((X, \tau)\). That is, \(f^{-1}(A)\) is minimal \(gsg\)-open set in \((X, \tau)\). Hence the inverse image of each maximal open set in \((Y, \sigma)\) is a minimal \(gsg\)-open set in \((X, \tau)\).

Conversely, let \(A\) be minimal closed set in \((Y, \sigma)\). Then \(X - A\) is maximal open set in \((Y, \sigma)\). By hypothesis, \(f^{-1}(X - A) = X - f^{-1}(A)\), is minimal \(gsg\)-open set in \((X, \tau)\). That is, \(f^{-1}(A)\) is maximal \(gsg\)-closed set in \((X, \tau)\). Hence \(f\) is minimal-maximal \(gsg\)-continuous.

Theorem 2.5.24 Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces. A map \(f : (X, \tau) \to (Y, \sigma)\) is maximal-minimal \(gsg\)-continuous if and only if the inverse image of each minimal open set in \((Y, \sigma)\) is a maximal \(gsg\)-open set in \((X, \tau)\).

**Proof:** The proof is similar to that of theorem 2.5.23.

Theorem 2.5.25 If \(f : (X, \tau) \to (Y, \sigma)\) is maximal \(gsg\)-irresolute, \((Y, \sigma)\) is a \(Max-T_{gsg}\) space and \(g : (Y, \sigma) \to (Z, \eta)\) is minimal-maximal \(gsg\)-continuous maps, then \(g \circ f : (X, \tau) \to (Z, \eta)\) is a minimal-maximal \(gsg\)-continuous map.

**Proof:** Let \(A\) be any minimal closed set in \((Z, \eta)\). Since \(g\) is minimal-maximal continuous, \(g^{-1}(A)\) is a maximal \(gsg\)-closed set in \((Y, \sigma)\). Since \((Y, \sigma)\) is a \(Max-T_{gsg}\) space, \(g^{-1}(A)\) is a maximal closed in \((Y, \sigma)\).
Again, since $f$ is maximal $gsg$- irresolute, $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is a maximal $gsg$-closed set in $(X, \tau)$. Hence $g \circ f$ is a minimal-maximal $gsg$-continuous.

**Theorem 2.5.26** If $f : (X, \tau) \to (Y, \sigma)$ is minimal $gsg$- irresolute, $(Y, \sigma)$ is $Min-T_{gsg}$ and $g : (Y, \sigma) \to (Z, \eta)$ is maximal-minimal $gsg$-continuous maps, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a maximal-minimal $gsg$-continuous.

**Proof:** The proof is similar to that of the theorem 2.5.25.

**Theorem 2.5.27** If $f : (X, \tau) \to (Y, \sigma)$ is maximal $gsg$-continuous and $(Y, \sigma)$ is a $Max-T_{gsg}$ space and $g : (Y, \sigma) \to (Z, \eta)$ is minimal-maximal $gsg$-continuous maps, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a minimal $gsg$-continuous.

**Proof:** The proof is similar to that of theorem 2.5.25 and making use of theorem 2.5.19.

**Theorem 2.5.28** If $f : (X, \tau) \to (Y, \sigma)$ is minimal $gsg$-continuous. $(Y, \sigma)$ is $Min-T_{gsg}$ space and $g : (Y, \sigma) \to (Z, \eta)$ is maximal-minimal $gsg$-continuous maps, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a maximal $gsg$-continuous.

**Proof:** The proof is similar to that of theorem 2.5.26 and making use of theorem 2.5.20.

**Theorem 2.5.29** If $f : (X, \tau) \to (Y, \sigma)$ is $gsg$-continuous and $(Y, \sigma)$ is a $T_{gsg}$-space and $g : (Y, \sigma) \to (Z, \eta)$ is minimal-maximal $gsg$-continuous maps, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a minimal $gsg$-continuous.
Proof: Let $A$ be any minimal closed set in $(Z, \eta)$. Since $g$ is minimal-maximal continuous, $g^{-1}(A)$ is a maximal gsg-closed set in $(Y, \sigma)$. That is, $g^{-1}(A)$ is gsg-closed set in $(Y, \sigma)$. Since $(Y, \sigma)$ is a $T_{gsg}$-space, $g^{-1}(A)$ is a closed in $(Y, \sigma)$. Again, since $f$ is gsg-continuous, $f^{-1}\left((g^{-1}(A))\right) = (g \circ f)^{-1}(A)$ is a gsg-closed set in $(X, \tau)$. Hence $g \circ f$ is a minimal gsg-continuous.

Theorem 2.5.30 If $f : (X, \tau) \to (Y, \sigma)$ is gsg-continuous, $(Y, \sigma)$ is $T_{gsg}$-space and $g : (Y, \sigma) \to (Z, \eta)$ is maximal-minimal gsg-continuous maps, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a maximal gsg-continuous.

Proof: The proof is similar to that of theorem 2.5.29.

Theorem 2.5.31 If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are minimal-maximal gsg-continuous maps and if $(Y, \sigma)$ is a $T_{min-gsg}$ space, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a minimal-maximal gsg-continuous.

Proof: Let $A$ be any minimal closed set in $(Z, \eta)$. Since $g$ is minimal-maximal gsg-continuous, $g^{-1}(A)$ is a maximal gsg-closed set in $(Y, \sigma)$. It follows that $g^{-1}(A)$ is gsg-closed subset of $(Y, \sigma)$. Since $(Y, \sigma)$ is $T_{min-gsg}$ space, $g^{-1}(A)$ is a minimal closed set in $(Y, \sigma)$. Again, since $f$ is minimal-maximal gsg-continuous, $f^{-1}\left((g^{-1}(A))\right) = (g \circ f)^{-1}(A)$ is a maximal gsg-closed set in $(X, \tau)$. Hence $g \circ f$ is a minimal-maximal gsg-continuous.

Theorem 2.5.32 If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are maximal-minimal gsg-continuous maps and if $(Y, \sigma)$ is a $T_{max-gsg}$ space, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a maximal-minimal gsg-continuous.

Proof: The proof is similar to that of the theorem 2.5.31.
2.6 CONCLUSION

In this chapter, the concepts of minimal \( gsg \)-closed, maximal \( gsg \)-closed, minimal \( gsg \)-open and maximal \( gsg \)-open sets in topological space are introduced and some of its basic topological structures are studied. Then continuous and irresolute mappings via the above sets are introduced and properties of these mapping are studied. The concepts of \( gsg \)-semi-maximal open sets and \( gsg \)-semi-minimal closed sets are also studied.