CHAPTER 6

SOFT $gsg$-CLOSED SET IN SOFT TOPOLOGICAL SPACES

6.1 INTRODUCTION

Molodtsov (1999) introduced the concept of soft set theory and started to develop the basics of the corresponding theory as a new approach for modeling uncertainties. Shabir & Naz (2011) introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. Many researchers extended the results of generalization of various soft closed sets in many directions.

The focus of this chapter 6 is to introduce soft generalized semi generalized closed sets in soft topological spaces which are defined over an initial universe with a fixed set of parameters. Further, some properties in the light of these defined sets are established.

**Definition 6.1.1** A soft set $(A, E)$ is called a **soft semi geneneralized closed set** (in short, soft $sg$-closed) in a soft topological space $(X, \tau, E)$ if $sCl(A, E) \subseteq (U, E)$ whenever $(A, E) \subseteq (U, E)$ and $(U, E)$ is soft semi open in $X$.

**Definition 6.1.2** a soft topological space $(X, \tau, E)$ is a **soft $T_\omega$-space** if every soft $\omega$-closed set in $(X, \tau, E)$ is soft closed in $(X, \tau, E)$. 
**Theorem 6.1.1** Every soft closed set is soft $sg$-closed in a soft topological space $(X, \tau, E)$.

**Proof:** Let $(A, E)$ be soft closed set in a soft topological space $(X, \tau, E)$, Let $(U, E)$ be any soft semi open set in a soft topological space $(X, \tau, E)$ such that $(A, E) \subseteq (U, E)$. Then $sCl(A, E) \subseteq Cl(A, E) = (A, E) \subseteq (U, E)$. Therefore $(A, E)$ is soft $sg$-closed in a soft topological space $(X, \tau, E)$.

**Theorem 6.1.2** Every soft semi closed set is soft $sg$-closed in a soft topological space $(X, \tau, E)$.

**Proof:** Let $(A, E)$ be soft semi closed set in a soft topological space $(X, \tau, E)$, and $(U, E)$ be any soft semi open set in a soft topological space $(X, \tau, E)$ such that $(A, E) \subseteq (U, E)$. Now $sCl(A, E) = (A, E) \subseteq (U, E)$. Therefore $(A, E)$ is soft $sg$-closed in a soft topological space $(X, \tau, E)$.

**Corollary 6.1.2** Every soft open (soft semi open) set is soft $sg$-open in a soft topological space $(X, \tau, E)$.

### 6.2 SOFT $gsg$-CLOSED SETS AND SOFT $gsg$-OPEN SETS

This section is devoted to define the concept of generalized semi generalized closed set in soft topological space. And a comparative study of soft $gsg$-closed sets with other types of soft closed sets in soft topological space is established.

**Definition 6.2.1** A soft set $(A, E)$ is called a **soft generalized semi generalized closed set** (in short, soft $gsg$-closed) in a soft topological spaces $(X, \tau, E)$ if $Cl(A, E) \subseteq (U, E)$ whenever $(A, E) \subseteq (U, E)$ and $(U, E)$ is soft $sg$-open in $X$. The complement of soft $gsg$-closed set **soft $gsg$-open set**.
**Proposition 6.2.1** Every soft closed is soft $g_{sg}$-closed in a soft topological space $(X, \tau, E)$.

**Proof:** Let $(A, E)$ be soft closed and $(U, E)$ be soft $s_{g}$-open such that $(A, E) \subseteq (U, E)$ in a soft topological space $(X, \tau, E)$. Consider $Cl(A, E) = (A, E) \subseteq (U, E)$. Therefore $(A, E)$ is soft $g_{sg}$-closed in a soft topological space $(X, \tau, E)$.

**Proposition 6.2.2** Every soft $g_{sg}$-closed is soft $g$-closed in a soft topological space $(X, \tau, E)$.

**Proof:** Let $(A, E)$ be soft $g_{sg}$-closed in a soft topological space $(X, \tau, E)$. Let $(U, E)$ be any soft open in $X$ such that $(A, E) \subseteq (U, E)$ in $(X, \tau, E)$. Since every soft open set is soft $s_{g}$-open, $Cl(A, E) \subseteq (U, E)$ such that $(A, E) \subseteq (U, E)$. Therefore $(A, E)$ is soft $g$-closed in a soft topological space $(X, \tau, E)$.

**Proposition 6.2.3** Every soft $g_{sg}$-closed set is soft $\omega$-closed in a soft topological space $(X, \tau, E)$.

**Proof:** Let $(A, E)$ be soft $g_{sg}$-closed and $(U, E)$ be any soft semi open set such that $(A, E) \subseteq (U, E)$ in a soft topological space $(X, \tau, E)$. Since every soft semi open set is soft $s_{g}$-open, $(U, E)$ is soft $s_{g}$-open. Therefore $(A, E)$ is soft $g_{sg}$-closed and $(A, E) \subseteq (U, E)$. Hence $(A, E)$ soft $\omega$-closed in a soft topological space $(X, \tau, E)$.

**Corollary 6.2.1** In a soft topological space $(X, \tau, E)$,

i. Every soft open is soft $g_{sg}$-open

ii. Every soft $g_{sg}$-open is soft $g$-open.

iii. Every soft $g_{sg}$-open set is soft $\omega$-open.
6.3 CHARACTERIZATION OF SOFT $gsg$-CLOSED SET AND
SOFT $gsg$-OPEN SET

In this section, some important characterizations of soft $gsg$-closed sets and soft $gsg$-open sets in soft topological spaces are established.

**Theorem 6.3.1** If $(A,E)$ and $(B,E)$ are soft $gsg$-closed sets in a soft topological space $(X,\tau,E)$ then $(A,E)\bigcup (B,E)$ is soft $gsg$-closed set in $(X,\tau,E)$.

**Proof:** Suppose $(A,E)$ and $(B,E)$ are soft $gsg$-closed set in $(X,\tau,E)$. Then $Cl(A,E) \subseteq (U,E)$ and $Cl(B,E) \subseteq (U,E)$ where $(A,E) \subseteq (U,E)$, $(B,E) \subseteq (U,E)$ and $(U,E)$ is soft $sg$-open in $(X,\tau,E)$. Consider $Cl((A,E)\bigcup (B,E)) = Cl(A,E) \bigcup Cl(B,E) \subseteq (U,E)$. Hence $(A,E)\bigcup (B,E)$ is soft $gsg$-closed set in a soft topological space $(X,\tau,E)$.

**Theorem 6.3.2** If a soft set $(A,E)$ is soft $gsg$-closed in a soft topological space $(X,\tau,E)$ then $Cl(A,E) - (A,E)$ contains only null soft closed set.

**Proof:** Let $(F,E)$ be a soft closed set in a soft topological space $(X,\tau,E)$ and $(F,E) \subseteq Cl(A,E) - (A,E)$. Then $(F,E) \subseteq Cl(A,E)$ and $(F,E) \subseteq X - (A,E)$. This implies $(A,E) \subseteq X - (F,E)$. Then $Cl(A,E) \subseteq X - (F,E)$, as $X - (F,E)$ as soft $sg$-open set. This implies $(F,E) \subseteq X - Cl(A,E)$. Therefore $(F,E) \subseteq Cl(A,E) \cap (X - Cl(A,E)) = \varnothing$. Hence $(F,E)$ is null soft closed set in a soft topological space $(X,\tau,E)$.

**Theorem 6.3.3** A soft $(A,E)$ is soft $gsg$-closed in a soft topological space $(X,\tau,E)$ if and only if $Cl(A,E) - (A,E)$ contains only null soft $sg$-closed set in $(X,\tau,E)$.
**Proof:** Let \((A, E)\) be a soft \(gsg\)-closed in a soft topological space \((X, \tau, E)\). Let \((F, E)\) be soft \(sg\)-closed set such that \((F, E) \not\subset Cl(A, E) - (A, E)\). Then \((F, E) \not\subset Cl(A, E)\) and \((F, E) \not\subset X - (A, E)\).

\[
\Rightarrow (A, E) \not\subset X - (F, E)
\]

\[
\Rightarrow Cl(A, E) \not\subset X - (F, E)
\]

\[
\Rightarrow (F, E) \not\subset X - Cl(A, E).
\]

Therefore \((F, E) \not\subset Cl(A, E) \cap (X - Cl(A, E)) = \varnothing\). Hence \((F, E)\) is null soft closed set in a soft topological space \((X, \tau, E)\).

Conversely, assume \(Cl(A, E) - (A, E)\) contains only null soft \(sg\)-closed set in \((X, \tau, E)\). Let \((U, E)\) be soft \(sg\)-open such that \((A, E) \subsetneq (U, E)\). Suppose \(Cl(A, E) \not\subset (U, E)\), then \(Cl(A, E) \cap (X - (U, E))\) is a non-null set. As \(Cl(A, E)\) and \(X - (U, E)\) is soft \(sg\)-closed, \(Cl(A, E) \cap (X - (U, E))\) is non null \(sg\)-closed which is contained in \(Cl(A, E) - (A, E)\). This is a contradiction. Therefore \(Cl(A, E) \subsetneq (U, E)\) and hence \((A, E)\) is a soft \(gsg\)-closed in a soft topological space \((X, \tau, E)\).

**Theorem 6.3.4** If \((A, E)\) is soft \(gsg\)-closed in a soft topological space \((X, \tau, E)\) and \((A, E) \subsetneq (B, E) \subsetneq Cl(A, E)\) then \((B, E)\) is soft \(gsg\)-closed in \((X, \tau, E)\).

**Proof:** Let \((U, E)\) be soft \(sg\)-open such that \((B, E) \subsetneq (U, E)\) in a soft topological space \((X, \tau, E)\). Since \((A, E)\) is soft \(gsg\)-closed in \((X, \tau, E)\), \(Cl(A, E) - (A, E)\) contains no non-null soft \(sg\)-closed set. \((B, E) \subsetneq Cl(A, E)\) implies \(Cl(B, E) \subsetneq Cl(A, E)\) and \(Cl(B, E) - (B, E) \subsetneq Cl(A, E) - (A, E)\). This implies \(Cl(B, E) - (B, E)\) does not contain any non-null soft \(sg\)-closed set. By the theorem 6.3.3, the set \((B, E)\) is soft \(gsg\)-closed.
Corollary 6.3.1 If \((A, E)\) is soft \(sg\)-open and soft \(gsg\)-closed then \((A, E)\) soft closed in a soft topological space \((X, \tau, E)\).

Theorem 6.3.5 In a soft topological space \((X, \tau, E)\), a soft set \((A, E)\) soft \(gsg\)-open if and only if \((F, E) \subseteq \text{Int}(A, E)\) where \((F, E)\) is soft \(sg\)-closed and \((F, E) \subseteq (A, E)\).

**Proof:** Assume that \((F, E) \subseteq \text{Int}(A, E)\), where \((F, E)\) is soft \(sg\)-closed and \((F, E) \subseteq (A, E)\). Then \(X - (F, E)\) is soft \(sg\)-open and \(X - (A, E) \subseteq X - (F, E)\). Now, \((F, E) \subseteq \text{Int}(A, E)\)

\[\Rightarrow (X - \text{Int}(A, E)) \subseteq X - (F, E)\]

\[\Rightarrow \text{Cl}(X - (A, E)) \subseteq X - (F, E).\]

This gives \(X - (A, E)\) is \(gsg\)-closed in \((X, \tau, E)\). Hence \((A, E)\) is \(gsg\)-open in \((X, \tau, E)\).

Conversely, \((A, E)\) soft \(gsg\)-open and \((F, E)\) is soft \(sg\)-closed with \((F, E) \subseteq (A, E)\). This implies \(X - (A, E)\) is soft \(gsg\)-closed and \(X - (F, E)\) is soft \(sg\)-open with \(X - (A, E) \subseteq X - (F, E)\).

Then \(\text{Cl}(X - (A, E)) \subseteq X - (F, E)\)

\[\Rightarrow X - \text{Int}(A, E) \subseteq X - (F, E)\]

\[\Rightarrow (F, E) \subseteq \text{Int}(A, E).\]

Hence proved.

Corollary 6.3.2 If \((A, E)\) and \((B, E)\) are soft \(gsg\)-open sets then \((A, E) \cap (B, E)\) is soft \(gsg\)-open in a soft topological space \((X, \tau, E)\).
Corollary 6.3.3 If \( Int(B, E) \subseteq (B, E) \subseteq (A, E) \) and if \((A, E)\) is soft \( gsg\)-open then \((B, E)\) is soft \( gsg\)-open in a soft topological space \((X, \tau, E)\).

Theorem 6.3.6 If a soft set \((A, E)\) is soft \( gsg\)-closed in a soft topological space \((X, \tau, E)\) then \( Cl(A, E) - (A, E)\) is soft \( gsg\)-open in \((X, \tau, E)\).

Proof: Suppose that \((A, E)\) is soft \( gsg\)-closed in \((X, \tau, E)\). Let \((F, E) \subseteq Cl(A, E) - (A, E)\) and \((F, E)\) is soft \( sg\)-closed. Then \((F, E)\) is null soft \( sg\)-closed set. Now, \((F, E) \subseteq Int(Cl(A, E) - (A, E))\) and hence by theorem 6.3.5, \( Cl(A, E) - (A, E)\) is soft \( gsg\)-open.

6.4 SOFT \( T_{gsg} \)-SPACE

In this section, soft \( T_{gsg} \)-space is introduced and compared with already existing soft spaces in soft topological spaces.

Definition 6.4.1 A soft topological space \((X, \tau, E)\) is called a soft \( T_{gsg} \)-space if every soft \( gsg\)-closed set in it is soft closed in \((X, \tau, E)\).

Theorem 6.4.1 Every soft \( T_{1/2} \)-space is soft \( T_{gsg} \)-space in a soft topological space \((X, \tau, E)\).

Proof: Let \((X, \tau, E)\) be a soft \( T_{1/2} \)-space and let \((A, E)\) be soft \( gsg\)-closed set in \((X, \tau, E)\). By proposition 6.2.2, \((A, E)\) is soft \( g\)-closed. Since \((X, \tau, E)\) is soft \( T_{1/2} \)-space, \((A, E)\) is soft closed in \((X, \tau, E)\). Hence \((X, \tau, E)\) is soft \( T_{gsg} \)-space.

Theorem 6.4.2 Every soft \( T_{\omega} \)-space is a soft \( T_{gsg} \)-space in a soft topological space \((X, \tau, E)\).
**Proof:** Let \((X, \tau, E)\) be a soft \(T_{\omega}\)-space and let \((A, E)\) be soft gsg-closed set in \((X, \tau, E)\). By proposition 6.2.3, \((A, E)\) is soft \(\omega\)-closed. Since \((X, \tau, E)\) is soft \(T_{\omega}\)-space, \((A, E)\) is soft closed in \((X, \tau, E)\). Hence \((X, \tau, E)\) is soft \(T_{gsg}\)-space.

6.5 CONCLUSION

In this chapter, soft \(gsg\)-closed sets are introduced and compared with already existing soft sets in soft topological spaces. A new soft \(T_{gsg}\)-space is also introduced and it is proved that every soft \(T_{1/2}\)-space and soft \(T_{\omega}\)-space is a soft \(T_{gsg}\)-space.