CHAPTER 5

PAIRWISE FUZZY $\alpha^*$-CONNECTEDNESS
BETWEEN FUZZY SETS

5.1 INTRODUCTION

The concept of fuzzy $\alpha^*$ set was introduced and studied by Uma et al (2005). The concept of pairwise fuzzy connectedness between fuzzy sets was studied by Thakur & Annamma Philip (1997).

Making use of the above concepts, the concept of pairwise fuzzy $\alpha^*$-connectedness between fuzzy sets is introduced in this chapter. Some interesting properties are investigated besides giving some examples. Throughout this chapter, $i, j = 1, 2$ where $i \neq j$. Here are a few known definitions which are to be used in this chapter.

**Definition 5.1.1 (Uma et al 2005)** Let $(X, \tau)$ be a fuzzy topological space and $\lambda$ be a fuzzy set in $X$. Then $\lambda$ is called a **fuzzy $\alpha^*$-set** if $\text{Int}(\lambda) = \text{Int}(\text{Cl}(\lambda))$.

**Definition 5.1.2 (Uma et al 2005)** Let $(X, \tau)$ be a fuzzy topological space and $\lambda$ be a fuzzy set in $X$. Then $\lambda$ is called a **fuzzy $\alpha^*$-closed** if $\text{Cl}(\lambda) = \text{Cl}(\text{Int}(\lambda))$.

**Definition 5.1.3 (Mageswari et al 1993)** A fuzzy topological space $(X, \tau)$ is said to be **fuzzy connected** between its fuzzy sets $\lambda$ and $\mu$ if and only if there is no fuzzy closed and fuzzy open set $\delta$ in $X$ such that $\lambda \leq \delta$ and $\delta \cap \mu$. 
Definition 5.1.4 (Thakur & Annamma Philip 1997) A fuzzy bitopological space \((X, \tau_1, \tau_2)\) is said to be **pairwise fuzzy connected** if it has no proper fuzzy set which is both \(\tau_i\) fuzzy open and \(\tau_j\) fuzzy closed, \(i, j = 1, 2, i \neq j\).

**Note 5.1.1** A fuzzy set \(\lambda\) which is both fuzzy \(\alpha^*\)-closed and fuzzy \(\alpha^*\)-open is called **fuzzy \(\alpha^*\)-clopen set** (in short, fuzzy \(\alpha^*\)CO).

### 5.2 Pairwise Fuzzy \(\alpha^*\)-Connectedness Between Fuzzy Sets

**Definition 5.2.1** A fuzzy bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise fuzzy \(\alpha^*\) connected between fuzzy sets \(\lambda\) and \(\mu\) if there is no fuzzy \((i,j)\)fuzzy \(\alpha^*\)CO \((\tau_i\)-fuzzy \(\alpha^*\) closed and \(\tau_j\)-fuzzy \(\alpha^*\) open) set \(\delta\) in \(X\) such that \(\lambda \leq \delta\) and \(\delta \leq \mu\).

**Remark 5.2.1** Pairwise fuzzy \(\alpha^*\)-connectedness between fuzzy sets \(\lambda\) and \(\mu\) is not equal to the fuzzy \(\alpha^*\) connectedness between \(\lambda\) and \(\mu\) of \((X, \tau_1)\) and \((X, \tau_2)\).

**Property 5.2.1** A fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise fuzzy \(\alpha^*\)-connected between fuzzy sets \(\lambda\) and \(\mu\) if and only if there is no \((i,j)\) fuzzy \(\alpha^*\)CO set \(\delta\) in \(X\) such that \(\lambda \leq \delta \leq 1 - \mu\).

**Property 5.2.2** If a fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise fuzzy \(\alpha^*\)-connected between fuzzy sets \(\lambda\) and \(\mu\) then \(\lambda \neq 0\) and \(\mu \neq 0\).

**Property 5.2.3** If a fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise fuzzy \(\alpha^*\)-connected between fuzzy sets \(\lambda\) and \(\mu\) if \(\lambda \leq \lambda_1\) and \(\mu \leq \mu_1\), then \((X, \tau_1, \tau_2)\) is pairwise fuzzy \(\alpha^*\) connected between \(\lambda_1\) and \(\mu_1\).
Proof: Suppose the fuzzy bitopological space $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected between fuzzy sets $\lambda_1$ and $\mu_1$. Then there is an $(i,j)$-fuzzy $\alpha'CO$ set $\delta$ in $X$ such that $\lambda_1 \leq \delta$ and $\delta q_\mu$. Clearly $\lambda \leq \delta$. Now claim that $\delta q_\mu$. If $\delta q_\mu$ then there exists a point $x \in X$ such that $\delta(x) + \mu(x) > 1$. Therefore, $\delta(x) + \mu_1(x) > \delta(x) + \mu(x) > 1$ and $\delta q_\mu$, a contradiction. Hence $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected between $\lambda$ and $\mu$.

Property 5.2.4 A fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy $\alpha'$-connected between fuzzy sets $\lambda$ and $\mu$ if and only if it is pairwise fuzzy $\alpha'$-connected between $\tau_i-\alpha'Cl(\lambda)$ and $\tau_j-\alpha'Cl(\mu)$.

Proof: Necessity: It follows by using property 5.2.3.

Sufficiency: Suppose the fuzzy bitopological space $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected between fuzzy sets $\lambda$ and $\mu$. Then there is an $(i,j)$-fuzzy $\alpha'CO$ set $\delta$ in $X$ such that $\lambda \leq \delta$ and $\delta q_\mu$. Since $\lambda \leq \delta$, $\tau_i-\alpha'Cl(\lambda) \leq \tau_i-\alpha'Cl(\delta) < \delta$ because $\delta$ is $\tau_i$-fuzzy $\alpha'$ closed.

Now, $\delta q_\mu \Rightarrow \delta \leq 1 - \mu$

$$\Rightarrow \delta \leq \tau_j-\alpha'Int(1-\mu)$$

$$\Rightarrow \delta \leq 1 - \tau_j-\alpha'Cl(\mu)$$

$$\Rightarrow \delta \leq \delta q_\tau_j-\alpha'Cl(\mu).$$

Hence $X$ is not pairwise fuzzy $\alpha'$-connected between the sets $\tau_i-\alpha'Cl(\lambda)$ and $\tau_j-\alpha'Cl(\mu)$, which is a contradiction. Hence the result.

Property 5.2.5 Let $(X, \tau_1, \tau_2)$ be a fuzzy bitopological space and let $\lambda$ and $\mu$ be two fuzzy sets in $(X, \tau_1, \tau_2)$. If $\lambda q_\mu$ then $(X, \tau_1, \tau_2)$ is pairwise fuzzy $\alpha'$-connected between the fuzzy sets $\lambda$ and $\mu$. 

Proof: If $\delta$ is any $(i,j)$ fuzzy $\alpha'$ CO set in $(X, \tau_1, \tau_2)$, such that $\lambda \leq \mu$. Then
\[
\lambda q \mu \Rightarrow \delta q \mu.
\]

Property 5.2.6 If a fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy $\alpha'$-connected neither between $\lambda$ and $\mu$, nor between $\lambda$ and $\delta$, then it is not pairwise fuzzy $\alpha'$-connected between $\lambda$ and $\mu \vee \delta$.

Proof: Since $(X, \tau_1, \tau_2)$ is pairwise fuzzy $\alpha'$-connected neither between the fuzzy sets $\lambda$ and $\mu$ nor between the fuzzy sets $\lambda$ and $\delta$, there exists $(i,j)$-fuzzy $\alpha'$ CO sets $\lambda_1$ and $\mu_1$ in $(X, \tau_1, \tau_2)$ such that $\lambda \leq \lambda_1, \lambda_1 \bar{q} \mu$ and $\lambda \leq \mu_1, \mu_1 \bar{q} \delta$. Put $\delta_1 = \lambda_1 \wedge \mu_1$. Then $\delta_1$ is $(i,j)$ fuzzy $\alpha'$ CO and $\lambda \leq \delta_1$. Now, it is claim that $\delta_1 \bar{q} (\mu \vee \delta)$. If $\delta_1 q (\mu \vee \delta)$ then there exists a point $x \in X$ such that $\delta_1(x) + (\mu \vee \delta)(x) > 1$. This implies that $\delta_1 q \mu$ or $\delta_1 q \delta$, a contradiction. Hence $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected between $\lambda$ and $\mu \vee \delta$.

Property 5.2.7 A fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy $\alpha'$-connected if and only if it is pairwise fuzzy $\alpha'$-connected between every pair of its non-zero fuzzy sets.

Proof: Necessity: Let $\lambda$ and $\mu$ be any pair of non-zero fuzzy sets of $(X, \tau_1, \tau_2)$. Suppose $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected between the fuzzy sets $\lambda$ and $\mu$. Then there is an $(i,j)$-fuzzy $\alpha'$ CO set $\delta$ in $(X, \tau_1, \tau_2)$ such that $\lambda \leq \delta$ and $\delta \bar{q} \mu$. Since $\lambda$ and $\mu$ are non-zero it follows that $\delta$ is a non-zero proper $(i,j)$-fuzzy $\alpha'$ CO set. Hence $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected.

Sufficiency: Suppose $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected. Then there exists a non zero proper $(i,j)$ fuzzy $\alpha'$ CO set $\delta$ on $X$. Consequently, $(X, \tau_1, \tau_2)$ is not pairwise fuzzy $\alpha'$-connected between $\delta$ and $1-\delta$, a contradiction.
Property 5.2.8 Let \((Y, (\tau_1)_Y, (\tau_2)_Y)\) be a subspace of fuzzy bitopological space \((X, \tau_1, \tau_2)\) and let \(\lambda, \mu\) be fuzzy set of \(Y\). If \((X, \tau_1, \tau_2)\) is pairwise fuzzy \(\alpha^c\)-connected between \(\lambda\) and \(\mu\) and \(\chi_Y\) is bifuzzy \(\alpha^c\) in \((X, \tau_1, \tau_2)\) then \((Y, (\tau_1)_Y, (\tau_2)_Y)\) is pairwise fuzzy \(\alpha^c\)-connected between \(\lambda\) and \(\mu\).

Proof: Suppose \((Y, (\tau_1)_Y, (\tau_2)_Y)\) is not pairwise fuzzy \(\alpha^c\)-connected between \(\lambda\) and \(\mu\) then there exists an \((i, j)\)-fuzzy \(\alpha^c\) set \(\delta\) in \(Y\) such that \(\lambda \leq \delta\) and \(\delta \subseteq \mu\). Since \(\chi_Y\) is bifuzzy \(\alpha^c\) and bifuzzy \(\alpha^c\) closed in \((X, \tau_1, \tau_2)\), \(\delta\) is \((i, j)\) fuzzy \(\alpha^c\) in \((X, \tau_1, \tau_2)\). Therefore \((X, \tau_1, \tau_2)\) is not pairwise fuzzy \(\alpha^c\)-connected between \(\lambda\) and \(\mu\). Hence the proof.

Property 5.2.9 Let \((Y, (\tau_1)_Y, (\tau_2)_Y)\) be a subspace of fuzzy bitopological space \((X, \tau_1, \tau_2)\) and let \(\lambda, \mu\) be fuzzy set of \(Y\). If \((Y, (\tau_1)_Y, (\tau_2)_Y)\) is pairwise fuzzy \(\alpha^c\)-connected between \(\lambda\) and \(\mu\) then \((X, \tau_1, \tau_2)\) is also pairwise fuzzy \(\alpha^c\)-connected between \(\lambda\) and \(\mu\).

Proof: Proof is similar to that of property 5.2.8.

5.3 CONCLUSION

The concept of pairwise fuzzy \(\alpha^c\)-connectedness between fuzzy sets is introduced and some interesting properties are investigated in this chapter.