Chapter 5

EXISTENCE OF MILD SOLUTIONS OF IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ALMOST SECTORIAL OPERATORS

5.1 Introduction

This chapter onwards we are going to introduce a new type of operator which is known as almost sectorial operator. To the best of our knowledge, the study of the existence of solutions using this operator is an untreated topic. This fact is the main motivation to our research in this field. In [258], F. Periago and B. Straub give a functional calculus for almost sectorial operators and using the semigroup of growth $1 + \gamma$ which is defined by this functional calculus, obtained the existence and uniqueness of classical solutions for cauchy problems of abstract evolution equations involving almost sectorial operators. Hernandez [266], first proved the existence of mild solutions for a class of abstract functional differential equations with almost sectorial operators. By the motivation of this paper, we extend the existence results to impulsive differential equations with almost sectorial operators using the semigroup of growth $\gamma$.

5.2 Preliminaries

Sectorial operators, that is, linear operators $A$ defined in Banach spaces, whose spectrum lies in a sector

$$S_w = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\text{arg}\lambda| \leq w \} \cup \{0\} \text{ for some } 0 \leq w \leq \frac{\pi}{2}$$

and whose resolvent satisfies an estimate

$$||(\lambda - A)^{-1}|| \leq C|\lambda|^{-1}, \quad \forall \ \lambda \in \mathbb{C} \setminus S_w,$$

(5.2.1)
have been studied extensively during the last 40 years, both in abstract settings and for their applications to partial differential equations. Many important elliptic differential operators belong to the class of sectorial operators, especially when they are considered in the Lebesgue spaces or in spaces of continuous functions (see [255] and [256], chapter 3). However, if we look at spaces of more regular functions such as the spaces of Holder continuous functions, we find that these elliptic operators do no longer satisfy the estimate (5.2.1) and therefore are not sectorial as was pointed out by Von Wahl (see [257], Ex.3.1.33, see [258]).

Nevertheless, for these operators estimates such as
\[ ||(\lambda - A)^{-1}|| \leq \frac{C}{|\lambda|^{1-\alpha}}, \quad \lambda \in \sum_{w,v} = \{ \lambda \in \mathbb{C} : |\arg(\lambda - w)| < v \} \]  
where \(\alpha \in (0, 1), w \in \mathbb{R} \) and \(v \in (\frac{\pi}{2}, \pi)\), can be obtained, (see [258]), which allows to define an associated "analytic semigroup" by means of the Dunford Integral
\[ T(t) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t}(\lambda - A)^{-1} d\lambda, \quad t > 0 \]  
where \(\Gamma_\theta = \{ te^{i\theta} : t \in \mathbb{R}\{0\}\}, \theta \in (v, \frac{\pi}{2})\).

In the literature, a linear operator \(A : D(A) \subset X \rightarrow X\) which satisfy the condition (5.2.2) is called almost sectorial and the operator family \(\{T(t), T(0) = I, t \geq 0\}\) is said to be the "semigroup of growth \(\alpha\)" generated by \(A\). The operator family \(T(t)_{t \geq 0}\) has properties similar at those of analytic semigroup which allow to study some classes of partial differential equations via the usual methods of semigroup theory. Concerning almost sectorial operators, semigroups of growth \(\alpha\) and applications to partial differential equations, we refer the reader to [258, 259, 260, 261, 262] and the references there in.

In this chapter, we prove the existence and uniqueness of mild solutions for a class of abstract impulsive functional differential equations of the form,
\[
\begin{align*}
x'(t) &= Ax(t) + f(t, x_t), \quad t \in [0, a], \quad t \neq t_k \\
\Delta x|_{t=t_k} &= I_k(x(t^-_k)), \quad t = t_k, \quad k = 1, 2, ..., m. \\
x_0 &= \phi \in \Omega \subset B
\end{align*}
\]  
where \(A : D(A) \subset X \rightarrow X\) is an almost sectorial operator, \((X, ||.||)\) is a Banach space, \(B\) is the phase, \(\Omega \subset B\) is open and \(f : [0, a] \times \Omega \rightarrow X\) is a suitable function. Here, \(0 < t_1 < t_2 < ... < t_m < t_{m+1} = a\), \(I_k \in C(X, X), \quad k = 1, 2, ..., m\). Let \(\Delta x|_{t=t_k} = x(t^+_k) - x(t^-_k), \quad x(t^+_k) \) and \(x(t^-_k)\) represent the right and left limits of \(x(t)\) at \(t = t_k\) respectively.

Here, we introduce some notations and technicalities. Let \((Z, ||.||_z)\) be a Banach space. In this paper, \(\mathcal{L}(Z, W)\) represents the space of bounded linear operators from \(Z\) into \(W\).
endowed with norm of operators denoted \( ||\cdot||_{\mathcal{L}(Z,W)} \), and we write \( \mathcal{L}(Z) \) and \( ||\cdot||_{\mathcal{L}(Z)} \) when \( Z = W \). In addition, \( B_l(z,Z) \) denotes the closed ball with center at \( z \in Z \) and radius \( l > 0 \) in \( Z \). As usual, \( C([c,d], Z) \) represents the space formed by all the continuous functions from \( I \) into \( Z \) endowed with the sup-norm denoted by \( ||\cdot||_{C([c,d],Z)} \) and \( L^p([c,d], X) \), \( p \geq 1 \), denotes the space formed by all the classes of Lebesgue-integrable functions from \([c,d]\) into \( X \) endowed with the norm

\[
||h||_{L^p([c,d],X)} = \left( \int_{[c,d]} ||h(s)||^p \, ds \right)^{\frac{1}{p}}.
\]

Throughout this paper, \((X, ||\cdot||)\) is a Banach space, \( A : D(A) \subset X \to X \) is an almost sectorial operator and \((T(t))_{t \geq 0}\) is the semigroup of growth \( \alpha \) generated by \( A \). For simplicity, next we assume \( w = 0 \). The next lemma consider some properties of the operator family \((T(t))_{t \geq 0}\).

**Lemma 5.2.1.** ([259, 260]): Under the above conditions, the followings properties are satisfied.

(a) The operator \( A \) is closed, \( T(t+s) = T(t)T(s) \) and \( AT(t)x = T(t)Ax \) for all \( t, s \in [0, \infty) \) and each \( x \in D(A) \).

(b) \( T(\cdot) \in C((0, \infty), X) \cap C^1((0, \infty), X) \) and \( \frac{d}{dt} T(t) = AT(t) \) for all \( t > 0 \).

(c) For \( n \in \mathbb{N} \cup \{0\} \), \( A^n T(\cdot) \in C((0, \infty), X) \) and there exists \( D_n > 0 \) and a constant \( \gamma > 0 \), which is independent of \( n \), such that \( ||A^n T(t)||_{\mathcal{L}(X)} \leq D_n e^{\gamma t} t^{-(n+\alpha)} \) for all \( t > 0 \).

In this section, we study the existence of mild solutions for the system (5.2.4). Let us define the solution of the equation (5.2.4).

**Definition 5.2.2.** A function \( u \in PC([-r, b], X), 0 < b \leq a \), is called a mild solution of the abstract system (5.2.4) on \([-r, b]\) if \( u_0 = \phi, u|_{[0,b]} \in PC((0,b],X) \) and

\[
u(t) = T(t)\phi(0) + \int_0^b T(t-s)f(s,u_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k^-)), \forall t \in [0,b]\] (5.2.5)

**Remark 5.2.1.** In the remainder of this chapter, \( \phi : [-r, 0] \to X \) is a given function and \( y : [-r, a] \to X \) is the function defined by \( y(\theta) = \phi(\theta) \) for \( \theta \leq 0 \) and \( y(t) = T(t)\phi(0) \) for \( t > 0 \). In addition, \( C_n, n \in \mathbb{N} \), are positive constants such that

\[
||A^n T(t)||_{\mathcal{L}(X)} \leq C_n t^{-(n+\alpha)}, \forall t \in (0,a],
\]
and for a bounded set \( B \subset X \), we use the notation \( \text{Diam}_X(B) \) for
\[
\text{Diam}_X(B) = \sup_{a,b \in B} ||a - b||.
\]

## 5.3 Existence Results

For the forthcoming results, we need the following assumptions, \( q \in \left( \frac{1}{1-\alpha}, \infty \right) \) or \( q = \infty \) and
\[
quotient{p}{p-1} \text{ for } q < \infty \quad \text{and} \quad q' = 1 \text{ if } q = \infty.
\]

\((H_1)\) The function \( f(\cdot, \psi) \) is strongly measurable on \( [0, a] \) for all \( \psi \in \Omega \) and \( f(t, \cdot) \in C(\Omega, X) \) for each \( t \in [0, a] \). There are \( m_f \in L^q([0, a], \mathbb{R}^+) \) and a non-decreasing function \( W_f \in C([0, \infty), (0, \infty)) \) such that
\[
||f(t, \psi)|| \leq m_f(t)W_f(||\psi||_B), \forall (t, \psi) \in [0, a] \times \Omega
\]

\((H_2)\) The function \( f \) is continuous and for all \( l > 0 \) with \( [0, l] \times B_l(\phi, B) \subset [0, a] \times \Omega \), there exists \( L_{f,l} \in L^q([0, a], \mathbb{R}^+) \) such that
\[
||f(s, \psi_1) - f(s, \psi_2)|| \leq L_{f,l}(s) ||\psi_1 - \psi_2||_B, \forall (s, \psi_i) \in [0, l] \times B_l(\phi, B)
\]

\((H_3)\) The function \( I_k \) are continuous and there exists positive constants \( L_k, k = 1, 2, ..., m \) such that
\[
||I_k(\psi)|| \leq L_k \forall \psi \in \Omega, k = 1, 2, ...m
\]

Next, we study the existence of mild solutions for (5.2.4) assuming that \( B = C([-r, 0], X) \). For simplicity, we write \( ||\cdot||_B \) in place of \( ||\cdot||_{C([-r, 0], X)} \). Now, we establish the existence result.

**Theorem 5.3.1.** Assume the conditions \((H_1)\) and \((H_3)\) are satisfied, \( T(\cdot)\phi(0) \in C([0, a], X) \) and \( T(t) \) is compact for all \( t > 0 \). Then there exists a mild solution of (5.2.4) on \([-r, b]\), for some \( 0 < b \leq a \).

**Proof.** Let \( 0 < b_1 < a \) and \( c > 0 \) such that \( B_{b_1}(\phi, B) \subset \Omega \) and \( W_f(||\psi||_B) \leq C, \forall \psi \in B_{b_1}(\phi, B) \). Since, \( \phi \in C([-r, 0], X) \) and \( T(\cdot)\phi(0) \in C([0, \infty), X) \), we have that the function \( t \rightarrow y_t \) belongs to \( C([0, a], B) \). Using this fact, we can select \( 0 < b < b_1 \) such that
\[
\sup_{s \in [0, b]} ||y_s - \phi||_B \leq \frac{b_1}{2}
\]
and

\[
CC_0 \frac{|m_f|_{L^\infty([0,b])}}{(1 - q'\alpha)^{\frac{1}{q'}}} + C_0 \sum_{k=1}^{m} L_k \leq \frac{b_1}{2}
\]

On the space

\[
B_{b_1^2}(0, s(b)) = \{ u \in PC([-r,b], X) : u_0 = 0, ||u||_{PC([0,b], X)} \leq \frac{b_1}{2} \}
\]

endowed with the norm $$||.||_{PC([0,b], X)}$$, we define the map

\[
\Gamma : B_{b_1^2}(0, s(b)) \rightarrow PC([-r,b], X)
\]

by $$(\Gamma u)_0 = 0$$ and

\[
\Gamma u(t) = \int_0^t T(t - s) f(s, u_s + y_s) ds + \sum_{0 < t_k < t} T(t - t_k) I_k(u(t_k^-)) , t \in [0,b]
\]

For $$(s, u) \in [0,b] \times B_{b_1^2}(0, s(b))$$,

\[
||u_s + y_s - \phi||_B \leq \sup_{\theta \in [0,s]} ||u(\theta)|| + ||y_s - \phi||_B \\
\leq \frac{b_1}{2} + \frac{b_1}{2} \\
\leq b_1
\]

which implies that $$u_s + y_s \in B_{b_1}(\phi, B)$$ and $$W_f(||u_s + y_s||_B) \leq C$$.

Now, from the properties of $$(T(t))_{t \geq 0}$$ and $$f$$, the Bochner’s criterion for integrable functions and the inequality,

\[
||T(t - s) f(s, u_s + y_s)|| \leq \frac{C_0 m_f(s) W(||u_s + y_s||_B)}{(t - s)^\alpha} \\
\leq \frac{C_0 C m_f(s)}{(t - s)^\alpha}
\]

We infer that the function $$s \rightarrow T(t - s) f(s, u_s + y_s)$$ is integrable on $$[0,t]$$ for all $$t \in [0,b]$$, which implies that $$\Gamma u \in PC([-r,b], X)$$ and $$\Gamma$$ is well defined.

On the another hand,

\[
||\Gamma u(t)|| \leq \int_0^t ||T(t - s) f(s, u_s + y_s)|| ds + \sum_{0 < t_k < t} ||T(t - t_k)|| ||I_k(u(t_k^-)) + y(t_k^-)|| \\
\leq CC_0 \sup_{t \in [0,b]} \int_0^t \frac{m_f(s)}{(t - s)^\alpha} ds + C_0 \sum_{k=1}^{m} L_k \\
\leq \frac{CC_0 |m_f|_{L^\infty([0,b])}}{(1 - q'\alpha)^{\frac{1}{q'}}} + C_0 \sum_{k=1}^{m} L_k
\]
It follows that $\Gamma u \in B_{\frac{r}{T}}(0, s(b))$ and $\Gamma B_{\frac{r}{T}}(0, s(b)) \subset B_{\frac{r}{T}}(0, s(b))$. By the application of the Lebesgue dominated convergence theorem proves that $\Gamma$ is continuous.

Next, we prove that $\Gamma$ is a compact map.

**Step 1:** The set $\Gamma B_{\frac{r}{T}}(0, S(b)) = \{ \Gamma u(t) : u \in B_{\frac{r}{T}}(0, S(b)) \}$ is relatively compact for all $t \in [-r, b]$.

Let $0 < \xi < t < b$. For $u \in B_{\frac{r}{T}}(0, S(b))$, we get

$$
\Gamma u(t) = T(\xi) \int_0^{t-\xi} T(t-s-\xi) f(s, x_s + y_s) ds + \int_{t-\xi}^t T(t-s) f(s, x_s + y_s) ds
$$

$$
+ T(\xi) \sum_{0 \leq t_k < t-\xi} T(t-\xi-t_k) I_k(u(t_k^-) + y(t_k^-)) + \sum_{t-\xi \leq t_k < t} T(t-t_k) I_k(u(t_k^-) + y(t_k^-))
$$

$$
\in T(\xi)B_{\frac{r}{T}}(0, X) + B_{l_\xi}(0, X)
$$

$$
+ T(\xi) \sum_{0 \leq t_k < t-\xi} T(t-\xi-t_k) I_k(u(t_k^-) + y(t_k^-)) + \sum_{t-\xi \leq t_k < t} T(t-t_k) I_k(u(t_k^-) + y(t_k^-))
$$

$$
\in T(\xi)B_{\frac{r}{T}}(0, X) + B_{l_\xi}(0, X)
$$

$$
+ T(\xi) \sum_{0 \leq t_k < t-\xi} T(t-\xi-t_k) L_k + \sum_{t-\xi \leq t_k < t} T(t-t_k) L_k
$$

where $l_\xi = CC_0 ||m_f||_{L_q([t-\xi,t])} \xi^{\frac{1}{T}} \alpha \frac{1}{1-q'\alpha}$

Since, $T(\xi)$ is compact and for $\xi \to 0$, we have $\Gamma B_{\frac{r}{T}}(0, S(b))(t)$ is relatively compact in $X$.

**Step 2:** The set $\Gamma B_{\frac{r}{T}}(0, S(b)) = \{ \Gamma u(t) : u \in B_{\frac{r}{T}}(0, S(b)) \}$ is equicontinuous.

Let $0 < 2\xi < t < b$. By noting that $T(.) \in C([\xi, b], L(X))$, we can select $0 < \delta < \xi$ such that

$$
||T(\xi+s) - T(\xi)||_{L(X)} \leq \xi, \forall 0 < s \leq \delta.
$$

Then, for $u \in B_{\frac{r}{T}}(0, S(b))$ and $0 < h < \delta$ such that $t + h < b$, we get

$$
||\Gamma u(t+h) - \Gamma u(t)|| \leq || \int_0^{t-2\xi} T(t+h-s) f(s, u_s + y_s) ds - \int_0^{t-2\xi} T(t-s) f(s, u_s + y_s) ds ||
$$

$$
+ || \int_{t-2\xi}^{t-h} T(t+h-s) f(s, u_s + y_s) ds || + || \int_{t-2\xi}^t T(t-s) f(s, u_s + y_s) ds ||
$$

$$
+ \sum_{0 \leq t_k \leq t} ||T(t+h-t_k) - T(t-t_k)|| ||I_k(u(t_k^-) + y(t_k^-))||
$$

$$
+ \sum_{t \leq t_k \leq t+h} ||T(t+h-t_k)|| ||I_k(u(t_k^-) + y(t_k^-))||
$$
Proof. For the proof of Theorem 5.2.4, there exists a fixed point $B$ such that $\Gamma B = B$. Then there exists a unique mild solution of $(5.2.4)$ on $[0, b]$. Let $0 < b < a$ and $C > 0$ such that $B_{b_1}(\phi, \mathcal{B}) \subseteq \Omega$ and $\|f(t, \psi)\| \leq C, \forall (t, \psi) \in [0, b_1] \times B_{b_1}(\phi, \mathcal{B})$. By using that the function $t \rightarrow y_t$ belongs to $C([0, a], \mathcal{B})$, we can select $0 < b < b_1$ such that

\[
\|y_s - \phi\|_{\mathcal{B}} \leq \frac{b_1}{2}
\]

and

\[
C_0 \left[ (b_1 + 1) \|L_{f, b_1}\|_{L_q([0, b])} \frac{b_1^{\frac{1}{q}} - \alpha}{(1 - q^\prime \alpha)^{\frac{1}{q^\prime}}} + \sup_{s \in [0, b]} \|f(s, \phi)\| \frac{b_1^{1-\alpha}}{1 - \alpha} + (1 + \frac{b_1}{2}) \sum_{k=1}^m L_k \right] < \min\{\frac{b_1}{2}, 1\}
\]

The right hand side of the above inequality tends to zero as $h \to 0$ and for $\xi$ sufficiently small, which proves that $\Gamma B_{b_1}(0, S(b))$ is right equicontinuous at $t \in (0, b)$. Arguing as above, we can show that $\Gamma B_{b_1}(0, S(b))$ is left equicontinuous at $t \in (0, b]$ and equicontinuous at $t \leq 0$. Thus, $\Gamma B_{b_1}(0, S(b))$ is equicontinuous on $[-r, b]$. Therefore, from the Schauder's fixed point theorem, there exists a fixed point $x$ of $\Gamma$, and by defining the function $u : [-r, b] \to X$ by $u = y + x$, we obtain a mild solution of $(5.2.4)$ on $[-r, b]$. \hfill $\Box$

Theorem 5.3.2. Assume the condition $(H_2)$ and $(H_3)$ are satisfied and $T(.)\phi(0) \in C([0, a], X)$. Then there exists a unique mild solution of $(5.2.4)$ on $[-r, b]$, for some $0 < b \leq a$.

Proof. Let $0 < b_1 < a$ and $C > 0$ such that $B_{b_1}(\phi, \mathcal{B}) \subseteq \Omega$ and $\|f(t, \psi)\| \leq C, \forall (t, \psi) \in [0, b_1] \times B_{b_1}(\phi, \mathcal{B})$. By using that the function $t \rightarrow y_t$ belongs to $C([0, a], \mathcal{B})$, we can select $0 < b < b_1$ such that

\[
\|y_s - \phi\|_{\mathcal{B}} \leq \frac{b_1}{2}
\]

and

\[
C_0 \left[ (b_1 + 1) \|L_{f, b_1}\|_{L_q([0, b])} \frac{b_1^{\frac{1}{q}} - \alpha}{(1 - q^\prime \alpha)^{\frac{1}{q^\prime}}} + \sup_{s \in [0, b]} \|f(s, \phi)\| \frac{b_1^{1-\alpha}}{1 - \alpha} + (1 + \frac{b_1}{2}) \sum_{k=1}^m L_k \right] < \min\{\frac{b_1}{2}, 1\}
\]

(5.3.1)
Let $\Gamma : B_{2b}^{1/2}(0, S(b)) \rightarrow PC([-r, b], X)$ be the operator introduced in the proof of the theorem [5.3.1]. Proceedings as in the proof of the theorem [5.3.1], it is easy to see that $\Gamma$ is well defined.

Next, we prove that $\Gamma$ is contraction on $B_{2b}^{1/2}(0, S(b))$.

For $(s, u) \in [0, b] \times B_{b}^{1/2}(0, S(b))$,

$$||u_s + y_s - \phi||_{B} \leq \sup_{\theta \in [0, s]} ||u(\theta)|| + ||y_s - \phi||_{B} \leq \frac{b_1}{2} + \frac{b_1}{2} \leq b_1,$$

from which we have that

$$||f(s, u_s + y_s)|| \leq C$$

Using the above, for $u \in B_{2b}^{1/2}(0, S(b))$, we find that

$$||\Gamma u(t)|| \leq C_0 \sup_{t \in [0, b]} \int_0^t \frac{||f(s, u_s + y_s) - f(s, \phi)||}{(t - s)^{\alpha}} ds + C_0 \int_0^t \frac{||f(s, \phi)||}{(t - s)^{\alpha}} ds + \sum_{0 \leq t_k \leq t} ||T(t - t_k)|| \cdot ||I_k(u(t_k^-) + y(t_k^-))||$$

$$\leq C_0 \left[ b_1 ||L_{f,b_1}||_{L_q([0,b])} \frac{b^{1-\alpha}}{(1 - q' \alpha)^{\frac{1}{q'}}} + \sup_{s \in [0,b]} ||f(s, \phi)|| \frac{b^{1-\alpha}}{1 - \alpha} \right] + C_0 \sum_{k=1}^m L_k$$

$$\leq C_0 \left[ b_1 ||L_{f,b_1}||_{L_q([0,b])} \frac{b^{1-\alpha}}{(1 - q' \alpha)^{\frac{1}{q'}}} + \sup_{s \in [0,b]} ||f(s, \phi)|| \frac{b^{1-\alpha}}{1 - \alpha} + \sum_{k=1}^m L_k \right]$$

which implies that $\Gamma u \in B_{2b}^{1/2}(0, s(b))$ and $\Gamma B_{2b}^{1/2}(0, s(b)) \subset B_{2b}^{1/2}(0, s(b))$, since $u$ is arbitrary.
Now consider,

$$||\Gamma u(t) - \Gamma v(t)|| \leq C_0 \int_0^t \frac{L_{f,b_1}(s)}{(t-s)^\alpha} ||u_s - v_s||_{C([-r,t_0],X)} ds$$

$$+ \sum_{0 \leq t_k \leq t} ||T(t - t_k)|| \left|\left| I_k(u(t_k)) - I_k(v(t_k)) \right|\right|$$

$$\leq C_0 ||L_{f,b_1}||_{L^q([0,b])} \frac{b^\frac{1}{q'} - \alpha}{(1 - q'\alpha)^\frac{1}{q'}} ||u - v||_{C([0,b],X)}$$

$$+ C_0 \sum_{k=1}^m L_k ||u - v||_{C([0,b],X)}$$

$$\leq C_0 \left[ ||L_{f,b_1}||_{L^q([0,b])} \frac{b^\frac{1}{q'} - \alpha}{(1 - q'\alpha)^\frac{1}{q'}} ||u - v||_{C([0,b],X)} \right.$$

$$\left. + \sum_{k=1}^m L_k ||u - v||_{C([0,b],X)} \right]$$

It follows that $\Gamma$ is a contraction on $B_{b_2}(0,s(b))$ and there exists a unique fixed point $x \in B_{b_2}(0,s(b))$ of $\Gamma$, and therefore by defining the function $u : [-r,b] \rightarrow X$ by $u = x + y$, we obtain a mild solution of (5.2.4) on $[-r,b]$. This completes the proof.

Remark 5.3.1. Using some minor modifications of theorem [5.3.1] and theorem [5.3.2], we can also prove the existence of mild solutions for the case , if $\mathcal{B} = L^p([-r,0],X)$ for some $p \in [1, \frac{1}{\alpha})$ and $||.||_\mathcal{B}$ denotes the norm in $L^p([-r,0],X)$.

5.4 Application

In this section, we apply our abstract results to an impulsive partial differential equation. To apply our results, we need to introduce the required technical tools. Let $\mathcal{U} \subset \mathbb{R}^n$ is an open bounded set with smooth boundary $\partial \mathcal{U}$, $\eta \in (0,1)$ and $X = C^\eta(\overline{\mathcal{U}}, \mathbb{R}^n)$ is the space formed by all the $\eta$ - Holder continuous functions from $\overline{\mathcal{U}}$ into $\mathbb{R}^n$ endowed with the norm

$$||\xi||_{C^\eta(\overline{\mathcal{U}},\mathbb{R}^n)} = ||\xi||_{C(\overline{\mathcal{U}},\mathbb{R}^n)} + ||\xi||_{C^\eta(\overline{\mathcal{U}},\mathbb{R}^n)}$$

where $||.||_{C(\overline{\mathcal{U}},\mathbb{R}^n)}$ is the sup-norm on $\overline{\mathcal{U}}$,

$$||\xi||_{C^\eta(\overline{\mathcal{U}},\mathbb{R}^n)} = \sup_{x,y \in \overline{\mathcal{U}}, x \neq y} \frac{|\xi(x) - \xi(y)|}{|x-y|^\eta}$$
and $|.|$ is the Euclidean norm in $\mathbb{R}^n$.

On the space $X$, we consider the operator $A : D(A) \subset X \to X$ given $Au = \Delta u$ with domain

$$D(A) = \{ u \in C^{2+n}(\overline{U}, \mathbb{R}^n) : u|_{\partial U} = 0 \}$$

From [259], we know that $A$ is an almost sectorial operator which verifies (5.2.5) with $\alpha = \frac{n}{2}$ and $A$ is not sectorial. In the remainder of this section, $(T(t))_{t \geq 0}$ represents the analytic semigroup of growth $\alpha$ generated by $A$.

Consider the impulsive partial differential system,

$$\begin{cases}
\frac{\partial}{\partial t} u(t, \xi) = \Delta u(t, \xi) + \int_{t-r}^{t} b(s-t)u(s, \xi)ds, \forall (t, \xi) \in [0, a] \times U, \\
u(t, \xi) = 0, \forall (t, \xi) \in [0, a] \times \partial U, \\
\Delta u(t, \xi) = \psi(t, \xi), \forall (t, \xi) \in [-r, 0] \times U, \\
\Delta u(t_k, \cdot) = \int_{t_{k-1}}^{t_k} P_k(t_k-s)u(s, \xi)ds
\end{cases}
$$

(5.4.1)

where $(t_k)_{k=1,2,\ldots,m}$ is a strictly increasing sequence of positive numbers.

To represent this system in the abstract form (5.2.4), we introduce the following functions:

(i) $f : [0, a] \times B \to X$ given by $f(t, \psi)(\xi) = \int_{t-r}^{t} b(s-t)\psi(s, \xi)ds$

(ii) $I_k : X \to X$ by, $I_k(\psi)(\xi) = \int_{-r}^{0} P_k(-s)\psi(s, \xi)ds$

We can transform (3.1) into the abstract system (1.1). Moreover, it is easy to see that if $b \in L^1([0, a], \mathbb{R})$ and $B = C([-r, 0], X)$, then $f \in C([0, a], \mathcal{L}(B, X))$ and

$$||f(t, \cdot)||_{\mathcal{L}(B, X)} \leq ||b||_{L^1([0, a])} \text{ for all } t \in [0, a]$$

and that each $I_k$ is a continuous function with

$$||I_kx - I_ky|| \leq L_k||x - y||, x, y \in X$$

In the next results, which are consequences of Theorem 2.2, we said that a function $u : [-r, b] \times \overline{U} \to \mathbb{R}^n$ is a mild solution of (5.4.1) on $[-r, b]$ if the function $u : [0, b] \to X$ given by $u(t)(\xi) = u(t, \xi)$ is a mild solution of the associated abstract system (5.3.1). Next, for $\theta \in [-r, 0]$, we denote by $\phi(\theta)$ the function $\phi(\theta) : \overline{U} \to \mathbb{R}^n$ given by $\phi(\theta)(\xi) = \psi(\theta, \xi)$.

**Proposition 5.4.1.** Assume $\mathcal{B} = C([-r, 0], X)$, the function $\theta \to \phi(\theta)$ belongs to $C([-r, 0], X)$ and $T(.)\phi(0) \in C([0, a], X)$. Then there exists a unique mild solution of (5.4.1) on $[-r, b]$, for some $0 < b \leq a$. 

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5.5 Conclusion

The operator family $T(t)$ is said to be the semigroup of growth $\alpha$ generated by $A$. If we look at spaces of more regular spaces such as Holder continuous functions, the elliptic operators do not satisfy the estimation of sectorial operator. In that case, we use almost sectorial operator to prove the existence results of mild solutions of first order impulsive differential system, using Schauder’s fixed point theorem and contraction mapping principle.