Chapter 5

Simultaneous Identification of Two Time Independent Coefficients in a Nonlinear Phase Field System

5.1 Introduction

In the mathematical literature, the free boundary problems arising from phase transitions have been studied for over a century. Most of the work is concerned with the classical Stefan problem [46] which incorporates the physics of latent heat and heat diffusion in a homogeneous medium. It should be recalled that the phase field models were first introduced by Caginalp [16] and Fix [44] and recently several authors rediscussed the problem and also improved the results from the thermodynamical point of view; see, for instance, the work by Penrose and Fife [86] for an exhaustive explanation of the underlying physics and in fact these works provide an extension of the enthalpy method for the Stefan problem with the advantage of making it possible to describe some rather fine physical phenomena which can take place during fusion-solidification processes. Moreover, in the recent years, the study of several variants of the model has been done and interesting results have been obtained in the directions of existence and regularity of solutions as well as of their dependence on the physical parameters (18, 17) and one can see [4] and [10] for the dynamical controllability of phase field models with one and two control forces respectively.
In this work, we consider the phase field model describing the phase transitions between two states, for example, solid or liquid, in pure material

\[
\begin{align*}
    u_t + \ell v_t - u_{xx} + a(x)u &= 0, \quad (x,t) \in \Omega_T = I \times (0,T], \\
v_t - v_{xx} + b(x)f(v) + c(x)u &= 0, \quad (x,t) \in \Omega_T, \\
u(x,0) &= \phi(x), \quad v(x,0) = \varphi(x), \quad x \in I, \\
u(0,t) &= u(1,t) = v(0,t) = v(1,t) = 0, \quad t \in (0,T],
\end{align*}
\]

(5.1.1)

where \( I = (0,1) \). The solution \( u \) denotes the temperature distribution of a material which occupies the region \( \Omega_T \) and can be in either of the two phases, solid or liquid (if the melting temperature is taken to be zero) and the smooth function \( v \) is called the phase field function. The interface is defined implicitly as the set of points for which \( v \) vanishes (see [16, 39]), where \( \ell \) is the latent heat and the initial data \( \phi(x) \) and \( \varphi(x) \), depending only on spatial variable \( x \), are sufficiently regular.

The coefficients \( a(x) \), \( b(x) \), \( c(x) \) are assumed to be sufficiently smooth and shall be kept independent of time \( t \) and the nonlinear function \( f \) is a polynomial or rational function of \( v \) which is sufficiently smooth. The problem of solving these equations on some space time domain provided that the coefficients, boundary conditions and initial condition is called the direct problem. These types of models have been analyzed in depth mathematically by several authors.

The main objective of this chapter is to obtain the stability estimate for an inverse problem of determining two spatial dependent coefficients \( a(x) \) and \( b(x) \) in the nonlinear phase field system from the following final time overdetermination data

\[
u(x,T) = m(x), \quad v(x,T) = n(x), \quad x \in I,
\]

(5.1.2)

where the functions \( m(x) \) and \( n(x) \) are given and satisfy the homogeneous Dirichlet boundary conditions.

However, over the past several decades, various methods have been employed to solve the inverse problems for phase field system. For example, Baranibalan et al. [9] established the stability estimate for the identification of two time independent coefficients in the linear phase field model from the arbitrary sub domain observation using Carleman estimate and discussed the inverse problem of retrieving spatially varying diffusive coefficient in the linear phase field system from single measurement data at arbitrary subdomain in [8]. Hoffman and Jiang [57] analyzed the phase field model for solidification with constant coefficient for the stability of
reconstruction of the source term using optimal control framework by considering the weighted quadratic cost function. Different from the above mentioned work, we establish the stability estimate for the simultaneous reconstruction of two spatially varying parameters in the nonlinear phase field system. The optimal control technique is the key step in establishing the stability estimate.

5.2 Optimal Control

Let us first convert the identification problem into an optimal control problem based on the final time overspecified output data and then prove the existence of the minimizer of the cost functional based on the assumption of the admissible parameter.

The following assumption on the parameters and the nonlinear functional are essential to prove the stability result.

**Assumption 5.2.1.** For $\alpha > 0$, we assume that the coefficients $a(x), b(x), c(x) \in C^\alpha(\bar{I})$, the initial data $\phi(x), \varphi(x) \geq 0, \phi(x), \varphi(x) \in C^{2,\alpha}(\bar{I})$ and the final time observation $m(x), n(x) \in L^2(I)$. Let $\Psi = \max_{x \in I} |\varphi(x)|$. We suppose that the function $f \in C^2[0, \Psi]$ satisfies

$$f(0) = 0, \quad f'(\cdot) > 0, \quad f''(\cdot) \leq 0 \quad \text{in } [0, \Psi]. \quad (5.2.1)$$

Physically it is very reasonable to search for the parameters $a(x)$ and $b(x)$ among all positive functions which are bounded below and bounded above by two roughly predicted fixed positive constants. Hence we define the admissible set $\mathcal{M}$ by

$$\mathcal{M} = \{a(x), b(x)|0 < a_0 \leq a \leq a_1, \quad 0 < b_0 \leq b \leq b_1, \quad \nabla a, \nabla b \in L^2(I)\} \quad (5.2.2)$$

and the optimal control problem can be stated as follows: Find $(\bar{a}(x), \bar{b}(x)) \in \mathcal{M} \times \mathcal{M}$ satisfying

$$\mathcal{J}(\bar{a}, \bar{b}) = \min_{a, b \in \mathcal{M}} \mathcal{J}(a, b), \quad (5.2.3)$$

where

$$\begin{align*}
\mathcal{J}(a, b) &= \frac{1}{2} \int_I \left( |u(x, T; a) - m(x)|^2 + |v(x, T; b) - n(x)|^2 \right) dx \\
&\quad + \frac{\varphi}{2} \int_I \left( |\nabla a|^2 + |\nabla b|^2 \right) dx, \quad (5.2.4)
\end{align*}$$
Chapter 5

and \((u, v)\) is the solution of the system \((5.1.1)\) for the coefficients \(a(x), b(x) \in \mathcal{M}\), \(a_0, a_1, b_0\) and \(b_1\) are given positive constants and \(\phi\) is the regularization parameter.

By the well known Schauder’s theory for parabolic equations, we can prove the following existence result (see [45, 47, 77, 90]).

**Theorem 5.2.1.** Let \(0 < \alpha < 1\) and the coefficients \(a(x), b(x), c(x) \in C^\alpha(\bar{I})\). Then the system \((5.1.1)\) has a unique solution \(u(x, t), v(x, t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega_T)\).

The following theorem proves the existence of the optimal control \(\bar{a}(x), \bar{b}(x) \in \mathcal{M}\) minimizing the cost functional \(J(a, b)\).

**Theorem 5.2.2.** Suppose \((u, v)\) is the solution of the system \((5.1.1)\). Then there exists a minimizer \(\bar{a}, \bar{b} \in \mathcal{M}\) of \(J(a, b)\) such that

\[
J(\bar{a}, \bar{b}) = \min_{a, b \in \mathcal{M}} J(a, b),
\]

where the admissible set \(\mathcal{M}\) and \(J(a, b)\) are the same as defined in \((5.2.2)\) and \((5.2.4)\).

**Proof.** We use the generalized Weierstrass theorem [112] in order to prove the existence of the solution of the optimal control problem. According to generalized Weierstrass theorem, we must show that the set \(\mathcal{M}\) is closed, bounded and convex and the functional \(J\) is lower semicontinuous. The boundedness and convexity of the admissible set \(\mathcal{M}\) is obvious from the definition. Now we have to show that the closedness of the admissible set \(\mathcal{M}\). From the structure of the construction of the cost functional \(J(a, b)\), note that the function \(J(a, b)\) is nonnegative and thus it has the greatest lower bound. From the definition of the greatest lower bound, it implies that there exists a sequence \((u_n, v_n, a_n, b_n)\) such that

\[
\inf_{a, b \in \mathcal{M}} J(a, b) \leq J(a_n, b_n) \leq \min_{a, b \in \mathcal{M}} J(a, b) + \frac{1}{n}, \quad n = 1, 2, \ldots
\]

where \(\{a_n, b_n\}\) are minimizing sequences of \(J(a, b)\) in \(\mathcal{M}\). From the cost functional, we observe that \(J(a_n, b_n) \leq C\), from which we easily deduce that

\[
\|\nabla a_n\|_{L^2(I)} + \|\nabla b_n\|_{L^2(I)} \leq C,
\]

where the constant \(C\) is independent of \(n\). Then the Sobolev imbeddings \(H^1(I) \subset C^\alpha(\bar{I})\), for \(0 < \alpha \leq \frac{1}{2}\), leads to

\[
\|a_n\|_{C^{\frac{1}{2}}(I)} + \|b_n\|_{C^{\frac{1}{2}}(I)} \leq C. \quad (5.2.5)
\]
Thus, by the existence of classical solutions of parabolic equations, we have
\[
\|u_n\|_{C^\frac{1}{2^+1}\left(\Omega_T\right)} + \|v_n\|_{C^\frac{1}{2^+1}\left(\Omega_T\right)} \leq C,
\]
and, for any \(\omega_T \Subset \Omega_T\), we also get
\[
\|u_n\|_{C^{2+\frac{1}{2^+1}}\left(\omega_T\right)} + \|v_n\|_{C^{2+\frac{1}{2^+1}}\left(\omega_T\right)} \leq C. \tag{5.2.6}
\]

The boundedness results (5.2.5) and (5.2.6) guarantee that there exists a subsequence \((u_{n_k}, v_{n_k}, a_{n_k}, b_{n_k})\) such that
\[
(a_{n_k}, b_{n_k}) \to (\bar{a}, \bar{b}) \in \left(C^{2\frac{1}{2}}(I)\right)^2 \text{ uniformly on } \left(C^{\alpha}(I)\right)^2,
\]
and
\[
\left(u_{n_k}, v_{n_k}\right) \to (\phi, \varphi) \text{ uniformly on } \left(C^{2^+\alpha, \frac{1}{2^+1}}(\Omega_T) \cap C^{2^+\alpha, 1+\alpha}_\text{loc}(\Omega_T)\right)^2.
\]

Hence, replacing \((u, v, a, b)\) in (5.1.1) by \((u_{n_k}, v_{n_k}, a_{n_k}, b_{n_k})\) and passing to the limit, we can see that \((\phi, \varphi, \bar{a}, \bar{b})\) satisfy the system (5.1.1). The closedness is proved by showing that the limit of every convergent sequence is an element of \(\mathcal{M}\). In order to show that the cost functional is lower semicontinuous, consider
\[
\int_I \left(|\nabla(a_{n_k} - \bar{a})|^2 + |\nabla(b_{n_k} - \bar{b})|^2\right) \, dx \geq 0,
\]
that is,
\[
\int_I (|\nabla a_{n_k}|^2 + |\nabla b_{n_k}|^2) \, dx \geq 2 \int_I (\nabla a_{n_k} \nabla \bar{a} + \nabla b_{n_k} \nabla \bar{b}) \, dx - \int_I (|\nabla \bar{a}|^2 + |\nabla \bar{b}|^2) \, dx.
\]
It follows that
\[
\lim_{k \to \infty} \int_I (|\nabla a_{n_k}|^2 + |\nabla b_{n_k}|^2) \, dx \\
\geq 2 \lim_{k \to \infty} \int_I (\nabla a_{n_k} \nabla \bar{a} + \nabla b_{n_k} \nabla \bar{b}) \, dx - \int_I (|\nabla \bar{a}|^2 + |\nabla \bar{b}|^2) \, dx \\
= \int_I (|\nabla \bar{a}|^2 + |\nabla \bar{b}|^2) \, dx.
\]
And so we obtain
\[
\lim_{k \to \infty} \mathcal{J}(a_{n_k}, b_{n_k}) \\
= \lim_{k \to \infty} \frac{1}{2} \int_I (|u_{n_k} - m(x)|^2 + |v_{n_k} - n(x)|^2) \, dx + \lim_{k \to \infty} \frac{\mathcal{Q}}{2} \int_I (|\nabla a_{n_k}|^2 + |\nabla b_{n_k}|^2) \, dx \\
= \mathcal{J}(\bar{a}, \bar{b}).
\]
From the above observations, we can easily deduce that the cost functional is lower semicontinuous, that is,

$$\min_{a, b \in M} J(a, b) \leq J(\bar{a}, \bar{b}) \leq \liminf_{n \to \infty} J(a_n, b_n) = \min_{a, b \in M} J(a, b),$$

hence

$$J(\bar{a}, \bar{b}) = \min_{a, b \in M} J(a, b);$$

Thus $\bar{(a, b)} := (a, b)$ is an optimal solution of the optimal control problem (5.2.2)-(5.2.4). Hence the proof is complete.

### 5.3 Necessary Conditions

In this section, we establish the optimality condition which is to be satisfied by each optimal control $(a, b)$. Suppose $(p, q)$ is the solution of the adjoint system associated with (5.1.1) of the form

\[
\begin{aligned}
-p_t - p_{xx} + ap + cq &= 0, \quad (x, t) \in \Omega_T, \\
-q_t - \ell_p - q_{xx} + bf'(v)q &= 0, \quad (x, t) \in \Omega_T, \\
 p(x, T) &= u(x, T) - m(x), \quad x \in I, \\
 q(x, T) &= v(x, T) - n(x), \quad x \in I, \\
 p(0, t) &= p(1, t) = q(0, t) = q(1, t) = 0, \quad t \in [0, T),
\end{aligned}
\]

(5.3.1)

where $m(x)$, $n(x)$ are the values of the solutions of the system (5.1.1) at final time $t = T$.

**Theorem 5.3.1.** Let $(a, b)$ be the solution of the optimal control problem (5.2.2). Then there exists a function $(u, v, p, q, \xi, \eta; a, b)$ satisfying

\[
\int_{\Omega_T} (p(a - k)u + q(b - l)f(v)) \, dx \, dt + \varphi \int_0^1 [\nabla a \cdot \nabla (k - a) + \nabla b \cdot \nabla (l - b)] \, dx \geq 0,
\]

for any $k, l \in \mathcal{M}$.

**Proof.** We set, for any $k, l \in \mathcal{M}$ and $0 \leq \delta \leq 1$,

$$a_\delta = (1 - \delta)a + \delta k \in \mathcal{M} \quad \text{and} \quad b_\delta = (1 - \delta)b + \delta l \in \mathcal{M}.$$
Then there exists a solution \((u_\delta, v_\delta)\) of the system \([5.1.1]\) with the coefficients \(a = a_\delta\) and \(b = b_\delta\) satisfying

\[
\mathcal{J}_\delta = \mathcal{J}(a_\delta, b_\delta) = \frac{1}{2} \int_I |u_\delta - m(x)|^2 + |v_\delta - n(x)|^2 \, dx + \frac{\varphi}{2} \int_I (|\nabla a_\delta|^2 + |\nabla b_\delta|^2) \, dx,
\]

where \(u_\delta = u(x, T; a_\delta)\) and \(v_\delta = v(x, T; b_\delta)\). Now taking the Fréchet derivative of \(\mathcal{J}_\delta\), we have

\[
\frac{d\mathcal{J}_\delta}{d\delta} \bigg|_{\delta=0} = \int_I \left( [u_\delta - m(x)] \frac{\partial u_\delta}{\partial \delta} \bigg|_{\delta=0} + [v_\delta - n(x)] \frac{\partial v_\delta}{\partial \delta} \bigg|_{\delta=0} \right) \, dx + \varphi \int_I [\nabla a \cdot \nabla (k - a) + \nabla b \cdot \nabla (l - b)] \, dx. \tag{5.3.2}\]

Moreover \((a, b)\) is the optimal solution and therefore

\[
\frac{d\mathcal{J}_\delta}{d\delta} \bigg|_{\delta=0} \geq 0. \tag{5.3.3}
\]

If we take \((\bar{u}_\delta, \bar{v}_\delta) = (\frac{\partial u_\delta}{\partial \delta}, \frac{\partial v_\delta}{\partial \delta})\), then \((\bar{u}_\delta, \bar{v}_\delta)\) satisfies the following system with the coefficients \((a_\delta, b_\delta)\):

\[
\begin{aligned}
(\bar{u}_\delta)_t + \ell(\bar{v}_\delta)_t - (\bar{u}_\delta)_{xx} + a_\delta \bar{u}_\delta + (k - a)u_\delta &= 0, \quad (x, t) \in \Omega_T, \\
(\bar{v}_\delta)_t - (\bar{v}_\delta)_{xx} + b_\delta f'(v_\delta)\bar{v}_\delta + (l - b)f(v_\delta) + c\bar{u}_\delta &= 0, \quad (x, t) \in \Omega_T, \\
\bar{u}_\delta(x, 0) &= \bar{v}_\delta(x, 0) = 0, \quad x \in I, \\
\bar{u}_\delta(0, t) &= \bar{u}_\delta(1, t) = \bar{v}_\delta(0, t) = \bar{v}_\delta(1, t) = 0, \quad t \in (0, T].
\end{aligned}
\]

Let us consider \(\xi = \bar{u}_\delta|_{\delta=0}\) and \(\eta = \bar{v}_\delta|_{\delta=0}\). Then \(\xi\) and \(\eta\) satisfy the following system

\[
\begin{aligned}
\xi_t + \ell \eta_t - \xi_{xx} + a(x) \xi &= (a - k)u, \quad (x, t) \in \Omega_T, \\
\eta_t - \eta_{xx} + b(x)f'(v)\eta + c(x)\xi &= (b - l)f(v), \quad (x, t) \in \Omega_T, \\
\xi(x, 0) &= \eta(x, 0) = 0, \quad x \in I, \\
\xi(0, t) &= \xi(1, t) = \eta(0, t) = \eta(1, t) = 0, \quad t \in (0, T],
\end{aligned} \tag{5.3.4}
\]

where \(u_\delta|_{\delta=0} = u\) and \(v_\delta|_{\delta=0} = v\). Now, from \([5.3.2]\) and \([5.3.3]\), we have

\[
\int_I ([u(x, T; a) - m(x)]\xi(x, T) + [v(x, T; b) - n(x)]\eta(x, T)) \, dx + \varphi \int_I [\nabla a \cdot \nabla (k - a) + \nabla b \cdot \nabla (l - b)] \, dx \geq 0.
\]
From (5.3.1), the last inequality leads to
\[
\int_I (p(x,T)\xi(x,T) + q(x,T)\eta(x,T)) \, dx \\
+ \varphi \int_I [\nabla a \cdot \nabla(k-a) + \nabla b \cdot \nabla(l-b)] \, dx \geq 0. \tag{5.3.5}
\]

Suppose \((p,q)\) is the solution of the system (5.3.1). Multiplying the first equation of (5.3.1) by \(\xi\) and using integration by parts and Green’s theorem, we have
\[
0 = \int_{\Omega_T} \xi(-p_t - p_{xx} + ap + cq) \, dx \, dt \\
= - \int_I [\xi p]_0^T \, dx + \int_{\Omega_T} p(\xi_t + \ell \eta_t - \xi_{xx} + a \xi) \, dx \, dt + \int_{\Omega_T} (cq \xi - \ell \eta p) \, dx \, dt.
\]

From the system (5.3.4), we get
\[
\int_0^1 \xi(x,T)p(x,T) \, dx = \int_{\Omega_T} p(a-k)u \, dx \, dt + \int_{\Omega_T} (cq \xi - \ell \eta p) \, dx \, dt. \tag{5.3.6}
\]

Similarly, from the second equation of (5.3.1) and (5.3.4), we have
\[
\int_0^1 \eta(x,T)q(x,T) \, dx = \int_{\Omega_T} q(b-l)f(v) \, dx \, dt + \int_{\Omega_T} (\ell \eta p - cq \xi) \, dx \, dt. \tag{5.3.7}
\]

Substituting the values of (5.3.6) and (5.3.7) in (5.3.5), we can easily complete the proof of Theorem 5.3.1.

5.4 Basic Lemmas and Main Result

In this section, we establish a stability estimate for the inverse problem consisting of retrieving two smooth coefficients \(a(x)\) and \(b(x)\) in the given nonlinear phase field system. The inequality estimates the discrepancy in the coefficients \(a(x)\) and \(\tilde{a}(x)\) and then between \(b(x)\) and \(\tilde{b}(x)\) of two materials with an upper bound given by some Sobolev norms of the solution at final time \(t = T\). The optimal control problem established in the previous section will be the key ingredient in the proof of such a stability estimate.

5.4.1 Basic Lemmas

In order to prove the main result, we need the following lemmas, which play the
crucial role in proving the stability estimate. Let \((\tilde{u}, \tilde{v})\) be the solution of the following system

\[
\begin{align*}
\tilde{u}_t + \ell \tilde{v}_t - \tilde{u}_{xx} + a(x)\tilde{u} &= 0, \quad (x, t) \in \Omega_T, \\
\tilde{v}_t - \tilde{v}_{xx} + b(x)f(\tilde{v}) + c(x)\tilde{u} &= 0, \quad (x, t) \in \Omega_T, \\
\tilde{u}(x, 0) &= \phi(x), \quad \tilde{v}(x, 0) = \varphi(x), \quad x \in I, \\
\tilde{u}(0, t) &= \tilde{u}(1, t) = \tilde{v}(0, t) = \tilde{v}(1, t) = 0, \quad t \in (0, T].
\end{align*}
\] (5.4.1)

Set \(U = u - \tilde{u}, \ V = v - \tilde{v}, \ \mathcal{A} = a - \tilde{a}\) and \(\mathcal{B} = b - \tilde{b}\) so that the subtraction of (5.4.1) from (5.1.1) yields

\[
\begin{align*}
U_t + \ell V_t - U_{xx} + aU &= -\mathcal{A}\tilde{u}, \quad (x, t) \in \Omega_T, \\
V_t - V_{xx} + bF_1(\tilde{v}, V) + cU &= -\mathcal{B}f(\tilde{v}), \quad (x, t) \in \Omega_T, \\
U(x, 0) &= 0, \quad V(x, 0) = 0, \quad x \in I, \\
U(0, t) &= U(1, t) = V(0, t) = V(1, t) = 0, \quad t \in (0, T],
\end{align*}
\] (5.4.2)

where we have considered

\[
f(v) - f(\tilde{v}) = \int_I \frac{d}{d\sigma} f(\tilde{v} + \sigma V) \, d\sigma = F_1(\tilde{v}, V) V
\] (5.4.3)

with \(F_1(\tilde{v}, V) = \int_I f_V(\tilde{v} + \sigma V) \, d\sigma\) is bounded by a positive real number \(M\).

**Lemma 5.4.1.** Let \((U, V)\) be the solution of the system (5.4.2). Then we have the following estimate:

\[
\max_{0 \leq t \leq T} \int_I (|U|^2 + |V|^2) \, dx 
\leq \exp(K_1 T) \left( \max_{x \in I} |\mathcal{A}|^2 \int_{\Omega_T} |\tilde{u}|^2 \, dxdt + 2r_1^2 \max_{x \in I} |\mathcal{B}|^2 \int_{\Omega_T} |\tilde{v}|^2 \, dxdt \right),
\]

where the constant \(K_1 = 2 + (1 + b_1 M)^2 + 2 \max_{x \in I} |c|^2\).

**Proof.** Multiply the first equation of (5.4.2) by \(U\) and integrate over \(I\) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|U\|^2_{L^2(I)} + \int_I |U_x|^2 \, dx + \int_I a|U|^2 \, dx
= -\int_I UV_t \, dx - \int_I \mathcal{A}U\tilde{u} \, dx
= -\int_I U[V_{xx} - bF_1 V - c U - \mathcal{B}f(\tilde{v})] \, dx - \int_I \mathcal{A}U\tilde{u} \, dx,
\]
where we have assumed that the latent heat $\ell$ as unity. Using the assumption on the coefficients $a$ and $b$, together with Cauchy’s inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|U\|_{L^2(I)}^2 + \frac{1}{2} \int_I |U_x|^2 \, dx + a_0 \int_I |U|^2 \, dx$$

$$\leq \frac{1}{2} \int_I |V_x|^2 \, dx + \left( \max_{x \in I} |c| + \frac{3}{2} \right) \int_I |U|^2 \, dx + \frac{b^2 M^2}{2} \int_I |V|^2 \, dx$$

$$+ \frac{1}{2} \max_{x \in I} |A|^2 \int_I |\tilde{u}|^2 \, dx + \frac{r^2}{2} \max_{x \in I} |B|^2 \int_I |\tilde{v}|^2 \, dx,$$

where we consider $|f(\tilde{v})| \leq r_1 |\tilde{v}|$. Similarly, from the second equation of (5.4.2) and the assumption on $b$, we have

$$\frac{1}{2} \frac{d}{dt} \|V\|_{L^2(I)}^2 + \int_I |V_x|^2 \, dx$$

$$\leq (1 + b_1 M) \int_I |V|^2 \, dx + \frac{1}{2} \max_{x \in I} |c|^2 \int_I |U|^2 \, dx + \frac{r^2}{2} \max_{x \in I} |B|^2 \int_I |\tilde{v}|^2 \, dx.$$

Now coupling the last two estimates, we get

$$\frac{d}{dt} \left[ \|U\|_{L^2(I)}^2 + \|V\|_{L^2(I)}^2 \right] + \int_I (|U_x|^2 + |V_x|^2) \, dx + a_0 \int_I |U|^2 \, dx$$

$$\leq K_1 \left( \|U\|_{L^2(I)}^2 + \|V\|_{L^2(I)}^2 \right) + \max_{x \in I} |A|^2 \int_I |\tilde{u}|^2 \, dx + 2 r_1^2 \max_{x \in I} |B|^2 \int_I |\tilde{v}|^2 \, dx,$$

whence it follows that

$$\frac{d}{dt} \left[ \exp(-K_1 t) \left( \|U\|_{L^2(I)}^2 + \|V\|_{L^2(I)}^2 \right) \right]$$

$$\leq \exp(-K_1 t) \left( \max_{x \in I} |A|^2 \int_I |\tilde{u}|^2 \, dx + 2 r_1^2 \max_{x \in I} |B|^2 \int_I |\tilde{v}|^2 \, dx \right).$$

Thus, integrating from 0 to $t$, we obtain

$$\|U\|_{L^2(I)}^2 + \|V\|_{L^2(I)}^2 \leq \exp(K_1 t) \left( \max_{x \in I} |A|^2 \int_{\Omega_t} \exp(-K_1 s) |\tilde{u}|^2 \, dx \, ds \right.$$

$$\left. + 2 r_1^2 \max_{x \in I} |B|^2 \int_{\Omega_t} \exp(-K_1 s) |\tilde{v}|^2 \, dx \, ds \right),$$

where $\Omega_t = I \times (0, t)$. The proof of Lemma (5.4.1) is thus complete. \( \square \)

Suppose that $(a, b)$ and $(\tilde{a}, \tilde{b})$ are the two minimizers of the control problem (5.2.3); $(u, v), (\tilde{u}, \tilde{v})$ are the solution of the system (5.1.1) and (5.4.1) respectively.
Lemma 5.4.2. Let \((P, Q)\) be the solution of the system \((5.4.4)\). Then there exists a constant \(C > 0\), independent of \(a_0, b_0\), such that

\[
\max_{0 \leq t \leq T} \int_I (|P|^2 + |Q|^2) \, dx \leq C \exp[2(K_2 + K_1)T] \left( \int_I (|m - \tilde{m}|^2 + |n - \tilde{n}|^2) \, dx \right)
+ (1 + T) \left( \max_{x \in \Omega_T} |A|^2 \int_{\Omega_T} (|\tilde{p}|^2 + |\tilde{u}|^2) \, dx \right.
+ \left| \int_{\Omega_T} (|\tilde{v}|^2 + r_1^2 |\tilde{v}|^2) \, dx \right) \right),
\]

where \(K_2 = K_1 + a_1^2\).

Proof. Multiply the first equation of \((5.4.4)\) by \(P\) and integrate over \(I\) to have

\[
-\frac{1}{2} \frac{d}{dt} \|P\|^2_{L^2(I)} + \int_I |P_x|^2 \, dx + a_0 \int_I |P|^2 \, dx = -\int_I cPQ \, dx - \int_I AP\tilde{p} \, dx.
\]

Applying Cauchy’s inequality, we get

\[
-\frac{1}{2} \frac{d}{dt} \|P\|^2_{L^2(I)} + \int_I |P_x|^2 \, dx + a_0 \int_I |P|^2 \, dx
\leq \int_I |P|^2 \, dx + \frac{1}{2} \max_{x \in I} |c|^2 \int_I |Q|^2 \, dx + \frac{1}{2} \max_{x \in I} |A|^2 \int_I |\tilde{p}|^2 \, dx.
\]

It should be noted that

\[
f'(v) - f'(\tilde{v}) = \int_I \frac{d}{d\sigma} (f'(\tilde{v} + \sigma V)) \, d\sigma
= F_2(\tilde{v}, V),
\]

where \(F_2(\tilde{v}, V) = \int_I f''(\tilde{v} + \sigma V) \, d\sigma\) is also bounded by a positive real number \(M\).

Now multiply the second equation of \((5.4.4)\) by \(Q\) and integrate over \(I\) to have

\[
-\frac{1}{2} \frac{d}{dt} \|Q\|^2_{L^2(I)} + \frac{1}{2} \int_I |Q_x|^2 \, dx + \int_I bf'(v)|Q|^2 \, dx
\leq \left( \frac{3}{2} + \max_{x \in I} |c|^2 + \frac{M^2 b_1^2}{2} \right) \int_I |Q|^2 \, dx + \frac{a_1^2}{2} \int_I |P|^2 \, dx + \frac{1}{2} \max_{x \in I} |A|^2 \int_I |\tilde{p}|^2 \, dx
+ \frac{1}{2} \max_{x \in I} |B|^2 \int_I |\tilde{Q}|^2 |f'(\tilde{v})|^2 \, dx + \frac{1}{2} \int_I |P_x|^2 \, dx + \frac{1}{2} \int_I |\tilde{Q}|^2 \, dx,
\]

and \((p, q), (\tilde{p}, \tilde{q})\) are the solutions of the adjoint system \((5.3.1)\). Now setting \(P = p - \tilde{p}\) and \(Q = q - \tilde{q}\), the adjoint system \((5.3.1)\) becomes

\[
\begin{aligned}
-P_t - P_{xx} + aP + cQ &= -A\tilde{p}, \quad (x, t) \in \Omega_T, \\
-Q_t - lP_t - Q_{xx} + b f'(v)Q - B\tilde{q} f'(\tilde{v}) - \tilde{q}b(f'(v) - f'(\tilde{v})), \quad (x, t) \in \Omega_T, \\
P(x, T) &= U(x, T) - (m - \tilde{m}), \quad x \in I, \\
Q(x, T) &= V(x, T) - (n - \tilde{n}), \quad x \in I, \\
P(0, t) &= P(1, t) = Q(0, t) = Q(1, t) = 0, \quad t \in [0, T).
\end{aligned}
\]
here also we assume that the latent heat $\ell$ as unity. Now, combining the above two estimates, we get
\[
-\frac{d}{dt} \left[ \|P\|_{L^2(I)}^2 + \|Q\|_{L^2(I)}^2 \right] + \int_I (|P_x|^2 + |Q_x|^2) \, dx \\
+ 2 \int_I b f'(v) |Q|^2 \, dx + 2a_0 \int_I |P|^2 \, dx \\
\leq K_2 \int_I (|P|^2 + |Q|^2) \, dx + \int_I |qV|^2 \, dx \\
+ 2 \max_{x \in I} |A|^2 \int_I |\bar{p}|^2 \, dx + \max_{x \in I} |B|^2 \int_I |\bar{q}|^2 |f'(\bar{v})|^2 \, dx.
\]
Thus, integrating with respect to $t$ over $t$ to $T$, we obtain
\[
\|P\|_{L^2(I)}^2 + \|Q\|_{L^2(I)}^2 \leq \exp(-K_2 t) \left( \int_{\tilde{\Omega}} \exp(K_2 s) |\tilde{q}V|^2 \, dx \, ds \right) \\
+ \exp(-K_2 t) \left( 2 \max_{x \in I} |A|^2 \int_{\tilde{\Omega}} \exp(K_2 s) |\tilde{p}|^2 \, dx \, ds \right) \\
+ \max_{x \in I} |B|^2 \int_{\tilde{\Omega}} \exp(K_2 s) |\tilde{q}|^2 |f'(\tilde{v})|^2 \, dx \, ds \right) \\
+ 2 \exp(K_2 (T-t)) \left( \int_I (|U(x,T)|^2 + |V(x,T)|^2) \, dx \\
+ \int_I (|m - \bar{m}|^2 + |n - \bar{n}|^2) \, dx \right),
\]
where we have used the assumptions \([5.2.1]\) and \([5.2.2]\). Further it is not difficult to conclude the proof by applying Lemma \([5.4.2]\). □

Moreover a direct computation gives the following estimate:

**Lemma 5.4.3.** Let $(p,q)$ be the solution of the system \([5.3.1]\). Then we have
\[
\max_{0 \leq t \leq T} \int_I (|p|^2 + |q|^2) \, dx \leq \exp(2K_3 T) \int_I (|u(x,T) - m(x)|^2 + |v(x,T) - n(x)|^2) \, dx,
\]
where $K_3 = 1 + a_1^2 + 3 \max_{x \in I} |c|^2$.

**Proof.** Multiplying the first equation of \([5.3.1]\) by $p$ and second equation of \([5.3.1]\) \linebreak by $q$ and integrating over $I$ to get (with $\ell = 1$)
\[
-\frac{d}{dt} \left[ \|P\|_{L^2(I)}^2 + \|Q\|_{L^2(I)}^2 \right] + \int_I (|p_x|^2 + |q_x|^2) \, dx \\
+ 2a_0 \int_I |p|^2 \, dx + 2 \int_I b f'(v) |q|^2 \, dx \\
\leq (1 + a_1^2) \int_I |p|^2 \, dx + (1 + 2 \max_{x \in I} |c| + \max_{x \in I} |c|^2) \int_I |q|^2 \, dx.
\]
Thus, by using Grönwall’s inequality, we obtain

$$-\frac{d}{dt}\left[\exp(K_3t)\left(\|p\|_{L^2(I)}^2 + \|q\|_{L^2(I)}^2\right)\right] \leq 0.$$ 

Now the integration upon $(t, T)$ concludes the proof. \hfill \Box

### 5.4.1 Stability

Now we prove the local stability estimate by using the lemmas proved in the previous section. The proof of this estimate follows certain ideas used for the reconstruction of the source term in the phase-field system \cite{57} and the local volatility in the Block-Scholes equation \cite{65}.

**Theorem 5.4.1.** Let $(u, v)$ and $(\tilde{u}, \tilde{v})$ be the solutions of the systems (5.1.1) and (5.4.1) respectively and suppose there exists a point $x_0 \in I$ such that $a(x_0) = \tilde{a}(x_0)$ and $b(x_0) = \tilde{b}(x_0)$. Then there exists an instant of time $T_0$ such that, for $T \geq T_0$, and a constant $C > 0$, independent of $a_0$ and $b_0$ (given in (5.2.2)), satisfying the following estimate

$$\max_{(0,1)} |A|^2 + \max_{(0,1)} |B|^2 \leq C \int_0^1 (|m - \tilde{m}|^2 + |n - \tilde{n}|^2) \, dx,$$

where $m, \tilde{m}, n$ and $\tilde{n}$ are the values of the solutions of the system (5.1.1) and (5.4.1) at final time $t = T$.

**Proof.** Let us start the proof by taking $k = \tilde{a}$, $l = \tilde{b}$ in the necessary condition. We have

$$\int_{\Omega_T} \left[ p(a - \tilde{a})u + q(b - \tilde{b})f(v) \right] \, dx \, dt + \phi \int_I [\nabla a \cdot \nabla (a - \tilde{a}) + \nabla b \cdot \nabla (b - \tilde{b})] \, dx \geq 0. \tag{5.4.6}$$

And, by taking $k = a$, $l = b$ when $a = \tilde{a}$, $b = \tilde{b}$, we also have

$$\int_{\Omega_T} \left[ \tilde{p}(\tilde{a} - a)\tilde{u} + \tilde{q}(\tilde{b} - b)f(\tilde{v}) \right] \, dx \, dt + \phi \int_I [\nabla \tilde{a} \cdot \nabla (a - \tilde{a}) + \nabla \tilde{b} \cdot \nabla (b - \tilde{b})] \, dx \geq 0, \tag{5.4.7}$$
where \((u, v), (\tilde{u}, \tilde{v})\) are the solutions of the systems (5.1.1) and (5.4.1) respectively and \((p, q), (\tilde{p}, \tilde{q})\) are the solutions of the corresponding adjoint system (5.3.1). Now coupling (5.4.6) and (5.4.7), we get

\[
\varphi \int_I \left( |\nabla A|^2 + |\nabla B|^2 \right) \, dx \\
\leq \int_{\Omega_T} A(pu - \tilde{p}u) \, dx \, dt + \int_{\Omega_T} B(qf(v) - \tilde{q}f(\tilde{v})) \, dx \, dt \\
= \int_{\Omega_T} A(pU + P\tilde{u}) \, dx \, dt + \int_{\Omega_T} B[qf(v) - f(\tilde{v}) + Qf(\tilde{v})] \, dx \, dt. \tag{5.4.8}
\]

Using the relation defined in (5.4.3) and applying Cauchy’s inequality to each of the right hand side integrals, we obtain

\[
\varphi \int_I \left( |\nabla A|^2 + |\nabla B|^2 \right) \, dx \\
\leq \frac{1}{2} \left( \max_{x \in I} |A|^2 \int_I (|p|^2 + |\tilde{u}|^2) \, dx + \max_{x \in I} |B|^2 \int_I (M^2|q|^2 + r_1^2|\tilde{v}|^2) \, dx \\
+ \int_{\Omega_T} (|U|^2 + |V|^2 + |P|^2 + |Q|^2) \, dx \, dt \right). \tag{5.4.9}
\]

From Lemmas 5.4.1 and 5.4.2 it is clear that

\[
\int_{\Omega_T} (|U|^2 + |V|^2 + |P|^2 + |Q|^2) \, dx \, dt \\
\leq CT \exp \left[ 2(K_2 + K_1)T \right] \left( (1 + T) \left( \max_{x \in I} |A|^2 \int_{\Omega_T} (|\tilde{p}|^2 + |\tilde{u}|^2) \, dx \, dt \\
+ \max_{x \in I} |B|^2 \int_{\Omega_T} (|\tilde{q}|^2 + r_1^2|\tilde{v}|^2) \, dx \, dt \right) + \int_I (|m - \tilde{m}|^2 + |n - \tilde{n}|^2) \, dx \right). \tag{5.4.10}
\]

Also, from Lemma 5.4.3 and an analogue of Lemma 5.4.1, there exists a constant \(\Gamma > 0\) such that

\[
\int_{\Omega_T} (|\tilde{p}|^2 + |\tilde{q}|^2) \, dx \, dt \leq T \exp[T(K_4 + 2K_3)]\Gamma \\
\text{and} \quad \int_{\Omega_T} (|\tilde{u}|^2 + |\tilde{v}|^2) \, dx \, dt \leq T \exp(K_4T)\Gamma, \tag{5.4.11}
\]

where \(K_4 = 1 + 2\max_{x \in I} |c|^2 + b_4^2r_1^4\). Moreover, taking \(A(x_0) = 0\) into account and applying Hölder’s inequality, we get

\[
|A(x)| = \left| \int_{x_0}^x (A(y))^\prime dy \right| = \left| \int_{x_0}^x \nabla A dy \right| \leq \left( \int_{x_0}^x |\nabla A|^2 dy \right)^{1/2}, \tag{5.4.12}
\]
so that
\[
\max_{x \in I} |A| \leq \|\nabla A\|_{L^2(I)}, \quad \forall x \in I.
\]
Combining the preceding estimates with (5.4.9), we arrive at
\[
\max_{x \in I} |A|^2 + \max_{x \in I} |B|^2 \leq C_T \left( \max_{x \in I} |A|^2 + \max_{x \in I} |B|^2 \right)
+ \frac{C T}{2\varphi} \exp[2T(K_1 + K_2)] \int_I \left( |(m - \tilde{m})|^2 + |(n - \tilde{n})|^2 \right) \, dx,
\]
where the constant \( C_T = \frac{C T^2}{2\varphi} (1 + T) \exp[2(K_1 + K_2 + K_3 + K_4)T] \Gamma \). Now choosing \( T_0 > 0 \) such that \( C T_0 < 1 \), we complete the proof.

**Remark 5.4.1.** From Theorem 5.4.1, we easily see that if the final measurements of the system (5.1.1) and (5.4.1) are equal, namely,
\[
u(x, T) = \tilde{\nu}(x, T) \quad \text{and} \quad v(x, T) = \tilde{v}(x, T),
\]
then the data \( a \) and \( b \) can be determined uniquely, that is, \( a = \tilde{a} \) and \( b = \tilde{b} \) in \( I \), for some small \( T_0 > 0 \). In fact, from (5.4.9)–(5.4.12), one indeed gets
\[
\int_I (|\nabla A|^2 + |\nabla B|^2) \, dx \leq C_T \int_I (|\nabla A|^2 + |\nabla B|^2) \, dx.
\]
Again, choosing \( T_0 > 0 \) such that \( C T_0 < 1 \), one concludes that
\[
\int_I (|\nabla A|^2 + |\nabla B|^2) \, dx \leq 0.
\]
Taking the assumptions \( A(x_0) = B(x_0) = 0 \) into account, we deduce that \( a(x) - \tilde{a}(x) \equiv 0 \) and \( b(x) - \tilde{b}(x) \equiv 0 \), for all \( x \in I \).

### 5.5 Summary

In this chapter, the stability result for the inverse problem of retrieving two smooth coefficients \( a(x) \) and \( b(x) \) in the nonlinear phase field system has been established. The inequality estimates the discrepancy in the coefficients \( a(x) \) and \( \tilde{a}(x) \) and then \( b(x) \) and \( \tilde{b}(x) \) with an upper bound given by some Sobolev norms of the solutions at \( t = T \). The stability result is achieved by transforming the inverse problem into an optimization problem in which cost functional based on the continuous final time output data is minimized over the admissible parameters. In chapter 6, we take up a system of three equations.