CHAPTER III
ωI - CONTINUOUS AND ωI - CLOSED FUNCTIONS IN IDEAL
TOPOLOGICAL SPACES

3.1 INTRODUCTION
Several authors working in the field of general topology have shown more
interest in studying the properties of generalizations of continuous and closed
functions. Strong and weak forms of continuous functions have been introduced
and studied by several topologists [73,12,13,125,90,1]. Sheik John [142]
introduced ω - continuous functions, ω – irresolute functions, strongly ω – continuous
functions and perfectly ω - continuous functions in topological spaces.

In this chapter, we introduce ωI - continuous functions, ωI – irresolute
functions, weakly ωI – continuous functions, strongly ωI – irresolute functions and
perfectly ωI - continuous functions in ideal topological spaces and discuss some of
their properties. At the end of this chapter, we introduce ωI - closed functions and
ωI - open functions in ideal topological spaces and explore certain characterizations
of these functions.

3.2 ωI -CONTINUOUS FUNCTIONS
In this section, we introduce ωI - continuous functions in ideal topological
spaces and we prove that the composition of two ωI - continuous functions need
not be ωI - continuous.

Definition 3.2.1 A function \( f : (X, \tau, I) \to (Y, \sigma) \) is called ωI - continuous if for
every closed set \( V \) of \( (Y, \sigma) \), \( f^{-1}(V) \in \omega I C(X, \tau, I) \).

Theorem 3.2.2 Every continuous function in \( (X, \tau) \) is ωI - continuous in \( (X, \tau, I) \).
The converse of the above Theorem is not true as seen from the following Example.

Example 3.2.3 Let \( X = Y = \{a, b, c, d\} \), \( \tau = \sigma = \{\phi, \{a, b\}, X\} \) and \( I = \{\phi, \{a\}\} \). Define \( f : (X, \tau, I) \to (Y, \sigma) \) by \( f(a) = a, f(b) = c, f(c) = b, f(d) = d \). Then \( f \) is ωI - continuous
but not continuous. Since \( A = \{c, d\} \) is closed in \( Y \) but \( f^{-1}(A) = \{b, d\} \) is not closed in
\( X \).
**Theorem 3.2.4** For a function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \), the following are equivalent:

(i) The function \( f \) is \( \omega I \) - continuous.

(ii) The inverse image of each closed set in \( Y \) is \( \omega I \) - closed in \( X \).

(iii) The inverse image of each open set in \( Y \) is \( \omega I \) - open in \( X \).

**Proof:** Follows from the Definition 3.2.1.

**Theorem 3.2.5** A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is \( \omega I \) - continuous if and only if

\[ f^{-1}(U) \text{ is } \omega I \text{ - open in } X \text{ for every open set } U \text{ in } (Y, \sigma). \]

**Proof: Necessity.** Let \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) be \( \omega I \) - continuous and \( U \) be an open set in \( (Y, \sigma) \). Then \( U^c \) is closed in \( (Y, \sigma) \) and since \( f \) is \( \omega I \) - continuous, \( f^{-1}(U^c) \) is \( \omega I \) - closed in \( (X, \tau, I) \). But \( f^{-1}(U^c) = (f^{-1}(U))^c \) and so \( f^{-1}(U) \) is \( \omega I \) - open in \( (X, \tau, I) \).

**Sufficiency.** Assume that \( f^{-1}(U) \) is \( \omega I \) - open in \( (X, \tau, I) \) for each open set \( U \) in \( (Y, \sigma) \). Let \( F \) be a closed set in \( (Y, \sigma) \). Then \( F^c \) is open in \( (Y, \sigma) \) and by assumption \( f^{-1}(F^c) \) is \( \omega I \) - open in \( (X, \tau, I) \). Since \( f^{-1}(F^c) = (f^{-1}(F))^c \), we have \( f^{-1}(F) \) is \( \omega I \) - closed in \( (X, \tau, I) \) and so \( f \) is \( \omega I \) - continuous.

**Theorem 3.2.6** A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is \( \omega I \) - continuous if and only if

\[ f : (X, \tau^{\omega I}, I) \rightarrow (Y, \sigma) \text{ is continuous}. \]

**Proof: Necessity.** Assume that \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is \( \omega I \) - continuous. Then \( f^{-1}(U) \in \tau^{\omega I} \) for every \( U \in \sigma \). Therefore, \( f : (X, \tau^{\omega I}, I) \rightarrow (Y, \sigma) \) is continuous.

**Sufficiency.** Assumes that \( f : (X, \tau^{\omega I}, I) \rightarrow (Y, \sigma) \) is continuous. Then \( f^{-1}(G) \in \tau^{\omega I} \) for every \( G \in \sigma \). Therefore, \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is \( \omega I \) - continuous.

**Definition 3.2.7** Let \( x \) be a point of \( (X, \tau, I) \) and \( W \) be a subset of \( (X, \tau, I) \). Then \( W \) is called an \( \omega I \) - neighborhood of \( x \) in \( (X, \tau, I) \) if there exists an \( \omega I \) - open set \( U \) of \( (X, \tau, I) \) such that \( x \in U \subseteq W \).

**Theorem 3.2.8** Let \( (X, \tau, I) \) be \( T \) - dense. Then, for a function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \), the following are equivalent:
(i) The function $f$ is $\omega I$ - continuous.

(ii) For each $x \in X$ and each open set $V$ in $Y$ with $f(x) \in V$, there exists an $\omega I$ - open set $U$ containing $x$ such that $f(U) \subseteq V$.

(iii) For each $x \in X$ and each open set $V$ in $Y$ with $f(x) \in V$, $f^{-1}(V)$ is an $\omega I$ -open neighborhood of $x$.

**Proof:** (i) $\Rightarrow$ (ii) Let $x \in X$ and let $V$ be an open set in $Y$ such that $f(x) \in V$. Since $f$ is $\omega I$ - continuous, $f^{-1}(V)$ is $\omega I$ - open in $X$. By putting $U = f^{-1}(V)$, we have $x \in U$ and $f(U) \subseteq V$.

(ii) $\Rightarrow$ (iii) Let $V$ be an open set in $(Y, \sigma)$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there exists an $\omega I$ - open set $U$ containing $x$ such that $f(U) \subseteq V$. So $x \in U \subseteq f^{-1}(V)$. Hence, $f^{-1}(V)$ is an $\omega I$ - open neighborhood of $x$.

(iii) $\Rightarrow$ (i) Let $V$ be an open set in $(Y, \sigma)$ and let $f(x) \in V$. Then by (iii), $f^{-1}(V)$ is an $\omega I$ - open neighborhood of $x$. Thus for each $x \in f^{-1}(V)$, there exists an $\omega I$ - open set $U_x$ containing $x$ such that $x \in U_x \subseteq f^{-1}(V)$. Hence $f^{-1}(V) \subseteq \bigcup_{x \in f^{-1}(V)} U_x$ and $f^{-1}(V)$ is $\omega I$ - open in $X$.

**Theorem 3.2.9** Let $f : (X, \tau, I) \to (Y, \sigma)$ be an $\omega I$ - continuous function and $A$ be any $\ast$- closed subset of $(X, \tau, I)$. Then the restriction $f \mid A : (A, \tau \mid A, I \mid A) \to (Y, \sigma)$ is $\omega I$ - continuous.

**Proof:** Let $F$ be any closed set in $(Y, \sigma)$. Since $f$ is $\omega I$ - continuous, $f^{-1}(F)$ is $\omega I$ - closed in $(X, \tau, I)$. Since $(f \mid A)^{-1}(F) = f^{-1}(F) \cap A$ and by Theorem 2.2.23, $f^{-1}(F) \cap A \in \omega I C(X, \tau)$. On the other hand, $(f \mid A)^{-1}(F) = f^{-1}(F) \cap A$ and $(f \mid A)^{-1} \in (A, \tau \mid A, I \mid A)$. This shows that $f \mid A : (A, \tau \mid A, I \mid A)$ is $\omega I$ - continuous.

**Theorem 3.2.10** Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function and $\{U_\alpha : \alpha \in \Delta\}$ be an open cover of a $T$ - dense space $X$. If the restriction function $f \mid U_\alpha$ is $\omega I$ - continuous for each $\alpha \in \Delta$, then $f$ is $\omega I$ - continuous.
Proof: Suppose $F$ is an arbitrary open set in $(Y, \sigma)$. Then for each $\alpha \in \Delta$, we have 
\[(f \mid U_\alpha)^{-1}(F) = f^{-1}(F) \cap U_\alpha.\] Because, $f \mid U_\alpha$ is $\omega_l$ - continuous, therefore, $f^{-1}(F) \cap U_\alpha$ is $\omega_l$ - open in $X$ for each $\alpha \in \Delta$. Since for each $\alpha \in \Delta$, $U_\alpha$ is open in $X$, by Theorem 2.3.7, $f^{-1}(F) \cap U_\alpha$ is $\omega_l$ - open in $X$. Now since $X$ is $T$ - dense, we have $f^{-1}(F)$ is $\omega_l$ - open in $X$. This implies $f$ is $\omega_l$ - continuous.

Definition 3.2.11 A space is $(X, \tau, I)$ called a $T^*_\omega$ - space if every $\omega_l$ - closed set in it is closed.

Theorem 3.2.12 Let $(X, \tau, I)$ be an ideal topological space, $(Z, \eta)$ be topological space and $(Y, \sigma, J)$ be a $T^*_\omega$ space. Then the composition $g \circ f : (X, \tau, I) \to (Z, \eta)$ of the $\omega_l$ - continuous functions $f : (X, \tau, I) \to (Y, \sigma, J)$ and $g : (Y, \sigma, J) \to (Z, \eta)$ is $\omega_l$ - continuous.

Proof: Let $F$ be any closed set of $(Z, \eta)$. Then $g^{-1}(F)$ is closed in $(Y, \sigma, J)$, since $g$ is $\omega_l$ - continuous and $(Y, \sigma, J)$ is a $T^*_\omega$ space. Since $g^{-1}(F)$ is closed in $(Y, \sigma, J)$ and $f$ is $\omega_l$ - continuous, $f^{-1}(g^{-1}(F))$ is $\omega_l$ - closed in $(X, \tau, I)$. But $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ and so $g \circ f$ is $\omega_l$ - continuous.

Theorem 3.2.13 Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be $\omega_l$ - continuous and $g : (Y, \sigma, J) \to (Z, \eta)$ be continuous. Then $g \circ f$ is $\omega_l$ - continuous.

Proof: Let $F$ be any closed set in $(Z, \eta)$. Since $g : (Y, \sigma, J) \to (Z, \eta)$ is continuous, $g^{-1}(F)$ is closed in $(Y, \sigma, J)$. Since $f : (X, \tau, I) \to (Y, \sigma, J)$ is $\omega_l$ - continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\omega_l$ - closed in $(X, \tau, I)$ and so $g \circ f$ is $\omega_l$ - continuous.

Remark 3.2.14 The composition of two $\omega_l$ - continuous functions need not be $\omega_l$ - continuous and this is shown by the following Example.

Example 3.2.15 Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$, $\eta = \{\emptyset, \{a\}, Z\}$, $I = \{\emptyset, \{a\}\}$ and $J = \emptyset$. Define $f : (X, \tau, I) \to (Y, \sigma, J)$ by $f(a) = f(c) = c$ and $f(b) = b$ and $g : (Y, \sigma, J) \to (Z, \eta)$ be the identity function. Then $f$ and $g$ are $\omega_l$ - continuous but their composition $g \circ f : (X, \tau, I) \to (Z, \eta)$ is not $\omega_l$ - continuous.
Since $A = \{b\}$ is closed in $(Z, \eta)$ but $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) = f^{-1}(g^{-1}\{b\}) = f^{-1}(\{b\}) = \{b\}$ which is not $\mathcal{dl}$ - closed in $(X, \tau, I)$.

**Theorem 3.2.16** If $(X, \tau, I)$ is a $T$- dense space and a function $f : (X, \tau, I) \to (Y, \sigma)$ is $\mathcal{dl}$ - continuous, then the graph function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$ is $\mathcal{dl}$ - continuous.

**Proof:** Let $f$ be $\mathcal{dl}$ - continuous. Now let $x \in X$ and $W$ be any open set in $X \times Y$ containing $g(x) = (x, f(x))$. Then there exists a basic open set $U \times V$ such that $g(x) \subseteq U \times V \subseteq W$. Since $f$ is $\mathcal{dl}$ - continuous, there exists a $\mathcal{dl}$ - open set $U_i$ in $X$ such that $x \in U_i \subseteq X$ and $f(U_i) \subseteq V$. Then by Theorem 2.3.7, $U_i \cap U$ is $\mathcal{dl}$ - open in $(X, \tau, I)$ and we have $x \in U_i \cap U \subseteq U$, then $g(U_i \cap U) \subseteq U \times V \subseteq W$. Since $(X, \tau, I)$ is $T$- dense, therefore by Theorem 3.2.8 $g$ is $\mathcal{dl}$ - continuous.

**Theorem 3.2.17** A function $f : (X, \tau, I) \to (Y, \sigma)$ is $\mathcal{dl}$ -continuous, if the graph function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$ is $\mathcal{dl}$ - continuous.

**Proof:** Suppose that $g$ is $\mathcal{dl}$ - continuous and let $V$ be an open set in $Y$ containing $f(x)$. Then $X \times V$ is an open set in $X \times Y$ and by the $\mathcal{dl}$ - continuity of $g$, there exists an $\mathcal{dl}$ - open set $U$ in $X$ containing $x$ such that $g(U) \subseteq X \times V$. Therefore, we obtain $f(U) \subseteq V$. This shows that $f$ is $\mathcal{dl}$ - continuous.

**Theorem 3.2.18** Let $\{X_\alpha : \alpha \in \Delta \}$ be any family of ideal topological spaces. If $f : (X, \tau, I) \to \prod_{\alpha \in \Delta} X_\alpha$ is a $\mathcal{dl}$ - continuous function, then $P_\alpha \circ f : X \to X_\alpha$ is $\mathcal{dl}$ - continuous for each $\alpha \in \Delta$, where $P_\alpha$ is the projection of $\prod X_\alpha$ onto $X_\alpha$.

**Proof:** We will consider a fixed $\alpha_0 \in \Delta$. Let $G_{\alpha_0}$ be an open set of $X_{\alpha_0}$. Then, $(P_{\alpha_0})^{-1}(G_{\alpha_0})$ is open in $\prod X_\alpha$. Since $f$ is $\mathcal{dl}$ -continuous, $f^{-1}((P_{\alpha_0})^{-1}(G_{\alpha_0})) = (P_{\alpha_0} \circ f)^{-1}(G_{\alpha_0})$ is $\mathcal{dl}$ - open in $X$. Thus, $P_\alpha \circ f$ is $\mathcal{dl}$ - continuous.
3.3. **ωI - IRRESOLUTE FUNCTIONS**

In this section, we introduce the concept of **ωI** - irresolute function in ideal topological spaces.

**Definition 3.3.1** A function \( f : (X, \tau, I) \to (Y, \sigma, J) \) is called **ωI** - irresolute if the inverse image of every **ωI** – closed set in \( (Y, \sigma, J) \) is **ωI** – closed set in \( (X, \tau, I) \).

**Remark 3.3.2** The following Examples show that the notion of irresolute and **ωI** - irresolute are independent of each other.

**Example 3.3.3** Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \), \( \sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\} \), \( I = \{\phi\} \) and \( J = \{\phi, \{a\}\} \). Then the identity function \( f : (X, \tau, I) \to (Y, \sigma, J) \) on \( X \) is **ωI** - irresolute but not irresolute.

**Example 3.3.4** Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \), \( \sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\} \), \( I = \{\phi, \{c\}\} \) and \( J = P(X) \). Then the function defined by \( f(a) = b, f(b) = a \) and \( f(c) = c \) is irresolute but it is not **ωI** - irresolute.

**Theorem 3.3.5** A function \( f : (X, \tau, I) \to (Y, \sigma, J) \) is **ωI** - irresolute if and only if \( f^{-1}(U) \) is **ωI** - open in \( (X, \tau, I) \) for every **ωI** - open set \( U \) in \( (Y, \sigma, J) \).

**Theorem 3.3.6** A function \( f : (X, \tau, I) \to (Y, \sigma, J) \) is **ωI** - irresolute then it is **ωI** - continuous.

The converse of the above Theorem is not true as seen from the following Example.

**Example 3.3.7** Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a\}, \{a, b\}, X\} \), \( \sigma = \{\phi, \{a\}, \{b, c\}, Y\} \), \( I = \{\phi\} \). Then the identity function \( f : (X, \tau, I) \to (Y, \sigma, J) \) on \( X \) is **ωI** - continuous but it is not **ωI** - irresolute as the inverse image of the **ωI** – closed set \( \{a, c\} \) in \( (Y, \sigma, J) \) is \( \{a, c\} \) which is not **ωI** – closed in \( (X, \tau, I) \).

**Theorem 3.3.8** Let \( (X, \tau, I) \) be an ideal space. \( (Y, \sigma, J) \) be a \( T_\omega^+ \) space and \( f : (X, \tau, I) \to (Y, \sigma, J) \) be any function. Then the following are equivalent.

(i) \( f \) is **ωI** - irresolute.

(ii) \( f \) is **ωI** - continuous.
Proof: (i) ⇒ (ii) Follows from Theorem 3.3.6.

(ii) ⇒ (i) Let $F$ be an $\omega I$ - closed set in $(Y, \sigma, J)$. Since $(Y, \sigma, J)$ is a $T^*_\omega$ space, $F$ is a closed set in $(Y, \sigma, J)$ and by hypothesis, $f^{-1}(F)$ is $\omega I$ - closed set in $(X, \tau, I)$. Therefore $f$ is $\omega I$ - irresolute.

**Theorem 3.3.9** For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:

(i) $f$ is $\omega I$ - irresolute,

(ii) The inverse image of each $\omega I$ - closed set in $(Y, \sigma, J)$ is $\omega I$ - closed in $(X, \tau, I)$.

(iii) For each $x \in X$ and each $V \in \omega IC(Y, \sigma)$ containing $f(x)$, there exists

$$U \in \omega IC(X, \tau)$$

containing $x$ such that $f(U) \subseteq V$.

**Definition 3.3.10** If $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be pre-semi-$I$ -open if and only if for all $A \in SIO(X)$, $f(A) \in SIO(Y)$.

**Lemma 3.3.11** If $f : (X, \tau, I) \to (Y, \sigma, J)$ is continuous and open, then $f$ is $I$ - irresolute.

**Proof:** Let $B \in SIO(Y)$ then there is an open set $V$ such that $V \subseteq B \subseteq cl^*(V)$. Also, $f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}(cl^*(V)) \subseteq cl^*(f^{-1}(V))$ and since $f$ is continuous $f^{-1}(V)$ is open. Thus, $f^{-1}(B)$ is semi-$I$ - open in $(X, \tau, I)$. This shows that $f$ is $I$ - irresolute.

**Lemma 3.3.12** If $f : (X, \tau, I) \to (Y, \sigma, J)$ is continuous and open. Let $A \in SIO(X)$. Then $f(A) \in SIO(Y)$.

**Lemma 3.3.13** If $f : (X, \tau, I) \to (Y, \sigma, J)$ is continuous and open, then $f$ is $I$ - irresolute and pre-semi-$I$ -open.

**Theorem 3.3.14** If $f : (X, \tau, I) \to (Y, \sigma, J)$ is bijective, pre-semi-$I$ - open and $\omega I$ - continuous, then $f$ is $\omega I$ - irresolute.

**Proof:** Let $A$ be $\omega I$ - closed set in $(Y, \sigma, J)$. Let $U$ be any semi-$I$ -open set in $(X, \tau, I)$ such that $f^{-1}(A) \subseteq U$. Then $A \subseteq f(U)$. Since $A$ is $\omega I$ - closed and $f(U)$ is semi-$I$ -open in $(Y, \sigma, J)$, $cl^*(A) \subseteq f(U)$ holds and hence $f^{-1}(cl^*(A)) \subseteq U$. Since $f$
is \( \omega_I \)-continuous and \( cl^*(A) \) is *-closed in \( (Y, \sigma, J) \), \( cl^*(f^{-1}(cl^*(A))) \subseteq U \) and so \( cl^*(f^{-1}(A)) \subseteq U \). Therefore, \( f^{-1}(A) \) is \( \omega_I \)-closed in \( (X, \tau, I) \) and hence \( f \) is \( \omega_I \)-irresolute.

The following Examples show that no assumption of Theorem 3.3.14 can be removed.

**Example 3.3.15** Let \( X = \{a, b, c\} = Y, \tau = \{\phi, \{a, b\}, X\}, \sigma = \{\phi, \{a\}, \{b, c\}, Y\} \) and \( I = \{\phi\} \).

Let \( f : (X, \tau, I) \to (Y, \sigma, I) \) defined by \( f(a) = f(c) = b \) and \( f(b) = c \). Then \( f \) is \( \omega_I \)-continuous and pre-semi-\( I \)-open but it is not bijective and so \( f \) is not \( \omega_I \)-irresolute.

**Example 3.3.16** Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\phi, \{a\}, \{a, b\}, X\}, \sigma = \{\phi, \{a\}, \{a, b\}, Y\} \) and \( I = \{\phi, \{c\}\} \). Then the identity function \( f : (X, \tau, I) \to (Y, \sigma, I) \) on \( X \) is \( \omega_I \)-continuous and bijective but not pre-semi-\( I \)-open and so \( f \) is not \( \omega_I \)-irresolute.

**Example 3.3.17** Let \( X = \{a, b, c\} = Y \), \( \tau = \{\phi, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{a, b\}, Y\} \) and \( I = \{\phi\} \). Define \( f : (X, \tau, I) \to (Y, \sigma, I) \) by \( f(a) = a \), \( f(b) = c \) and \( f(c) = b \). Then \( f \) is bijective and pre-semi-\( I \)-open but not \( \omega_I \)-continuous and so \( f \) is not \( \omega_I \)-irresolute.

**Theorem 3.3.18** If \( f : (X, \tau, I) \to (Y, \sigma, J) \) is bijective, *-closed and \( I \)-irresolute, then the inverse function \( f^{-1} : (Y, \sigma, J) \to (X, \tau, I) \) is \( \omega_I \)-irresolute.

**Proof:** Let \( A \) be \( \omega_I \)-closed in \( (X, \tau, I) \). Let \( (f^{-1})^{-1}(A) = f(A) \subseteq U \) where \( U \) is semi-\( I \)-open in \( (Y, \sigma, J) \). Then, \( A \subseteq f^{-1}(U) \) holds. Since \( f^{-1}(U) \) is semi-\( I \)-open in \( (X, \tau, I) \) and \( A \) is \( \omega_I \)-closed in \( (X, \tau, I) \), \( cl^*(A) \subseteq f^{-1}(U) \) and hence \( f(cl^*(A)) \subseteq U \).

Since \( f \) is *-closed and \( cl^*(A) \) is *-closed in \( (X, \tau, I) \), \( f(cl^*(A)) \) is *-closed in \( (Y, \sigma, J) \) and so \( f(cl^*(A)) \) is \( \omega_I \)-closed in \( (Y, \sigma, J) \). Therefore \( cl^*(f(cl^*(A))) \subseteq U \) and hence \( cl^*(f(A)) \subseteq U \). Thus \( f(A) \) is \( \omega_I \)-closed in \( (Y, \sigma, J) \) and so \( f^{-1} \) is \( \omega_I \)-irresolute.
Theorem 3.3.19 Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) be \( \omega I \)-irresolute function and let \( A \) be any \(*\)-closed subset of \((X, \tau, I)\). Then the restriction \( f \mid_A : (A, \tau \mid_A, I \mid_A) \to (Y, \sigma, J) \) is \( \omega I \)-irresolute.

3.4 STRONGLY \( \omega I \)-IRRESOLUTE FUNCTIONS, PERFECT AND WEAK FORMS OF \( \omega I \)-CONTINUOUS FUNCTIONS

In this section, we introduce the concepts of weakly \( \omega I \)-continuous functions, strongly \( \omega I \)-irresolute functions and perfectly \( \omega I \)-continuous functions in ideal topological spaces.

Definition 3.4.1 A function \( f : (X, \tau, I) \to (Y, \sigma, J) \) is called weakly \( \omega I \)-continuous if for each \( x \in X \) and each open set \( V \) in \( Y \) containing \( f(x) \), there exists an \( \omega I \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq \text{cl}^I(V) \).

Remark 3.4.2

(i) Every weakly \( I \)-continuous function is weakly \( \omega I \)-continuous.

(ii) Every \( \omega I \)-continuous function is weakly \( \omega I \)-continuous.

Remark 3.4.3 \( \omega I \)-continuity and weakly \( I \)-continuity are independent of each other.

Example 3.4.4 Let \( X = Y = \{ a, b, c, d \} \) with \( \tau = \sigma = \{ \phi, \{ a, b \}, X \} \) and \( I = \{ \phi, \{ a \} \} \). Define \( f : (X, \tau, I) \to (Y, \sigma) \) by \( f(a) = a, f(b) = c, f(c) = b, f(d) = d \). Then \( f \) is \( \omega I \)-continuous but not weakly \( I \)-continuous. Since \( f(b) = c \) and a closed set \( A = \{ c, d \} \) containing \( f(b) \), the only closed set containing \( \{ b \} \) is \( U = X \) and \( f(U) \not\subseteq \text{cl}^I(A) = \{ c, d \} \).

Example 3.4.5 Let \( X = \{ a, b, c, d \} \), with \( \tau = \{ \phi, \{ a, b, c \}, X \} \) and \( I = \{ \phi, \{ a \}, \{ b \}, \{ a, b \} \} \). Let \( Y = \{ 1, 2, 3, 4 \} \), \( \sigma = \{ \phi, \{ 1, 2 \}, Y \} \) and \( J = \{ \phi, \{ 3 \}, \{ 4 \}, \{ 3, 4 \} \} \). Define \( f : (X, \tau, I) \to (Y, \sigma, J) \) by \( f(a) = 1, f(b) = 3, f(c) = 2, f(d) = 4 \). Then \( f \) is weakly \( I \)-continuous but not \( \omega I \)-continuous. Since \( V = \{ 1, 2 \} \) is open in \( Y \) but \( f^{-1}(V) = \{ a, c \} \) is not \( \omega I \)-open in \( X \).
Definition 3.4.6 A function $f: (X, \tau) \to (Y, \sigma, J)$ is said to be strongly $\omega I$-irresolute if for every $\omega I$-open set $V$ in $(Y, \sigma, J)$, $f^{-1}(V)$ is open in $(X, \tau)$.

Theorem 3.4.7 If $f: (X, \tau, I) \to (Y, \sigma)$ is strongly $\omega I$-irresolute, then it is continuous.

The converse of the above Theorem is not true as seen from the following Example.

Example 3.4.8 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\sigma = \{\emptyset, \{a, b\}, Y\}$ and $J = \{\emptyset, \{a\}\}$. Define a function $f: (X, \tau) \to (Y, \sigma, J)$ by $f(a) = f(b) = a$ and $f(c) = b$. Then $f$ is continuous. But $f$ is not strongly $\omega I$-irresolute, since for the $\omega I$-open set $A = \{a\}$ in $(Y, \sigma, J)$, $f^{-1}(A) = \{a, b\}$ is not open in $(X, \tau)$.

Theorem 3.4.9 Let $(X, \tau, I)$ be an ideal space. $(Y, \sigma, J)$ be a $T_{\omega I}^*$-space and $f: (X, \tau, I) \to (Y, \sigma, J)$ be any function. Then the following are equivalent:

(i) $f$ is strongly $\omega I$-irresolute.
(ii) $f$ is continuous.

Proof: (i) $\Rightarrow$ (ii) Follows from Theorem 3.4.7.

(ii) $\Rightarrow$ (i) Let $U$ be any $\omega I$-open set in $(Y, \sigma, J)$. Since $(Y, \sigma, J)$ is a $T_{\omega I}^*$-space, $U$ is open set in $(Y, \sigma, J)$ and by hypothesis, $f^{-1}(U)$ is open set in $(X, \tau, I)$. Therefore $f$ is strongly $\omega I$-irresolute.

Theorem 3.4.10 If $f: (X, \tau) \to (Y, \sigma, J)$ is strongly continuous, then it is strongly $\omega I$-irresolute.

The converse of the above Theorem is not true as seen from the following Example.

Example 3.4.11 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a, b\}, Y\}$ and $J = \{\emptyset, \{a\}\}$. Let $f = (X, \tau) \to (Y, \sigma, J)$ be the identity function. Then $f$ is strongly $\omega I$-irresolute, but $f$ is not strongly continuous.

Theorem 3.4.12 A function is $f: (X, \tau) \to (Y, \sigma, J)$ strongly $\omega I$-irresolute if and only if the inverse image of every $\omega I$-closed set in $(Y, \sigma, J)$ is closed in $(X, \tau)$.

Definition 3.4.13 An ideal topological space $(X, \tau, I)$ is called an $\omega I$-space if every subset in it is $\omega I$-closed. That is, $\tau^{\omega I} = P(X)$.  

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Example 3.4.14 Let \( X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \), \( Y = \{p, q\} \) and \( \sigma \) be the discrete topology on \( Y \), and \( J = \{\emptyset\} \). Then the topological space \((X, \tau, I)\) is an \( \omega I \) - space because \( \tau^{\omega I} = P(X) \).

Theorem 3.4.15 Let \((X, \tau)\) be a discrete topological space, \((Y, \sigma, J)\) be an \( \omega I \) - space and \( f : (X, \tau) \to (Y, \sigma, J) \) be any function. Then the following are equivalent:

(i) \( f \) is strongly continuous.

(ii) \( f \) is strongly \( \omega I \) - irresolute.

Proof: (i) \( \Rightarrow \) (ii) Follows from Theorem 3.4.10.

(ii) \( \Rightarrow \) (i) Let \( U \) be any \( \omega I \) - open set in \((Y, \sigma, J)\). Since \((Y, \sigma, J)\) is an \( \omega I \) - space, \( U \) is an \( \omega I \) - open subset of \((Y, \sigma, J)\) and by hypothesis, \( f^{-1}(U) \) is open in \((X, \tau)\). But \((X, \tau)\) is a discrete topological space and so \( f^{-1}(U) \) is also closed in \((X, \tau)\). That is, \( f^{-1}(U) \) is both open and closed in \((X, \tau)\) and so \( f \) is strongly continuous.

Theorem 3.4.16 Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) be any function and both \((X, \tau, I)\) and \((Y, \sigma, J)\) be \( T_{\omega}^* \) - spaces. Then the following are equivalent:

(i) \( f \) is strongly \( \omega I \) - irresolute.

(ii) \( f \) is continuous.

(iii) \( f \) is \( \omega I \) - irresolute.

(iv) \( f \) is \( \omega I \) - continuous.

Theorem 3.4.17 Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) and \( g : (Y, \sigma, J) \to (Z, \eta, K) \) are strongly \( \omega I \) - irresolute, where \( I, J \) and \( K \) are ideals on \( X, Y \) and \( Z \) respectively. Then their composition \( g \circ f : (X, \tau, I) \to (Z, \eta, K) \) is also strongly \( \omega I \) - irresolute.

Proof: Let \( U \) be an \( \omega I \) - open set in \((Z, \eta, K)\). Since \( g \) is strongly \( \omega I \) - irresolute, \( g^{-1}(U) \) is open in \((Y, \sigma, J)\). Since \( g^{-1}(U) \) is open, it is \( \omega I \) - open in \((Y, \sigma, J)\). As \( f \) is also strongly \( \omega I \) - irresolute \( f^{-1}(U) = (g \circ f)^{-1}(U) \) is open in \((X, \tau, I)\) and so \( (g \circ f) \) is strongly \( \omega I \) - irresolute.

Theorem 3.4.18 Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) and \( g : (Y, \sigma, J) \to (Z, \eta, K) \) be any two
functions, where $I, J$ and $K$ are ideals on $X, Y$ and $Z$ respectively. Then their composition $g \circ f : (X, \tau, I) \to (Z, \eta, K)$ is

(i) strongly $\omega \ell$ - irresolute if $f$ is continuous and $g$ is strongly $\omega \ell$ - irresolute.

(ii) $\omega \ell$ - irresolute if $f$ is $\omega \ell$ - continuous ($f$ is $\omega \ell$ - irresolute) and $g$ is strongly $\omega \ell$ - irresolute.

(iii) strongly $\omega \ell$ - irresolute, if $f$ is irresolute and $g$ is strongly continuous.

(iv) continuous if $f$ is strongly $\omega \ell$ - irresolute and $g$ is $\omega \ell$ - continuous.

**Theorem 3.4.19** Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a strongly $\omega \ell$ - irresolute function and $A$ is $\ast$- closed, then the restriction $f \mid_A : (A, \tau \mid_A, I \mid_A) \to (Y, \sigma, J)$ is $\omega \ell$ - continuous.

**Definition 3.4.20** A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is called perfectly $\omega \ell$ - continuous if the inverse image of every $\omega \ell$ - open set in $(Y, \sigma, J)$ is both open and closed in $(X, \tau)$.

**Theorem 3.4.21** If $f : (X, \tau, I) \to (Y, \sigma, J)$ is perfectly $\omega \ell$ - continuous then it is strongly $\omega \ell$ - irresolute.

**Proof:** Since $f : (X, \tau, I) \to (Y, \sigma, J)$ is perfectly $\omega \ell$ - continuous, $f^{-1}(U)$ is both open and closed in $(X, \tau)$ for every $\omega \ell$ - open set $U$ in $(Y, \sigma, J)$. Therefore, $f$ is strongly $\omega \ell$ - irresolute.

The converse of the above Theorem is not true as seen from the following Example.

**Example 3.4.22** Let $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, $\sigma = \{\emptyset, \{a, b\}, Y\}$ and $J = \{\emptyset, \{a\}\}$. Define $f : (X, \tau) \to (Y, \sigma, J)$ be the identity function. Then $f$ is strongly $\omega \ell$ - irresolute. But $f$ is not perfectly $\omega \ell$ - continuous.

**Theorem 3.4.23** If $f : (X, \tau) \to (Y, \sigma, J)$ is strongly continuous, then it is perfectly $\omega \ell$ - continuous.

**Proof:** Since $f : (X, \tau) \to (Y, \sigma, J)$ is strongly continuous, $f^{-1}(U)$ is both open and closed in $(X, \tau)$, for every $\omega \ell$ - open set $U$ in $(Y, \sigma, J)$. Therefore, $f$ is perfectly $\omega \ell$ - continuous.

The converse of the above Theorem is not true as seen from the following Example.
Example 3.4.24 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $\sigma = \{Y, \phi, \{a\}\}$ and $J = \{\phi\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma, J)$ be the identity function. Then $f$ is perfectly $\omega I$-continuous, but not strongly continuous.

Remark 3.4.25 From the above definitions, for a function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$, we obtain the following implications:

Perfectly $\omega I$-continuity
\[ \Rightarrow \]

strongly $\omega I$-irresolute $\Rightarrow$ continuity $\Rightarrow$ $\omega I$-continuity
\[ \Rightarrow \]

Weakly $I$-continuity $\Rightarrow$ weakly $\omega I$-continuity

Theorem 3.4.26 Let $(X, \tau)$ be a discrete topological space, $(Y, \sigma, J)$ be any ideal topological space and $f : (X, \tau) \rightarrow (Y, \sigma, J)$ be a function. Then the following are equivalent:

(i) $f$ is perfectly $\omega I$-continuous.

(ii) $f$ is strongly $\omega I$-irresolute.

Proof: (i) $\Rightarrow$ (ii) Follows from Theorem 3.4.21.

(ii) $\Rightarrow$ (i) Let $U$ be any $\omega I$-open set in $(Y, \sigma, J)$. By hypothesis $f^{-1}(U)$ is open in $(X, \tau)$. Since $(X, \tau)$ is a discrete space, $f^{-1}(U)$ is also closed in $(X, \tau)$. That is, $f^{-1}(U)$ is both open and closed in $(X, \tau)$ and so $f$ is perfectly $\omega I$-continuous.

Theorem 3.4.27 Let $(X, \tau)$ be a discrete topological space and $(Y, \sigma, J)$ be an $\omega I$-space and $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a function. Then the following are equivalent:

(i) $f$ is strongly continuous.

(ii) $f$ is strongly $\omega I$-irresolute.

(iii) $f$ is perfectly $\omega I$-continuous.

Theorem 3.4.28 A function $f : (X, \tau) \rightarrow (Y, \sigma, J)$ is perfectly $\omega I$-continuous if and only if the inverse image of every $\omega I$-closed set in $(Y, \sigma, J)$ is both open and closed in $(X, \tau)$.
Theorem 3.4.29 Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) and \( g : (Y, \sigma, J) \to (Z, \eta, K) \) are perfectly \(\omega I\) - continuous, where \(I, J\) and \(K\) are ideals on \(X, Y\) and \(Z\) respectively. Then their composition \( g \circ f : (X, \tau, I) \to (Z, \eta, K) \) is also perfectly \(\omega I\) - continuous.

Theorem 3.4.30 Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) and \( g : (Y, \sigma, J) \to (Z, \eta, K) \) be two functions, where \(I, J\) and \(K\) are ideals on \(X, Y\) and \(Z\) respectively. Then their composition \( g \circ f : (X, \tau, I) \to (Z, \eta, K) \) is

(i) \(\omega I\) - continuous if \(f\) is \(\omega I\) - continuous and \(g\) is strongly continuous.

(ii) \(\omega I\) - irresolute if \(f\) is \(\omega I\) - continuous (or \(f\) is \(\omega I\) - irresolute) and \(g\) is perfectly \(\omega I\) - continuous.

(iii) strongly \(\omega I\) - irresolute if \(f\) is continuous (or \(f\) is strongly \(\omega I\) - continuous) and \(g\) is perfectly \(\omega I\) - continuous.

(iv) perfectly \(\omega I\) - continuous if \(f\) is perfectly \(\omega I\) - continuous and \(g\) is strongly continuous.

Theorem 3.4.31 Let \( f : (X, \tau, I) \to (Y, \sigma) \) be a perfectly \(\omega I\) - continuous function and \(A\) is \(*\) - closed then the restriction \( f \mid A : (A, \tau \restriction A, I \restriction A) \to (Y, \sigma) \) is \(\omega I\) - continuous.

3.5 \(\omega I\) - OPEN AND \(\omega I\) - CLOSED FUNCTIONS

In this section, we introduce a new class of functions called \(\omega I\) - closed function, \(\omega I\) - open functions, \(\omega^* I\) - closed functions and \(\omega^* I\) - open functions in ideal topological space and obtain some of their characterizations of these functions.

Definition 3.5.1 A function \( f : (X, \tau) \to (Y, \sigma, J) \) is said to be \(\omega I\) - open (resp. closed) if the image of every open (resp. closed) in \((X, \tau)\) is \(\omega I\) - open (resp. \(\omega I\) - closed) in \((Y, \sigma, J)\).

Example 3.5.2 Let \(X = Y = \{a, b, c\}\) with \(\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}\), \(\sigma = \{\phi, \{a\}, \{b, c\}, Y\}\) and \(J = \{\phi, \{a\}\}\). Let \( f : (X, \tau) \to (Y, \sigma, J) \) be the identity function. Then \(f\) is an \(\omega I\) - closed function.
Theorem 3.5.3 A function \( f:(X,\tau) \to (Y,\sigma,J) \) is \( \omega l \) - closed if and only if \( \omega l - cl(f(A) \subseteq f(cl(A)) \) for every subset \( A \) of \( (X,\tau) \).

**Proof:** Necessity. Suppose that \( f \) is \( \omega l \) - closed and \( A \subseteq X \). Then \( f(cl(A)) \) is \( \omega l \) - closed in \( (Y,\sigma,J) \). We have \( f(A) \subseteq f(cl(A)) \) and by Lemma 2.3.17 and Lemma 2.3.18, \( \omega l - cl(f(A) \subseteq \omega l - cl(f(cl(A))) = f(cl(A)) \).

**Sufficiency.** Let \( A \) be any closed set in \( (X,\tau) \). Then \( f(A) = f(cl(A)) \) is \( \omega l \) - closed in \( (X,\tau) \). We have \( f(A) \subseteq f(cl(A)) \) and by Lemma 2.3.17 and Lemma 2.3.18, \( f(A) = \omega l - cl(f(A)) \). That is, \( f(A) \) is \( \omega l \) - closed by Theorem 2.3.17 and hence \( f \) is \( \omega l \) - closed.

Theorem 3.5.4 Let \( f: (X,\tau) \to (Y,\sigma,J) \) be a function such that \( \omega l - cl(f(A) \subseteq f(cl(A)) \) for every subset \( A \subseteq X \). Then the image \( f(A) \) of a closed set \( A \) in \( (X,\tau) \) is \( \tau_{\omega l} \) - closed in \( (Y,\sigma,J) \).

**Proof:** Let \( A \) be a closed set in \( (X,\tau) \). Then by hypothesis, \( \omega l - cl(f(A) \subseteq f(cl(A)) = f(A) \) and so \( \omega l - cl(f(A) = f(A) \). Therefore, \( f(A) \) is \( \tau_{\omega l} \) - closed in \( (Y,\sigma,J) \).

Theorem 3.5.5 A function \( f:(X,\tau) \to (Y,\sigma,J) \) is \( \omega l \) - closed if and only if for each subset \( S \) of \( (Y,\sigma,J) \) and for each open set \( U \) containing \( f^{-1}(S) \) there is an \( \omega l \) - open set \( V \) of \( (Y,\sigma,J) \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**Proof:** Necessity. Suppose that \( f \) is \( \omega l \) - closed. Let \( S \subseteq Y \) and \( U \) be open set of \( (X,\tau) \) such that \( f^{-1}(S) \subseteq U \). Then \( V = f(U^c)^c \) is an \( \omega l \) - open set containing \( S \) such that \( f^{-1}(V) \subseteq U \).

**Sufficiency.** Let \( F \) be a closed set of \( (X,\tau) \). Then \( f^{-1}(f(F)^c) \subseteq F^c \) and \( F^c \) is open. By assumption, there exists an \( \omega l \) - open set \( V \) of \( (Y,\sigma,J) \) such that \( (f(F))^c \subseteq V \) and \( f^{-1}(V) \subseteq F^c \) and so \( F \subseteq (f^{-1}(V))^c \). Hence, \( V^c \subseteq f(F) \subseteq f((f^{-1}(V))^c) \subseteq V^c \) which implies \( f(F) = V^c \). Since \( V^c \) is \( \omega l \) - closed, \( f(F) \) is \( \omega l \) - closed and therefore \( f \) is \( \omega l \) - closed.

Theorem 3.5.6 If \( f:(X,\tau,I) \to (Y,\sigma,J) \) is \( I \) - irresolute \( \omega l \) - closed and \( A \) is an \( \omega l \) - closed subset of \( (X,\tau,I) \) , then \( f(A) \) is an \( \omega l \) - closed set.
Proof: Let $U$ be a semi-$I$-open set in $(Y, \sigma, J)$ such that $f(A) \subseteq U$. Since $f$ is $I$-irresolute, $f^{-1}(U)$ is a semi-$I$-open set containing $A$. Hence $cl^*(A) \subseteq f^{-1}(U)$ as $A$ is $\omega I$-closed in $(X, \tau, I)$. Since $f$ is $\omega I$-closed, $f(cl^*(A))$ is an $\omega I$-closed set contained in the semi-$I$-open set $U$, which implies that $cl^*(f(cl^*(A))) \subseteq U$ and hence $cl^*(f(A)) \subseteq U$. Therefore, $f(A)$ is an $\omega I$-closed set.

The following Example shows that the composition of two $\omega I$-closed functions need not be $\omega I$-closed.

Example 3.5.7 Let $X = Y = Z = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$, $\eta = \{\phi, \{a, c\}, Z\}$, $J = \{\phi\}$ and $K = \{\phi, \{c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma, J)$ be the identity function and define $g : (Y, \sigma, J) \rightarrow (Z, \eta, K)$ by $g(a) = g(b) = b$ and $g(c) = a$. Then both $f$ and $g$ are $\omega I$-closed function but their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta, K)$ is not an $\omega I$-closed function, since for the closed set $\{c\}$ in $(X, \tau)$, $(g \circ f)(\{c\}) = \{a\}$, which is not an $\omega I$-closed set in $(Z, \eta, K)$.

Corollary 3.5.8 Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be $\omega I$-closed and $g : (Y, \sigma, J) \rightarrow (Z, \eta, K)$ be $\omega I$-closed and $I$-irresolute where $I, J$ and $K$ are ideals on $X, Y$ and $Z$ respectively, then their composition $g \circ f : (X, \tau, I) \rightarrow (Z, \eta, K)$ is $\omega I$-closed.

Proof: Let $A$ be any closed set of $(X, \tau, I)$. Then by hypothesis $f(A)$ is an $\omega I$-closed set of $(Y, \sigma, J)$. Since $g$ is both $\omega I$-closed and $I$-irresolute by Theorem 3.5.6, $g(f(A)) = (g \circ f)(A)$ is $\omega I$-closed in $(Z, \eta, K)$ and therefore, $g \circ f$ is $\omega I$-closed.

Theorem 3.5.9 Let $f : (X, \tau) \rightarrow (Y, \sigma, J)$ and $g : (Y, \sigma, J) \rightarrow (Z, \eta, K)$ be $\omega I$-closed functions and $(Y, \sigma, J)$ be a $T^*_\omega$ space. Then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta, K)$ is $\omega I$-closed.

Proof: Let $A$ be any closed set of $(X, \tau)$. Then by assumption $f(A)$ is an $\omega I$-closed set of $(Y, \sigma, J)$. Since $(Y, \sigma, J)$ be a $T^*_\omega$ space, $f(A)$ is closed in $(Y, \sigma, J)$ and again by assumption $g(f(A))$ is $\omega I$-closed in $(Z, \eta, K)$. That is, $(g \circ f)(A)$ is $\omega I$-closed.
closed in \((Z, \eta, K)\) and therefore, \(g \circ f\) is \(\omega I\) - closed.

**Theorem 3.5.10** Let \(f : (X, \tau) \to (Y, \sigma, J)\) be a closed function and \(g : (Y, \sigma, J) \to (Z, \eta, K)\) be an \(\omega I\) - closed functions, then their composition \(g \circ f : (X, \tau) \to (Z, \eta, K)\) is \(\omega I\) - closed.

**Theorem 3.5.11** For any bijection \(f : (X, \tau) \to (Y, \sigma, J)\), the following statements are equivalent:

\(i\) \(f^{-1} : (Y, \sigma, J) \to (X, \tau)\) is \(\omega I\) - continuous.

\(ii\) \(f(U)\) is \(\omega I\) - open in \(Y\) for every open set \(U\) in \(X\).

\(iii\) \(f(U)\) is \(\omega I\) - closed in \(Y\) for every closed set \(U\) in \(X\).

**Proof:** \((i) \Rightarrow (ii)\) Let \(U\) be an open set of \((X, \tau)\). By assumption \((f^{-1})^{-1}(U) = f(U)\) is \(\omega I\) - open in \((Y, \sigma, J)\) and so \(f\) is \(\omega I\) - open.

\((ii) \Rightarrow (iii)\) Let \(F\) be closed set of \((X, \tau)\). Then \(F^c\) is open in \((X, \tau)\). By assumption \(f(F^c)\) is \(\omega I\) - open in \((Y, \sigma, J)\). That is, \(f(F^c) = (f(F))^c\) is \(\omega I\) - open in \((Y, \sigma, J)\) and therefore, \(f(F)\) is \(\omega I\) - closed in \((Y, \sigma, J)\). Hence, \(f\) is \(\omega I\) - closed.

\((iii) \Rightarrow (i)\) Let \(F\) be closed set of \((X, \tau)\). By assumption \(f(F)\) is \(\omega I\) - closed in \((Y, \sigma, J)\). But \(f(F) = (f^{-1})^{-1}(F)\) and therefore, \(f^{-1}\) is \(\omega I\) - continuous.

In the next two Theorems, we obtain various characterizations of \(\omega I\) - open functions.

**Theorem 3.5.12** Let \((Y, \sigma, J)\) be \(T\) - dense. Then for a function \(f : (X, \tau) \to (Y, \sigma, J)\) the following statements are equivalent:

\(i\) \(f\) is an \(\omega I\) - open function.

\(ii\) For a subset \(A\) of \((X, \tau)\), \(f(\text{int}(A)) \subseteq \omega I - \text{int}(f(A))\).

\(iii\) For each \(x \in X\) and for each neighbourhood \(U\) of \(x\) in \((X, \tau, I)\), there exists an \(\omega I\) - neighbourhood \(W\) of \(f(x)\) in \((Y, \sigma, J)\) such that \(W \subseteq f(U)\).

**Proof:** \((i) \Rightarrow (ii)\) Suppose \(f\) is \(\omega I\) - open. Let \(A \subseteq X\). Then \(\text{int}(A)\) is open in \((X, \tau, I)\) and so \(f(\text{int}(A))\) is \(\omega I\) - open in \((Y, \sigma, J)\). We have \(f(\text{int}(A)) \subseteq f(A)\). Therefore, by
Lemma 2.3.15, \( f(\text{int}(A)) \subseteq \omega l - \text{int}(f(A)) \).

(ii) \( \Rightarrow \) (iii) Suppose (ii) holds. Let \( x \in X \) and \( U \) be an arbitrary neighbourhood of \( x \) in \( (X, \tau, I) \). Then there exists an open set \( G \) such that \( x \in G \subseteq U \). By assumption, \( f(G) = f(\text{int}(G)) \subseteq \omega l - \text{int}(f(G)) \). This implies \( f(G) = \omega l - \text{int}(f(G)) \). By lemma 2.3.15, we have \( f(G) \) is \( \omega l \) - open in \( (Y, \sigma, J) \). Further, \( f(x) \in f(G) \subseteq f(U) \), and so (iii) holds, by taking \( W = f(G) \).

(iii) \( \Rightarrow \) (i) Suppose (iii) holds. Let \( U \) be any open set in \( (X, \tau), x \in U \) and \( f(x) = y \). Then \( y \in f(U) \), and for each \( y \in f(U) \), by assumption, there exists an \( \omega l \) - neighbourhood \( W_y \) of \( y \) in \( (Y, \sigma, J) \) such that \( W_y \subseteq f(U) \). Since \( W_y \) is an \( \omega l \) - neighbourhood of \( y \), there exists an \( \omega l \) - open set \( V_y \) in \( (Y, \sigma, J) \) such that \( y \in V_y \subseteq W_y \). Therefore, \( f(U) = \cup \{ V_y : y \in f(U) \} \) is an \( \omega l \) - open set in \( (Y, \sigma, J) \).

Thus, \( f \) is an \( \omega l \) - open function.

**Theorem 3.5.13** A function \( f : (X, \tau) \rightarrow (Y, \sigma, J) \) is \( \omega l \) - open if and only if for any subset \( S \) of \( (Y, \sigma, J) \) and for any closed set \( F \) containing \( f^{-1}(S) \), there exists an \( \omega l \) - closed set \( K \) of \( (Y, \sigma, J) \) containing \( S \) such that \( f^{-1}(K) \subseteq F \).

**Corollary 3.5.14** A function \( f : (X, \tau) \rightarrow (Y, \sigma, J) \) is \( \omega l \) - open if and only if \( f^{-1}(\omega l - \text{cl}(B)) \subseteq \text{cl}(f^{-1}(B)) \) for each subset \( B \) of \( (Y, \sigma, J) \).

**Proof:** **Necessity.** Suppose that \( f \) is \( \omega l \) - open. Then for any \( B \subseteq Y \), \( f^{-1}(B) \subseteq \text{cl}(f^{-1}(B)) \), by Theorem 3.5.13, there exists an \( \omega l \) - closed set \( K \) of \( X \) such that \( B \subseteq K \) and \( f^{-1}(K) \subseteq \text{cl}(f^{-1}(B)) \). Therefore, \( f^{-1}(\omega l - \text{cl}(B)) \subseteq f^{-1}(K) \subseteq \text{cl}(f^{-1}(B)) \), since \( K \) is an \( \omega l \) - closed set in \( (Y, \sigma, J) \).

**Sufficiency.** Let \( S \) be any subset of \( (Y, \sigma, J) \) and \( F \) be any closed set containing \( f^{-1}(S) \). Put \( K = \omega l - \text{cl}(S) \) then \( K \) is an \( \omega l \) - closed set and \( S \subseteq K \). By assumption \( f^{-1}(K) = f^{-1}(\omega l - \text{cl}(S)) \subseteq \text{cl}(f^{-1}(S)) \subseteq F \) and therefore, by Theorem 3.5.13, \( f \) is \( \omega l \) - open.
Finally in this section, we define another new class of functions called $\omega^* I$-closed functions which are stronger than $\omega I$-closed functions.

**Definition 3.5.15** A function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is said to be $\omega^* I$-closed if the image of every $\omega I$-closed in $(X, \tau, I)$ is $\omega I$-closed in $(Y, \sigma, J)$.

**Example 3.5.16** Let $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$, $I = \{\emptyset, \{b\}\}$ and $J = \{\emptyset, \{a\}\}$. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be the identity function. Then the function $f$ is an $\omega^* I$-closed function.

**Remark 3.5.17** Since every closed set is $\omega I$-closed set we have every $\omega^* I$-closed function is an $\omega I$-closed function.

The converse is not true in general as seen from the following Example.

**Example 3.5.18** Let $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, Y\}$ and $I = J = \{\emptyset, \{a\}\}$. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be the identity function. Then the function $f$ is an $\omega I$-closed function but not an $\omega^* I$-closed function, since $\{a, b\}$ is an $\omega I$-closed set in $(X, \tau, I)$ but its image under $f$ is $\{a, b\}$ which is not $\omega I$-closed in $(Y, \sigma, J)$.

**Theorem 3.5.19** A function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is $\omega^* I$-closed if and only if $\omega I - \text{cl}(f(A)) \subseteq f(\text{cl}(A))$ for every subset $A$ of $(X, \tau, I)$.

Analogous to $\omega^* I$-closed function we can also define $\omega^* I$-open function.

**Theorem 3.5.20** For any bijective function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$, the following statements are equivalent:

(i) $f^{-1} : (Y, \sigma, J) \rightarrow (X, \tau, I)$ is $\omega I$-irresolute.

(ii) $f(U)$ is $\omega^* I$-open in $Y$ for every open set $U$ in $X$.

(iii) $f(U)$ is $\omega^* I$-closed in $Y$ for every closed set $U$ in $X$.

**Theorem 3.5.21** If $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is $I$-irresolute and $\omega I$-closed, then it is an $\omega^* I$-closed function.
**Theorem 3.5.22** Let \((X, \tau, I)\) be an ideal topological space and \(S \subseteq X\) be \(\omega I\)-closed. If \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) is a continuous and *-closed function, then \(f(S)\) is an \(\omega I\)-closed set in \((Y, \sigma, J)\).

**Proof:** Suppose that \(S \subseteq X\) is a \(\omega I\)-closed set and \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) is a continuous *-closed function. Let \(f(S) \subseteq K\) where \(K\) is a semi-\(I\)-open in \((Y, \sigma, J)\).

It follows that \(S \subseteq f^{-1}(K)\). Since \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) is a continuous function and \(S\) is an \(\omega I\)-closed set, then we have \(cl^I(S) \subseteq f^{-1}(K)\). Moreover, we have \(f(cl^I(S)) \subseteq f(f^{-1}(K)) \subseteq K\). Since \(f\) is *-closed function, then \(cl^I(f(S)) \subseteq cl^I(f(cl^I(S))) = f(cl^I(S)) \subseteq K\). It follows that \(cl^I(f(S)) \subseteq K\) and hence \(f(S)\) is an \(\omega I\)-closed set in \((Y, \sigma, J)\).

**Theorem 3.5.23** Let \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) be a continuous and *-closed surjective function. If \((X, \tau, I)\) is an \(\omega I\)-Alexandroff ideal space, then \((Y, \sigma, J)\) is an \(\omega I\)-Alexandroff ideal space.

**Proof:** Suppose that \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) be a continuous and *-closed surjective function. Let \((X, \tau, I)\) be an \(\omega I\)-Alexandroff ideal space and \(\{M_i : i = 1, 2, 3, \ldots n\}\) be a family of closed sets in \((Y, \sigma, J)\). Since \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) is a continuous function, then \(K = \bigcup_{i=1,2,3} f^{-1}(M_i)\) is an \(\omega I\)-closed set in \((X, \tau, I)\). We take \(M = \bigcup M_i\).

It follows from theorem 3.5.22, that \(f(K) = f(\bigcup_{i=1,2,3} f^{-1}(M_i)) = M\) is an \(\omega I\)-closed set. Thus, \((Y, \sigma, J)\) is an \(\omega I\)-Alexandroff ideal space.