Chapter 7

Controllability of Impulsive Second Order Delay Integrodifferential Systems

7.1 Introduction

The investigation of functional differential equations with infinite delay in an abstract admissible phase space was initiated by Hale and Kato [54] and Kappel and Schappacher [78]. The method of using admissible phase spaces enables one to treat a large class of functional differential equations with infinite delay in the same time and obtain general results. Most of the practical systems are nonlinear in nature and hence the study of nonlinear systems is important. For the basic theory on evolution system, one can refer the book [134].

During the past decades, many papers have been published on the existence and controllability of nonlinear systems in which the authors effectively used the fixed point technique. For example, Balachandran et al. [7, 16] discussed the controllability of second order integrodifferential systems in Banach spaces. Since many control systems arising from realistic models heavily depend on infinite delay, there is an increasing interest to study the controllability of partial functional differential and integrodifferential systems with infinite delay. The controllability result for nonlinear impulsive systems combined with infinite delay has been studied by many authors [31, 117]. Bahuguna et al. [4] studied the non-autonomous nonlinear integro-differential equations with infinite delay. Recently Henriquez [59] investigated the
existence of solutions of non-autonomous second order functional differential equations with infinite delay. However the results obtained there are only in connection with infinite delay and with impulsive effects absent. But no result has been reported regarding the controllability of non-autonomous second order impulsive systems. This fact is the main motivation to study these equations. In this chapter, we concentrate on the case with infinite delay and impulsive effects and establish sufficient conditions for the controllability of second order impulsive differential and integrodifferential evolution systems with infinite delay by relying on fixed-point techniques.

7.2 Preliminaries

In this section, we review some fundamental concepts, notations, and properties required to establish our main results. Nowadays there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

\[ x''(t) = A(t)x(t) + f(t), \quad 0 \leq s, t \leq a, \quad (7.2.1) \]
\[ x(s) = v, \quad x'(s) = w, \quad (7.2.2) \]

where \( A(t) : D(A(t)) \subseteq X \to X, \ t \in J = [0, a] \) is a closed densely defined operator and \( f : J \to X \) is an appropriate function. Equations of this type have been considered in many works, one can refer \[93, 113\] and the references therein. In the most of works, the existence of solutions to the problem \( (7.2.1)-(7.2.2) \) is related to the existence of an evolution operator \( S(t, s) \) for the homogeneous equation

\[ x''(t) = A(t)x(t), \quad 0 \leq s, t \leq a. \quad (7.2.3) \]

Let us assume that the domain of \( A(t) \) is a subspace \( D \) dense in \( X \) and independent of \( t \) and, for each \( x \in D \), the function \( t \mapsto A(t)x \) is continuous. The fundamental solution for the second-order evolution equation \( (7.2.3) \) has been developed by Kozak [85]. The concept of evolution operators is defined in Definition 6.2.1 in the Section 6.2. Throughout this chapter, we assume that there exists an evolution operator \( S(t, s) \) associated with the operators \( A(t) \). To abbreviate the text, we introduce the operator \( C(t, s) = -\frac{\partial S(t, s)}{\partial s} \). In addition, we set \( N \) and \( \tilde{N} \) for positive constants such that

\[ \sup_{0 \leq s, t \leq a} \|S(t, s)\| \leq N \quad \text{and} \quad \sup_{0 \leq s, t \leq a} \|C(t, s)\| \leq \tilde{N}. \]

Further we denote by \( N_1 \) a positive constant such that

\[ \|S(t + h, s) - S(t, s)\| \leq N_1|h|, \]
for all \( s, t, t + h \in [0, a] \). Assuming that \( f : J \to X \) is an integrable function, the mild solution \( x : [0, a] \to X \) of the problem (7.2.1)-(7.2.2) is given by

\[
x(t) = C(t, s)v + S(t, s)w + \int_s^t S(t, \tau)f(\tau)d\tau.
\]

In the literature, several techniques have been discussed to establish the existence of the evolution operator \( S(\cdot, \cdot) \). In particular, often studied situation is that \( A(t) \) is the perturbation of an operator \( A \) that generates a cosine operator function. For this reason, below we review some essential properties of the theory of cosine functions.

Let \( A : D(A) \subseteq X \to X \) be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators \( (C(t))_{t \in \mathbb{R}} \) on the Banach space \( X \). We denote by \( (S(t))_{t \in \mathbb{R}} \) the sine function associated with \( (C(t))_{t \in \mathbb{R}} \) which is defined by

\[
S(t)x = \int_0^t C(s)xds, \quad x \in X \text{ and } t \in \mathbb{R}.
\]

The existence of solutions for the second order abstract Cauchy problem

\[
\begin{align*}
x''(t) &= Ax(t) + h(t), \quad 0 \leq t \leq a, \quad (7.2.4) \\
x(s) &= v, \quad x'(s) = w, \quad (7.2.5)
\end{align*}
\]

where \( h : J \to X \) is an integrable function, has been discussed in \[137\]. Similarly the existence of solutions of semilinear second order abstract Cauchy problems has been treated in \[138\]. We only mention here that the function \( x(\cdot) \) given by

\[
x(t) = C(t - s)v + S(t - s)w + \int_s^t S(t - \tau)h(\tau)d\tau, \quad 0 \leq t \leq a, \quad (7.2.6)
\]

is called a mild solution of (7.2.4)-(7.2.5) and that, when \( v \in \tilde{E}, x(\cdot) \) is continuously differentiable and

\[
x'(t) = AS(t - s)v + C(t - s)w + \int_s^t C(t - \tau)h(\tau)d\tau \quad 0 \leq t \leq a.
\]

In addition, if \( v \in D(A), w \in \tilde{E} \) and \( f \) is a continuously differentiable function, then the function \( x(\cdot) \) is a solution of the initial value problem (7.2.4)-(7.2.5).

Assume now that \( A(t) = A + \tilde{B}(t) \) where \( \tilde{B} : \mathbb{R} \to \mathcal{L}(\tilde{E}, X) \) is a map such that

the function \( t \mapsto \tilde{B}(t)x \) is continuously differentiable in \( X \), for each \( x \in \tilde{E} \). It has been established by Serizawa \[128\] that, for each \( (v, w) \in D(A) \times \tilde{E} \), the non-autonomous abstract Cauchy problem

\[
\begin{align*}
x''(t) &= (A + \tilde{B}(t))x(t), \quad t \in \mathbb{R}, \quad (7.2.7) \\
x(0) &= v, \quad x'(0) = w, \quad (7.2.8)
\end{align*}
\]
has a unique solution $x(\cdot)$ such that the function $t \mapsto x(t)$ is continuously differentiable in $\tilde{E}$. It is clear that the same argument allows us to conclude that (7.2.7) with the initial condition (7.2.5) has a unique solution $x(\cdot, s)$ such that the function $t \mapsto x(t, s)$ is continuously differentiable in $\tilde{E}$. It follows, from (7.2.6), that
\[
x(t, s) = C(t-s)v + S(t-s)w + \int_s^t S(t-\tau)\tilde{B}(\tau)x(\tau, s)d\tau.
\]
In particular, for $v = 0$, we have
\[
x(t, s) = S(t-s)w + \int_s^t S(t-\tau)\tilde{B}(\tau)x(\tau, s)d\tau. \tag{7.2.9}
\]
Consequently
\[
\|x(t, s)\|_1 \leq \|S(t-s)\|_{\mathcal{L}(X, \tilde{E})}\|w\| + \int_s^t \|S(t-\tau)\|_{\mathcal{L}(X, \tilde{E})}\|\tilde{B}(\tau)\|_{\mathcal{L}(\tilde{E}, X)}\|x(\tau, s)\|_1 d\tau,
\]
and, applying the Gronwall-Bellman lemma, we infer that
\[
\|x(t, s)\|_1 \leq \tilde{M}\|w\|, \quad s, t \in J. \tag{7.2.10}
\]
We define the operator $S(t, s)w = x(t, s)$. It follows from the previous estimate that $S(t, s)$ is a bounded linear map on $\tilde{E}$. Since $\tilde{E}$ is dense in $X$, we can extend $S(t, s)$ to $X$. We keep the notation $S(t, s)$ for this extension. It is well known that, except in the case dim($X$) $< \infty$, the cosine function $C(t)$ cannot be compact for all $t \in \mathbb{R}$. By contrast, for the cosine functions that arise in specific applications, the sine function $S(t)$ is very often a compact operator for all $t \in \mathbb{R}$. This motivates the result [59, Theorem 1.2]. Assumptions for the impulsive conditions and the phase space $\mathcal{B}$ are defined in Section 4.3.

**Lemma 7.2.1.** A set $G \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if and only if each set $\tilde{G}_k$, $k = 0, 1, \ldots, m$, is relatively compact in $C([t_k, t_{k+1}]; X)$.

Now we include that some of our proofs are based on the following well-known result [52, Theorem 6.5.4].

**Lemma 7.2.2.** Let $D$ be a closed convex subset of a Banach space $X$ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set
\[
\{x \in D : x = \lambda F(x), \ 0 < \lambda < 1\}
\]
is unbounded or the map $F$ has a fixed point in $D$. 

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The terminology and notations are those generally used in functional analysis. By \( \sigma(A) \) (respectively, \( \rho(A) \)), we denote the spectrum (respectively, the resolvent set) of a linear operator \( A \).

In what follows, the notation \( \mathcal{G}(a) \) stands for the space
\[
\mathcal{G}(a) = \{ y : (-\infty, a] \to X : y|_J \in PC, \ y_0 = 0 \}
\]
endowed with the sup norm. In addition, we denote by \( \tilde{\phi} : (-\infty, a] \to X \) the function defined by \( \tilde{\phi}_0 = \phi \) and \( \tilde{\phi}(t) = C(t, 0)\phi(0) + S(t, 0)\zeta \), for \( t \geq 0 \). Also \( c_1 = \sup_{0 \leq t \leq a} \| \tilde{\phi}_t \|_B \) and \( \| y_s + \tilde{\phi}_s \|_B \leq \rho = K_a r + c_1 \).

### 7.3 Impulsive Delay Evolution Systems

Consider the following second-order impulsive evolution system with infinite delay of the form

\[
x''(t) = A(t)x(t) + Bu(t) + f(t, x_t), \\
t \in J = [0, a], \ t \neq t_k, \ k = 1, 2, \ldots, m, \\
x_0 = \phi, \ x'(0) = \zeta, \\
\Delta x(t_k) = I_k(x_{t_k}), \ k = 1, 2, \ldots, m, \\
\Delta x'(t_k) = J_k(x_{t_k}), \ k = 1, 2, \ldots, m,
\]

where \( \phi \in \mathcal{B} \) and \( \zeta \in X \). The control function \( u(\cdot) \) is given in \( L^2(J, U) \), a Banach space of admissible control functions with \( U \) as a Banach space and \( B : U \to X \) as a bounded linear operator; for \( t \in J \), \( x_t \) represents the function \( x_t : (-\infty, 0] \to X \) defined by \( x_t(\theta) = x(t+\theta), \ -\infty < \theta \leq 0 \) which belongs to some abstract phase space \( \mathcal{B} \) defined axiomatically; \( f : J \times \mathcal{B} \to X, \ I_k : \mathcal{B} \to X, \ J_k : \mathcal{B} \to X \) are appropriate functions and will be specified later. \( 0 < t_1 < \ldots < t_n < a \) are fixed numbers and the symbol \( \Delta \xi(t) \) represents the jump of a function \( \xi \) at \( t \) defined by \( \Delta \xi(t) = \xi(t^+) - \xi(t^-) \). Throughout the text, we assume that \( A(\cdot) \) generates an evolution operator \( S(t, s) \).

In the following definition, we introduce the concept of a mild solution for system (7.3.1)-(7.3.4).

**Definition 7.3.1.** A function \( x : (-\infty, a] \to X \) is called a mild solution of the abstract Cauchy problem (7.3.1)-(7.3.4), if \( x_0 = \phi \in \mathcal{B}, \ x|_J \in PC \), the impulsive
conditions \( \triangle x(t_k) = I_k(x_{t_k}), \triangle x'(t_k) = J_k(x_{t_k}), \ k = 1, 2, \ldots, m, \) are satisfied and the following integral equation

\[
x(t) = C(t, 0)\phi(0) + S(t, 0)\zeta + \int_0^t S(t, s)\left[Bu(s) + f(s, x_s)\right]ds + \sum_{0 < t_k < t} C(t, t_k)I_k(x_{t_k}) + \sum_{0 < t_k < t} S(t, t_k)J_k(x_{t_k}), \ 0 \leq t \leq a,
\]

is verified.

Motivated by this definition, we introduce the following assumptions:

(H1) The function \( f : J \times \mathcal{B} \to X \) is continuous and there exists constants \( L_f > 0, \tilde{L}_f > 0, \) for \( \psi_1, \psi_2 \in \mathcal{B} \) such that

\[
\|f(t, \psi_1) - f(t, \psi_2)\| \leq L_f\|\psi_1 - \psi_2\|_B
\]

and \( \tilde{L}_f = \sup_{t \in J} \|f(t, 0)\|. \)

(H2) \( B \) is a continuous operator from \( U \) to \( X \) and the linear operator \( W : L^2(J, U) \to X, \) defined by

\[
Wu = \int_0^a S(a, s)Bu(s)ds,
\]

has a bounded inverse operator \( W^{-1} \) which takes values in \( L^2(J, U)/\ker W \) and there exists positive constant \( M \) such that \( \|BW^{-1}\| \leq M_1. \)

(H3) The impulsive functions satisfy the following conditions:

(i) The maps \( I_k : \mathcal{B} \to X, \ k = 1, 2, \ldots, m, \) are continuous and there exist constants \( L_I > 0, \tilde{L}_I > 0, \) for \( \psi_1, \psi_2 \in \mathcal{B} \) such that

\[
\|I_k(\psi_1) - I_k(\psi_2)\| \leq L_I\|\psi_1 - \psi_2\|
\]

and \( \tilde{L}_I = \|I_k(0)\|. \)

(ii) The maps \( J_k : \mathcal{B} \to X, \ k = 1, 2, \ldots, m, \) are continuous and there exist constants \( L_J > 0, \tilde{L}_J > 0, \) for \( \psi_1, \psi_2 \in \mathcal{B} \) such that

\[
\|J_k(\psi_1) - J_k(\psi_2)\| \leq L_J\|\psi_1 - \psi_2\|
\]

and \( \tilde{L}_J = \|J_k(0)\|. \)
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(H4) Let \( aN \left[ L_f(K_a r + c_1) + \tilde{L}_f \right] + aNA_0 + \sum_{k=1}^{m} \left( \tilde{N} L_I + N L_J \right) \left[ K_a r + \| \tilde{\phi}_k \| \right] + \sum_{k=1}^{m} \left( \tilde{N} \tilde{L}_I + N \tilde{L}_J \right) \leq r \), for some \( r > 0 \).

(H5) Let \( \mu = K_a(1 + aN M_1) \left[ aN L_f + \sum_{k=1}^{m} (\tilde{N} L_I + N L_J) \right] < 1 \) be such that \( 0 \leq \mu < 1 \).

**Definition 7.3.2.** The system \((7.3.1)-(7.3.4)\) is said to be controllable on the interval \( J \), if for every \( x_0 = \phi \in \mathcal{B} \), \( x'(0) = \zeta \) and \( z_1 \in X \), there exists a control \( u \in L^2(J, U) \) such that the mild solution \( x(\cdot) \) of \((7.3.1)-(7.3.4)\) satisfies \( x(a) = z_1 \).

The following result is an immediate application of the contraction principle of Banach. To simplify the text, we let \( K_a = \sup_{0 \leq t \leq a} K(t) \).

**Theorem 7.3.1.** If the hypotheses (H1)-(H5) are satisfied, then the impulsive second order system \((7.3.1)-(7.3.4)\) is controllable on \( J \).

**Proof.** Using the assumption (H2), we define the control function

\[
 u(t) = W^{-1} \left[ z_1 - C(a, 0)\phi(0) - S(a, 0)\zeta - \int_0^a S(a, s)f(s, x_s)ds - \sum_{k=1}^{m} C(a, t_k)I_k(x_{t_k}) - \sum_{k=1}^{m} S(a, t_k)J_k(y_{t_k}) \right] (t).
\]

We now show that, when using this control, the operator \( \Gamma \) on the space \( \mathcal{G}(a) \) defined by \((\Gamma y)_0 = 0 \) and

\[
(\Gamma y)(t) = \int_0^t S(t, s)f(s, y_s + \tilde{\phi}_s)ds + \int_0^t S(t, \eta)BW^{-1} \left[ z_1 - C(a, 0)\phi(0) - S(a, 0)\zeta - \int_0^a S(a, s)f(s, y_s + \tilde{\phi}_s)ds - \sum_{k=1}^{m} C(a, t_k)I_k(y_{t_k} + \tilde{\phi}_{t_k}) - \sum_{k=1}^{m} S(a, t_k)J_k(y_{t_k} + \tilde{\phi}_{t_k}) \right] d\eta + \sum_{0 < t_k < t} C(t, t_k)I_k(y_{t_k} + \tilde{\phi}_{t_k})
\]

has a fixed point \( x(\cdot) \). This fixed point is then a mild solution of the system \((7.3.1)-(7.3.4)\). Clearly \( (\Gamma x)(a) = z_1 \) which means that the control \( u \) steers the system from the initial state \( \phi \) to \( z_1 \) in time \( a \), provided we obtain a fixed point of the operator \( \Gamma \).
which implies that the system is controllable. From the assumptions, it is easy to see that \( \Gamma \) is well defined and continuous. For convenience, let us take

\[
\|Bu(s)\| \leq M_1 \left[ \|z_1\| + \bar{N}\|\phi(0)\| + N\|\zeta\| + aN \left[ L_f(K_1r + c_1) + \bar{L}_f \right] + \bar{N} \sum_{k=1}^{m} \tilde{L}_I \right.
\]

\[
+ L_I(K_1r + \|\tilde{\phi}_k\|) + N \sum_{k=1}^{m} \left[ L_f(K_1r + \|\tilde{\phi}_k\|) + \tilde{L}_f \right] = A_0.
\]

First we show that \( \Gamma \) maps \( B_r(0, G(a)) \) into \( B_r(0, G(a)) \). To this end, from the definition of the operator \( \Gamma \) in (7.3.5) and our hypotheses, we obtain

\[
\|(\Gamma y)(t)\| \leq \int_0^t \|S(t, s)\| \left[ \|f(s, y_s + \tilde{\phi}_s) - f(s, 0)\| + \|f(s, 0)\| \right] ds
\]

\[
+ \int_0^t \|S(t, \eta)\| \|Bu(\eta)\| d\eta + \sum_{k=1}^{m} \|C(t, t_k)\| \left[ \|I_k(y_{t_k} + \tilde{\phi}_{t_k}) - I_k(0)\| + \|I_k(0)\| \right] + \sum_{k=1}^{m} \|S(t, t_k)\| \left[ \|J_k(y_{t_k} + \tilde{\phi}_{t_k}) - J_k(0)\| + \|J_k(0)\| \right] \leq aN \left[ L_f(K_1r + c_1) + \bar{L}_f \right] + aNA_0 + \sum_{k=1}^{m} \left( \bar{N}\tilde{L}_I + N\tilde{L}_J \right)
\]

\[
+ \sum_{k=1}^{m} \left( \bar{N}\tilde{L}_I + N\tilde{L}_J \right) \left[ K_1r + \|\tilde{\phi}_{t_k}\| \right] \leq r,
\]

for \( y \in G(a) \) and \( t \in J \). Hence \( \|\Gamma y\|_a \leq r \). Therefore \( \Gamma \) maps \( B_r(0, G(a)) \) into itself.

Now, for \( y, z \in B_r(0, G(a)) \), we have

\[
\|(\Gamma y)(t) - (\Gamma z)(t)\|
\]

\[
\leq \int_0^t \|S(t, s)\| \|f(s, y_s + \tilde{\phi}_s) - f(s, z_s + \tilde{\phi}_s)\| ds + \int_0^t \|S(t, \eta)\| \|BW^{-1}\| + \sum_{k=1}^{m} \|C(a, t_k)\|
\]

\[
\times \left[ \int_0^a \|S(t, s)\| \|f(s, y_s + \tilde{\phi}_s) - f(s, z_s + \tilde{\phi}_s)\| ds + \sum_{k=1}^{m} \|S(a, t_k)\| \|J_k(y_{t_k} + \tilde{\phi}_{t_k}) - I_k(z_{t_k} + \tilde{\phi}_{t_k})\| + \sum_{k=1}^{m} \|S(a, t_k)\| \|J_k(y_{t_k} + \tilde{\phi}_{t_k}) - I_k(z_{t_k} + \tilde{\phi}_{t_k})\|
\]

\[
- J_k(z_{t_k} + \tilde{\phi}_{t_k}) \right] d\eta + \sum_{0 < t_k < t} \|C(t, t_k)\| \|I_k(y_{t_k} + \tilde{\phi}_{t_k}) - I_k(z_{t_k} + \tilde{\phi}_{t_k})\| + \sum_{0 < t_k < t} \|S(t, t_k)\| \|J_k(y_{t_k} + \tilde{\phi}_{t_k}) - J_k(z_{t_k} + \tilde{\phi}_{t_k})\|
\]
\[ N \int_0^t L_f \| y_s - z_s \| ds + aNM_1 \left[ N \int_0^a L_f \| y_s - z_s \| ds + \sum_{k=1}^m \tilde{N}L_I \| y_{t_k} - z_{t_k} \| \right] + \sum_{k=1}^m NL_f \| y_{t_k} - z_{t_k} \| \leq K_a(1 + aNM_1) \left[ aNL_f + \sum_{k=1}^m (\tilde{N}L_I + NL_I) \right] \| y - z \| \]
\[ \leq \mu \| y - z \|, \]

which implies that \( \Gamma \) is a contraction on \( B_r(0, \mathcal{G}(a)) \). Hence, by Banach fixed point theorem, \( \Gamma \) has a unique fixed point \( y \) in \( \mathcal{G}(a) \). Defining \( x(t) = y(t) + \tilde{o}(t), \ -\infty < t \leq a \), we obtain that \( x(\cdot) \) is a mild solution of the problem \((7.3.1)-(7.3.4)\) and the proof is complete. \( \blacksquare \)

We use the condition stated below instead of \((H1)\) to avoid the Lipschitz continuity of \( f \) used in Theorem 7.3.1.

**(A1)** The function \( f : J \times \mathcal{B} \to X \) satisfies the following conditions:

(i) For each \( t \in J \), the function \( f(t, \cdot) : \mathcal{B} \to X \) is continuous and the function \( t \to f(t, x_t) \) is strongly measurable.

(ii) There exists an integrable function \( p : J \to [0, \infty) \) and a continuous non-decreasing function \( \Omega : [0, \infty) \to (0, \infty) \) such that

\[ \| f(t, \psi) \| \leq p(t)\Omega(\| \psi \|_B), \ (t, \psi) \in J \times \mathcal{B}. \]

**Theorem 7.3.2.** Assume that \( f \) satisfies the conditions \((A1)\) and \((H2)\). Also let the following conditions hold:

(a) The set \( U(r,t) = \{ S(t,s)f(s,\psi) : s \in [0,a], \ \psi \in B_r(0, \mathcal{G}(a)) \} \) is relatively compact in \( X \), for each \( r > 0 \) and \( t > 0 \).

(b) The functions \( I_k : \mathcal{B} \to X \) are completely continuous and there exist positive constants \( \alpha^i_k, \ i = 1, 2, \) such that \( \| I_k(\psi) \| \leq \alpha^i_k \| \psi \|_B + \alpha^2_k, \ k = 1, 2, \ldots, m, \) for every \( \psi \in \mathcal{B} \).

(c) The functions \( J_k : \mathcal{B} \to X \) are completely continuous and there exist positive constants \( \beta^i_k, \ i = 1, 2, \) such that \( \| J_k(\psi) \| \leq \beta^i_k \| \psi \|_B + \beta^2_k, \ k = 1, 2, \ldots, m, \) for every \( \psi \in \mathcal{B} \).
(d) The constant
\[ \mu = K_a \sum_{k=1}^{m} (\tilde{N}_{\alpha_k} + N_{\beta_k}) < 1 \] and
\[ \int_0^\infty \frac{ds}{\Omega(s)} > \frac{K_a N}{1 - \mu} \int_0^a p(s)ds, \]
where
\[ c = \frac{1}{1 - \mu} \left[ a K_a N M_1 M^* + c_1 + K_a \sum_{k=1}^{m} (\tilde{N}_{\alpha_k}^2 + N_{\beta_k}^2) \right] \]
and
\[ M^* = \left[ \| z_1 \| + \tilde{N} \| \phi(0) \| + N \| \zeta \| + N \int_0^a p(s) \Omega(\rho) ds \right. \]
\[ + \tilde{N} \sum_{k=1}^{m} (\alpha_k(\rho) + \alpha_k^2) + N \sum_{k=1}^{m} (\beta_k(\rho) + \beta_k^2) \].

Then the system (7.3.1)-(7.3.4) is controllable on \( J \).

**Proof.** We define the map \( \Gamma \) on the space \( G(a) \) as in equation (7.3.5). To prove the controllability of the problem (7.3.1)-(7.3.4), we show that the operator \( \Gamma \) has a fixed point. This fixed point is then a mild solution of the system (7.3.1)-(7.3.4). From the assumptions, it is easy to see that \( \Gamma \) is well defined and continuous.

In order to apply Lemma 7.2.2, we need to obtain an a priori bound for the solutions of the integral equation \( y = \lambda \Gamma(y), \lambda \in (0,1) \). To this end, let \( y^\lambda \) be a solution of \( \lambda \Gamma(y) = y, \lambda \in (0,1) \). Using the notation \( v^\lambda(t) = \sup_{0 \leq s \leq t} \| y^\lambda_s + \tilde{\phi}_s \|_{B,a} \leq K_a \| y^\lambda \|_s + \| \tilde{\phi}_s \|_{B,a} \), we observe that
\[
\| y^\lambda(t) \| \leq N \int_0^t p(s) \Omega(v^\lambda(s)) ds + a N M_1 M^* \]
\[ + \sum_{0 < t_k < t} (\tilde{N}_{\alpha_k}^1 + N_{\beta_k}^1)v^\lambda(t_k) + \sum_{0 < t_k < t} (\tilde{N}_{\alpha_k}^2 + N_{\beta_k}^2). \]
Hence it follows that
\[
v^\lambda(t) \leq K_a N \int_0^t p(s) \Omega(v^\lambda(s)) ds + a K_a N M_1 M^* + \sup_{0 \leq s \leq t} \| \tilde{\phi}_s \|_{B,a} \]
\[ + K_a \sum_{k=1}^{m} (\tilde{N}_{\alpha_k}^2 + N_{\beta_k}^2) + \mu v^\lambda(t), \]
which yields
\[
v^\lambda(t) \leq c + \frac{K_a N}{1 - \mu} \int_0^t p(s) \Omega(v^\lambda(s)) ds. \]
Denoting by \( w_\lambda(t) \) the right-hand side of the previous inequality, we see that

\[
w'_\lambda(t) \leq \frac{K_a N}{1 - \mu} [p(t) \Omega(w_\lambda(t))],
\]

and subsequently, upon integrating over \([0, t]\), we obtain

\[
\int_t^w \frac{ds}{\Omega(s)} \leq \frac{K_a N}{1 - \mu} \int_0^t p(s) ds \leq \frac{K_a N}{1 - \mu} \int_0^a p(s) ds < \int_t^\infty \frac{ds}{\Omega(s)}.
\]

This estimate permits us to conclude that the set of functions \( \{ w_\lambda : \lambda \in (0, 1) \} \) is bounded and, in turn, that \( \{ y^\lambda : \lambda \in (0, 1) \} \) is bounded in \( \mathcal{G}(a) \). Next we show that \( \Gamma \) is completely continuous.

To clarify this proof, we decompose \( \Gamma \) in the form \( \Gamma = \Gamma_1 + \Gamma_2 \) where

\[
\Gamma_1 y(t) = \int_0^t S(t, s) \left[ f(s, y_s + \tilde{\phi}_s) + Bu(s) \right] ds,
\]

and \( \Gamma_2 y(t) = \sum_{0 < t_k < t} C(t, t_k) \int_k(y_{t_k} + \tilde{\phi}_{t_k}) + \sum_{0 < t_k < t} S(t, t_k) J_k(y_{t_k} + \tilde{\phi}_{t_k}), \quad t \in J. \)

Using the hypotheses, conditions \((b), (c)\) and Lemma 7.2.1, we obtain that \( \Gamma_1 \) is continuous and that \( \Gamma_2 \) is completely continuous. In order to use the Ascoli-Arzela theorem, we prove that \( \Gamma_1 \) takes bounded sets into relatively compact ones. As above, \( B_r = B_r(0, \mathcal{G}(a)) \) and \( t \in J, \| y_t + \tilde{\phi}_t \|_B \leq K_a r + c_1 \), and we denote by \( \rho \) the right-hand side of the above expression. And also

\[
\|(Bu)(s)\| \leq M_1 \left[ \|z_1\| + \bar{N}\|\phi(0)\| + N\|\zeta\| + N \int_0^a p(s) \Omega(\rho) ds + \sum_{k=1}^m (\bar{N}\alpha_k^1 + N\beta_k^1)\rho + \sum_{k=1}^m (\bar{N}\alpha_k^2 + N\beta_k^2) \right] = B_0.
\]

From the mean value theorem, we see that

\[
\Gamma_1 y(t) \in t \text{ co}(\{ S(t, s) f(s, \psi) : s \in [0, a], \| \psi \|_B \leq \rho \}),
\]

which implies that the set \( \{ \Gamma_1 y(t) : y \in B_r(0, \mathcal{G}(a)) \} \) is relatively compact, for each \( t \in J \). Here \( \text{co}(K) \) denotes the convex hull of the set \( K \). Moreover, from

\[
\Gamma_1 y(t + h) - \Gamma_1 y(t) = \int_0^t \left[ S(t + h, s) - S(t, s) \right] \left[ f(s, y_s + \tilde{\phi}_s) + (Bu)(s) \right] ds + \int_t^{t+h} S(t + h, s) \left[ f(s, y_s + \tilde{\phi}_s) + (Bu)(s) \right] ds
\]
and using that \( S(\cdot, s) \) verifies a Lipschitz condition, we obtain that
\[
\| \Gamma_1 y(t + h) - \Gamma_1 y(t) \| \leq |h| N_1 \int_0^a [p(s) \Omega(\rho) + B_0] ds + N \int_t^{t+h} [p(s) \Omega(\rho) + B_0] ds,
\]
which implies that \( \| \Gamma_1 y(t + h) - \Gamma_1 y(t) \| \to 0 \) as \( h \to 0 \) uniformly for \( y \in B_r(0, G(a)) \).

From this, we infer that \( \{ \Gamma_1 y(t) : y \in B_r(0, G(a)) \} \) is relatively compact in \( G(a) \) and consequently that \( \Gamma_1 \) is completely continuous. This completes the proof of the assertion that the map \( \Gamma \) is completely continuous.

By an application of Lemma 7.2.2, we conclude that there exists a fixed point \( y \) of \( \Gamma \). It is clear that the function \( x = y + \tilde{\phi} \) is a mild solution of the system (7.3.1)-(7.3.4). This completes the proof. \( \blacksquare \)

### 7.4 Impulsive Delay Integrodifferential Evolution Systems

Consider the following second-order delay integrodifferential evolution system with impulses of the form
\[
x''(t) = A(t)x(t) + Bu(t) + \int_0^t e(t-s)g(s, x_s)ds,
\]
\( t \in J = [0, a] \), \( t \neq t_k \), \( k = 1, 2, \ldots, m \), \( \Delta x(t_k) = I_k(x_{t_k}) \), \( k = 1, 2, \ldots, m \), \( \Delta x'(t_k) = J_k(x_{t_k}) \), \( k = 1, 2, \ldots, m \)

where \( A, B, I_k \) and \( J_k \) are defined as in equations (7.3.1)-(7.3.4). Here \( g : J \times \mathcal{B} \to \mathcal{X} \), \( e : J \to \mathcal{X} \), \( 0 \leq s \leq t \leq a \) are appropriate functions. Further we assume the following conditions:

(H6) The function \( g : J \times \mathcal{B} \to \mathcal{X} \) is continuous and there exist constants \( L_g > 0 \), \( \tilde{L}_g > 0 \), for \( \psi_1, \psi_2 \in \mathcal{B} \) such that
\[
\| g(t, \psi_1) - g(t, \psi_2) \| \leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}
\]
and \( \tilde{L}_g = \sup_{t \in J} \| g(t, 0) \| \).
(H7) The real-valued function \( e \) is continuous on \( J \) and there exists a positive constant \( e_T \) such that \( |e(t)| \leq e_T \), for \( t \in J \).

(H8) Let \( aN e_T \left[ aL_g(K_a r + c_1) + a\tilde{L}_g \right] + aN A_0 + \sum_{k=1}^{m} \left( \tilde{N} L_I + NL_J \right) \left[ K_a r + \|\tilde{\phi}_k\| \right] \) + \sum_{k=1}^{m} \left( \tilde{N} \tilde{L}_I + N \tilde{L}_J \right) \leq r, \) for some \( r > 0 \).

(H9) Let \( \nu = K_a(1 + aN M_1) \left[ a^2 N e_T L_g + \sum_{k=1}^{m} (\tilde{N} L_I + NL_J) \right] < 1 \) be such that \( 0 \leq \nu < 1 \).

Let us start by defining what we mean by a mild solution of problem (7.4.1)-(7.4.4).

**Definition 7.4.1.** A function \( x : (-\infty, a] \rightarrow X \) is called a mild solution of the abstract Cauchy problem (7.4.1)-(7.4.4), if \( x_0 = \phi \in \mathcal{B} \), \( x|_{J} \in \mathcal{PC} \), the impulsive conditions \( \Delta x(t_k) = I_k(x_{t_k}), \Delta x'(t_k) = J_k(x_{t_k}), k = 1, 2, \ldots, m \), are satisfied and the following integral equation

\[
x(t) = C(t, 0)\phi(0) + S(t, 0)\zeta + \int_0^t S(t, s) \left[ Bu(s) + \int_0^s e(s - \tau)g(\tau, x_\tau)d\tau \right] ds
+ \sum_{0 < t_k < t} C(t, t_k)I_k(x_{t_k}) + \sum_{0 < t_k < t} S(t, t_k)J_k(x_{t_k}), \quad 0 \leq t \leq a,
\]

(7.4.5)
is verified.

**Definition 7.4.2.** The system (7.4.1)-(7.4.4) is said to be controllable on the interval \( J \), if for every \( x_0 = \phi \in \mathcal{B} \), \( x'(0) = \zeta \) and \( z_1 \in X \), there exists a control \( u \in L^2(J, U) \) such that the mild solution \( x(\cdot) \) of (7.4.1)-(7.4.4) satisfies \( x(a) = z_1 \).

**Theorem 7.4.1.** Suppose that (H2), (H3) and (H6)-(H9) are satisfied. Then the impulsive second order integrodifferential system (7.4.1)-(7.4.4) is controllable on \( J \).

**Proof.** Using hypothesis (H2), for an arbitrary function \( x(\cdot) \), we define the control

\[
u(t) = W^{-1} \left[ z_1 - C(a, 0)\phi(0) - S(a, 0)\zeta - \int_0^a S(a, s) \left( \int_0^s e(s - \tau)g(\tau, x_\tau)d\tau \right) ds \right.
\]

\[- \sum_{k=1}^{m} C(a, t_k)I_k(x_{t_k}) - \sum_{k=1}^{m} S(a, t_k)J_k(x_{t_k}) \right](t).
\]

We define the nonlinear operator \( \Phi \) on the space \( \mathcal{G}(a) \) by \( (\Phi y_0) = 0 \) and

\[
(\Phi y)(t) = \int_0^t S(t, s) \left( \int_0^s e(s - \tau)g(\tau, y_\tau + \tilde{\phi}_\tau)d\tau \right) ds + \int_0^t S(t, \eta)BW^{-1} \left[ z_1 - C(a, 0)\phi(0) - S(a, 0)\zeta - \int_0^a S(a, s) \left( \int_0^s e(s - \tau)g(\tau, y_\tau + \tilde{\phi}_\tau)d\tau \right) ds \right.
\]

\[- \sum_{k=1}^{m} C(a, t_k)I_k(x_{t_k}) - \sum_{k=1}^{m} S(a, t_k)J_k(x_{t_k}) \right](t).
\]
\[ - \sum_{k=1}^{m} C(a, t_k) I_k(y_{t_k} + \tilde{\phi}_{t_k}) - \sum_{k=1}^{m} S(a, t_k) J_k(y_{t_k} + \tilde{\phi}_{t_k}) \left( \eta \right) d\eta \]
\[ + \sum_{0<t_k<t} C(t, t_k) I_k(y_{t_k} + \tilde{\phi}_{t_k}) + \sum_{0<t_k<t} S(t, t_k) J_k(y_{t_k} + \tilde{\phi}_{t_k}), \quad t \in J. \quad (7.4.7) \]

Note that the control (7.4.6) transfers the system (7.4.5) from the initial state to the final state provided that the operator \( \Phi \) has a fixed point. This fixed point is then a mild solution of the system (7.4.1)-(7.4.4). So, if the operator \( \Phi \) has a fixed point, then the system is controllable. It follows from the assumptions that, for each \( y \in \mathcal{G}(a) \), the function \( \Phi y \) is well defined and continuous. For convenience, let us take

\[ \|Bu(s)\| \]
\[ \leq M_1 \left[ \|z_1\| + \tilde{N}\|\phi(0)\| + N\|\zeta\| + aN e_T [aL_g(K_o r + c_1) + a \tilde{L}_g] \right] \]
\[ + \tilde{N} \sum_{k=1}^{m} [L_I(K_o r + \|\tilde{\phi}_{t_k}\|) + \tilde{L}_I] + N \sum_{k=1}^{m} [L_J(K_o r + \|\tilde{\phi}_{t_k}\|) + \tilde{L}_J] \] = \tilde{\Lambda}_0.

Now we show that \( \Phi \) maps \( B_r(0, \mathcal{G}(a)) \) into \( B_r(0, \mathcal{G}(a)) \). For \( y \in \mathcal{G}(a) \),

\[ \| (\Phi y)(t) \| \leq \int_0^t \int_0^s \| S(t, s) \| \| e(s - \tau) \| \left[ \| g(\tau, y_{\tau} + \tilde{\phi}_{\tau}) - g(\tau, 0) \| + \| g(\tau, 0) \| \right] d\tau ds \]
\[ + \int_0^t \| S(t, \eta) \| \| Bu(\eta) \| d\eta + \sum_{k=1}^{m} \| C(t, t_k) \| \left[ \| I_k(y_{t_k} + \tilde{\phi}_{t_k}) - I_k(0) \| + \| I_k(0) \| \right] \]
\[ + \sum_{k=1}^{m} \| S(t, t_k) \| \left[ \| J_k(y_{t_k} + \tilde{\phi}_{t_k}) - J_k(0) \| + \| J_k(0) \| \right] \]
\[ \leq aN e_T [aL_g(K_o r + c_1) + a \tilde{L}_g] + aN \tilde{\Lambda}_0 + \sum_{k=1}^{m} \left( \tilde{N} \tilde{L}_I + N \tilde{L}_J \right) \]
\[ + \sum_{k=1}^{m} \left( \tilde{N} L_I + N L_J \right) \left[ K_o r + \|\tilde{\phi}_{t_k}\| \right] \]
\[ \leq r. \]

Thus \( \Phi \) maps \( B_r(0, \mathcal{G}(a)) \) into itself. Now, for \( y, z \in B_r(0, \mathcal{G}(a)) \), we have

\[ \| (\Phi y)(t) - (\Phi z)(t) \| \]
\[ \leq \int_0^t \int_0^s \| S(t, s) \| \| e(s - \tau) \| \| g(\tau, y_{\tau} + \tilde{\phi}_{\tau}) - g(\tau, z_{\tau} + \tilde{\phi}_{\tau}) \| d\tau ds + \int_0^t \| S(t, \eta) \| d\eta \]
\[ \times \| BW^{-1} \| \left[ \int_0^a \int_0^a \| S(t, s) \| \| e(s - \tau) \| \| g(\tau, y_{\tau} + \tilde{\phi}_{\tau}) - g(\tau, z_{\tau} + \tilde{\phi}_{\tau}) \| d\tau ds \right] \]
\[
+ \sum_{k=1}^{m} \left\| C(a, t_k) \right\| \left\| I_k(y_{tk} + \tilde{t}_{tk}) - I_k(z_{tk} + \tilde{t}_{tk}) \right\| + \sum_{k=1}^{m} \left\| S(a, t_k) \right\|
\]
\[
\times \left\| J_k(y_{tk} + \tilde{t}_{tk}) - J_k(z_{tk} + \tilde{t}_{tk}) \right\| d\eta + \sum_{0 < t_k < t} \left[ \left\| C(t, t_k) \right\| \left\| I_k(y_{tk} + \tilde{t}_{tk}) \right\| - I_k(z_{tk} + \tilde{t}_{tk}) \right]
\]
\[
\leq aNe_T \int_{0}^{s} L_g \| y_{\tau} - z_{\tau} \| d\tau + aNM_1 \left[ aNe_T \int_{0}^{s} L_g \| y_{\tau} - z_{\tau} \| d\tau \right]
\]
\[
+ \sum_{k=1}^{m} NL_I \| y_{tk} - z_{tk} \| + \sum_{k=1}^{m} NL_J \| y_{tk} - z_{tk} \|
\]
\[
+ \sum_{k=1}^{m} NL_I \| y_{tk} - z_{tk} \|
\]
\[
\leq K_a (1 + aNM_1) \left[ a^2 Ne_T L_g + \sum_{k=1}^{m} (NL_I + NL_J) \right] \| y - z \|
\]
\[
\leq \nu \| y - z \|_a.
\]

Now, from the assumptions, we find that \( \Phi \) is a contraction operator on \( B_r(0, G(a)) \). It follows, from the Banach fixed point theorem, that the operator \( \Phi \) has a fixed point \( y \in G(a) \), so that \( x = y + \tilde{\phi} \) is a mild solution of the problem (7.4.1)-(7.4.4). This completes the proof.

Now we assume that the condition \( (H6) \) is replaced by the following one.

**A2** The function \( g : J \times B \to X \) satisfies the following conditions:

(i) For each \( t \in J \), the function \( g(t, \cdot) : B \to X \) is continuous and the function \( t \to g(t, x_t) \) is strongly measurable.

(ii) There exists an integrable function \( q : J \to [0, \infty) \) and a continuous non-decreasing function \( \Omega : [0, \infty) \to (0, \infty) \) such that

\[
\| g(t, \psi) \| \leq q(t)\Omega_1(\| \psi \|_B), \quad (t, \psi) \in J \times B.
\]

**Theorem 7.4.2.** Assume that the function \( g \) verify the assumption \( (A2) \) and \( (H2) \) is satisfied. Also let the following conditions be fulfilled:

(a) The set \( U(r, t) = \{S(t, s)f(s, \psi) : s \in [0, a], \ psi \in B_r(0, G(a))\} \) is relatively compact in \( X \), for each \( r > 0 \) and \( t > 0 \).
(b) The functions $I_k : \mathcal{B} \to X$ are completely continuous and there exist positive constants $\alpha_k^i$, $i = 1, 2$, such that $\|I_k(\psi)\| \leq \alpha_k^1\|\psi\|_B + \alpha_k^2$, $k = 1, 2, \ldots, m$, for every $\psi \in \mathcal{B}$.

(c) The functions $J_k : \mathcal{B} \to X$ are completely continuous and there exist positive constants $\beta_k^i$, $i = 1, 2$, such that $\|J_k(\psi)\| \leq \beta_k^1\|\psi\|_B + \beta_k^2$, $k = 1, 2, \ldots, m$, for every $\psi \in \mathcal{B}$.

(d) The constant

$$
\mu = 1 - K_a \sum_{k=1}^{m} (\tilde{N}\alpha_k^1 + N\beta_k^1) > 0
$$

and $\mu \neq 0$ and

$$
c(1 - \mu) = \left[ aK_a, NM_1, N^* + c_1 + K_a \sum_{k=1}^{m} (\tilde{N}\alpha_k^2 + N\beta_k^2) \right]
$$

and $N^* = \left[ \|z_1\| + \tilde{N}\|\phi(0)\| + N\|\zeta\| + aNe_T \int_0^a q(s)\Omega_1(\rho)ds \right. + \tilde{N}\sum_{k=1}^{m} (\alpha_k^1(\rho) + \alpha_k^2) + N\sum_{k=1}^{m} (\beta_k^1(\rho) + \beta_k^2) \right].$

Then the impulsive system (7.4.1)-(7.4.4) is controllable on $J$.

**Proof.** We now show that the operator $\Phi : \mathcal{G}(a) \to \mathcal{G}(a)$ defined as in equation (7.4.7) has a fixed point. This fixed point is then a mild solution of the system (7.4.1)-(7.4.4). From the assumptions, it is easy to see that $\Phi$ is well defined and continuous. Now we obtain a priori estimate for the solutions of the integral equation

$$
y(\lambda) = \lambda \Phi(y(\lambda)),$$

where $\lambda \in (0, 1)$ and let $y^\lambda(\cdot)$ be a solution of $y = \lambda \Phi(y)$. Using the notation $v^\lambda(t) = \sup_{0 \leq s \leq t} \|y^\lambda_s + \phi_s\|_B \leq K_a \|y^\lambda\| + \|\phi_s\|_{\mathcal{B}, a}$ and applying the definition of $\Phi$, we find that

$$
\|y^\lambda(t)\| \leq aNe_T \int_0^t q(s)\Omega_1(v^\lambda(s))ds + aNM_1N^*
$$

$$
+ \sum_{0 < t_k < t} (\tilde{N}\alpha_k^1 + N\beta_k^1)v^\lambda(t_k) + \sum_{0 < t_k < t} (\tilde{N}\alpha_k^2 + N\beta_k^2)
$$

which implies that

$$
v^\lambda(t) \leq aK_aNe_T \int_0^t q(s)\Omega_1(v^\lambda(s))ds + aK_aNM_1N^* + \sup_{0 \leq s \leq t} \|\phi_s\|_B
$$

$$
+ K_a \sum_{k=1}^{m} (\tilde{N}\alpha_k^2 + N\beta_k^2) + \mu v^\lambda(t).
$$
Consequently

\[ v^\lambda(t) \leq c + \frac{aK_aNe_T}{1-\mu} \int_0^t q(s)\Omega_1(v^\lambda(s))ds. \]

Denoting by \( \sigma_\lambda(t) \) the right-hand side of the above expression, we get

\[ \sigma_\lambda'(t) \leq \frac{aK_aNe_T}{1-\mu} [q(t)\Omega_1(\sigma_\lambda(t))]. \]

Integrating this expression, we get

\[ \int_c^{\sigma_\lambda(t)} \frac{ds}{\Omega_1(s)} \leq \frac{aK_aNe_T}{1-\mu} \int_0^a q(s)ds < \int_c^{\infty} \frac{ds}{\Omega_1(s)}, \]

which implies that the functions \( \sigma_\lambda(\cdot) \) are bounded. Thus the functions \( y^\lambda(\cdot) \) are also bounded on \( \mathcal{G}(a) \).

Next we prove that \( \Phi \) is completely continuous. For this purpose, we introduce the decomposition \( \Phi = \sum_{i=1}^2 \Phi_i \), where

\[ \Phi_1 y(t) = \int_0^t S(t,s) \left[ \int_0^s e(s-\tau)g(\tau, y_\tau + \tilde{\phi}_\tau) d\tau + Bu(s) \right] ds, \]

and

\[ \Phi_2 y(t) = \sum_{0 < t_k < t} C(t,t_k)I_k(y_{t_k} + \tilde{\phi}_{t_k}) + \sum_{0 < t_k < t} S(t,t_k)J_k(y_{t_k} + \tilde{\phi}_{t_k}), \ t \in J. \]

It is clear that the operators \( \Phi_i \) are continuous. In a similar way, the conditions (b), (c) and Lemma 7.2.1 imply that the map \( \Phi_2 \) is completely continuous. To complete this part of the proof, it remains to prove that \( \Phi_1 \) is relatively compact in \( \mathcal{G}(a) \). Next we observe that, for every \( y \in B_r = B_r(0, \mathcal{G}(a)) \), \( \|y_\tau + \tilde{\phi}_\tau\|_B \leq \rho = K_ar + c_1 \) and

\[ \|(Bu)(s)\| \leq M_1 \left[ \|z_1\| + \tilde{N}\|\phi(0)\| + N\|\zeta\| + aNe_T \int_0^a q(s)\Omega_1(\rho)ds + \sum_{k=1}^m (\tilde{N}\alpha_k + N\beta_k)\rho + \sum_{k=1}^m (\tilde{N}\alpha_k^2 + N\beta_k^2) \right] = \tilde{B}_0. \]

Applying now the mean value theorem, we write

\[ \Phi_1 y(t) \in t \co \{S(t,s)g(s,\psi) : s \in [0, a], \|\psi\|_B \leq \rho\}. \]

As a result, we conclude that the set \( \{\Phi_1 y(t) : y \in B_r(0, \mathcal{G}(a))\} \) is relatively compact, for each \( t \in J \). Moreover, from the estimate \( \Phi_1 y(t+h) - \Phi_1 y(t) \)

\[ = \int_0^t [S(t+h,s) - S(t,s)] \left[ (Bu)(s) + \int_0^s e(s-\tau)g(\tau, y_\tau + \tilde{\phi}_\tau)d\tau \right] ds \]

\[ + \int_t^{t+h} S(t+h,s) \left[ (Bu)(s) + \int_0^s e(s-\tau)g(\tau, y_\tau + \tilde{\phi}_\tau)d\tau \right] ds \]
and using that $S(\cdot,s)$ verifies a Lipschitz condition, we obtain that
\begin{align*}
\|\Phi_1 y(t+h) - \Phi_1 y(t)\| & \leq |h|N_1 \int_0^a [e_T q(s) \Omega_1(\rho) + \tilde{B}_0] ds \\
& \quad + N \int_t^{t+h} [e_T q(s) \Omega_1(\rho) + \tilde{B}_0] ds,
\end{align*}
which implies that $\|\Phi_1 y(t+h) - \Phi_1 y(t)\| \to 0$ as $h \to 0$ uniformly, for $y \in B_r(0, \mathcal{G}(a))$.
The application of Ascoli-Arzela Theorem completes the proof that $\{\Phi_1 y : y \in B_r(0, \mathcal{G}(a))\}$ is a relatively compact set in $\mathcal{G}(a)$. As a consequence of the above proofs, we conclude that $\Phi$ is a completely continuous map.

Finally, from Lemma 7.2.2 we infer the existence of a fixed point $y$ of $\Phi$. Let
$$x(t) = y(t) + \tilde{\phi}(t), \quad t \in (-\infty, a].$$
Then $x$ is a fixed point of the operator $\Phi$ which is a mild solution of the system (7.4.1)-(7.4.4). Hence the system is controllable on $J$.

7.5 Example

In this section, we apply our abstract results on a concrete wave equation with impulsive conditions. Following the equations (7.2.7)-(7.2.8), here we consider $A(t) = A + \tilde{B}(t)$ where $A$ is the infinitesimal generator of a cosine function $C(t)$ with associated sine function $S(t)$ and $\tilde{B}(t) : D(\tilde{B}(t)) \to X$ is a closed linear operator with $D \subseteq D(\tilde{B}(t))$, for all $t \in J$. Examples of phase spaces are constructed in [71].

Now we introduce the required technical framework. Let $X = L^2(T, \mathbb{C})$, where the group $T$ is defined as the quotient $\mathbb{R}/2\pi \mathbb{Z}$. We use the identification between functions on $T$ and $2\pi$-periodic functions on $\mathbb{R}$. Specifically in what follows we denote, by $L^2(T, \mathbb{C})$, the space of $2\pi$-periodic 2-integrable functions from $\mathbb{R}$ into $\mathbb{C}$. Similarly $H^2(T, \mathbb{C})$ denotes the Sobolev space of $2\pi$-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(T, \mathbb{C})$.

We consider the operator $Ax(\xi) = x''(\xi)$ with domain $D(A) = H^2(T, \mathbb{C})$. It is well known that $A$ is the infinitesimal generator of a strongly continuous cosine function $C(t)$ on $X$. Further $A$ has a discrete spectrum, the eigenvalues being $-n^2$ for $n \in \mathbb{Z}$ with corresponding normalized eigenvectors $w_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}$, $n \in \mathbb{Z}$, and the set $\{w_n : n \in \mathbb{Z}\}$ is an orthonormal basis of $X$. In particular
$$Ax = \sum_{n \in \mathbb{Z}} -n^2 < x, w_n > w_n,$$
for $x \in D(A)$. The cosine function $C(t)$ is given by
\[
C(t)x = \sum_{n \in \mathbb{Z}} \cos(nt) <x, w_n > w_n, \ t \in \mathbb{R},
\]
with the associated sine function
\[
S(t)x = t <x, w_0 > w_0 + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\sin(nt)}{n} <x, w_n > w_n, \ t \in \mathbb{R}.
\]
It is clear that $\|C(t)\| \leq 1$, for all $t \in \mathbb{R}$. Thus $C(\cdot)$ is uniformly bounded on $\mathbb{R}$.

Consider the second-order impulsive Cauchy problem with control $\hat{\mu}(t, \cdot)$
\[
\frac{\partial^2}{\partial t^2} z(t, \tau) = \frac{\partial^2}{\partial \tau^2} z(t, \tau) + b(t) \frac{\partial}{\partial t} z(t, \tau) + \hat{\mu}(t, \tau) + \int_{-\infty}^{t} \sigma(t-s) z(s, \tau) ds, \quad (7.5.1)
\]
for $t \geq 0$, $0 < \tau < 2\pi$, subject to the initial conditions
\[
\begin{align*}
  z(t, 0) &= z(t, 2\pi) = 0, \ t \geq 0, \\
  z(\theta, \tau) &= \phi(\theta, \tau), \ \frac{\partial}{\partial \tau} z(0, \tau) = w(\tau), \ \theta \in (-\infty, 0], \ 0 \leq \tau \leq 2\pi, \\
  \triangle z(t_k)(\tau) &= \int_{-\infty}^{t_k} \gamma_k(t_k-s) z(s, \tau) ds, \ k = 1, 2, \ldots, m, \\
  \triangle z'(t_k)(\tau) &= \int_{-\infty}^{t_k} \bar{\gamma}_k(t_k-s) z(s, \tau) ds, \ k = 1, 2, \ldots, m,
\end{align*}
\]
where we assume that $\sigma : [0, \infty) \to \mathbb{R}$ and $b : [0, \infty) \to \mathbb{R}$ are continuous functions and that the function $\phi(\theta, \cdot) \in \mathcal{B}$, for $\theta \leq 0$, with the identification $\phi(\theta)(\tau) = \phi(\theta, \tau)$, for functions $\phi : (-\infty, 0) \to X$ and $0 < t_1 < \ldots < t_m < a$. We fix $a > 0$ and set $\beta = \sup_{0 \leq t \leq a} |b(t)|$.

We have to show that there exists a control $\hat{\mu}$ which steers (7.5.1) from any specified initial state to the final state in the Banach space $X$.

We take $\tilde{B}(t)x(\tau) = b(t)x'(\tau)$ defined on $H^1(\mathbb{T}, \mathbb{C})$. It is easy to see that $A(t) = A + \tilde{B}(t)$ is a closed linear operator. Initially we will show that $A + \tilde{B}(t)$ generates an evolution operator. It is well known that the solution of the scalar initial value problem
\[
\begin{align*}
  q''(t) &= -n^2 q(t) + p(t), \\
  q(s) &= 0, \quad q'(s) = q_1,
\end{align*}
\]
is given by
\[ q(t) = \frac{q_1}{n} \sin n(t-s) + \frac{1}{n} \int_s^t \sin n(t-\tau)p(\tau)d\tau. \]

Therefore the solution of the scalar initial value problem
\[ q''(t) = -n^2 q(t) + n b(t)q(t), \quad q(s) = 0, \quad q'(s) = 1, \] (7.5.2) (7.5.3)
satisfies the integral equation
\[ q(t) = \frac{q_1}{n} \sin n(t-s) + i \int_s^t \sin n(t-\tau)b(\tau)q(\tau)d\tau. \]

Applying the Gronwall-Bellman lemma, we affirm that
\[ |q(t)| \leq \frac{|q_1|}{n} e^{\beta (t-s)}, \] (7.5.4)
for \( s \leq t \). We denote, by \( q_n(t,s) \), the solution of (7.5.2)-(7.5.3). We define
\[ S(t,s)x = \sum_{n\in\mathbb{Z}} q_n(t,s) < x, w_n > w_n. \]

It follows from the estimate (7.5.4) that \( S(t,s) : X \to X \) is well defined and satisfies the conditions of Definition 6.2.1.

To treat this system, we assume that the functions \( \sigma, \gamma_k, \tilde{\gamma}_k \) satisfy the following conditions:

(i) The function \( \sigma(\cdot) \) is continuous, \( L_f = \left( \int_{-\infty}^0 \frac{\sigma^2(-\theta)}{\rho(\theta)} d\theta \right)^{\frac{1}{2}} < \infty \).

(ii) The functions \( \gamma_k, \tilde{\gamma}_k \) are continuous, \( L_I = \left( \int_{-\infty}^0 \frac{\gamma_k^2(-\theta)}{\rho(\theta)} d\theta \right)^{\frac{1}{2}} < \infty \) and
\[ L_J = \left( \int_{-\infty}^0 \frac{\tilde{\gamma}_k^2(-\theta)}{\rho(\theta)} d\theta \right)^{\frac{1}{2}} < \infty \] for every \( k = 1, \ldots, m \).

Assume that the bounded linear operator \( B : U \subset J \to X \) is defined by
\[ (Bu)(t)(\tau) = \hat{\mu}(t,\tau), \quad \tau \in [0,2\pi]. \]

Define the operators \( f : J \times \mathcal{B} \to X \) and \( I_k, J_k : \mathcal{B} \to X \) by
\[ f(\psi)(\tau) = \int_{-\infty}^0 \sigma(-\theta)\psi(\theta,\tau)d\theta, \]
\[ I_k(\psi)(\tau) = \int_{-\infty}^0 \gamma_k(-\theta)\psi(\theta,\tau)d\theta, \]
\[ J_k(\psi)(\tau) = \int_{-\infty}^0 \tilde{\gamma}_k(-\theta)\psi(\theta,\tau)d\theta. \]
Further the linear operator $W$ is given by

$$(Wu)(\tau) = \sum_{n=1}^{\infty} \int_{0}^{2\pi} \frac{1}{n} \sin ns(\hat{\mu}(s, \tau), w_n)w_n ds, \quad \tau \in [0, 2\pi].$$

Assume that this operator has a bounded inverse $W^{-1}$ in $L^2(J, U)/\ker W$. With the choice of $A$, $\tilde{B}$, $B$, $W$, $f$, $I_k$ and $J_k$, (7.3.1)-(7.3.4) is the abstract formulation of (7.5.1). Moreover the functions $f$, $I_k$ and $J_k$, $k = 1, 2, \ldots, m$ are bounded linear operators with $\|f\| \leq L_f$, $\|I_k\| \leq L_I$ and $\|J_k\| \leq L_J$. Hence the second order impulsive evolution system (7.5.1) is controllable on $J$. 