Chapter 5

Neutral Delay Difference Systems

5.1 Introduction

Consider the neutral delay difference system of the form

\[ \Delta (X(n) + pX(n - k)) + Q(n)X(n - \ell) = 0, n \in \mathbb{N}(n_0), \quad (5.1.1) \]

where \( \mathbb{N}(n_0) = \{n_0, n_0 + 1, n_0 + 2, \cdots \} \), \( n_0 \) is a nonnegative integer, \( p \in \mathbb{R}, X \in \mathbb{R}^m \). and \( Q \) is continuous \( m \times m \) matrix for \( n \in \mathbb{N}(n_0) \).

Let \( \theta = \max\{k, \ell\} \). By a solution of system (5.1.1), we mean a real sequence \( \{X(n)\} \) defined for all \( \mathbb{N}(n_0 - \theta) \) and satisfies the system (5.1.1) for all \( n \in \mathbb{N}(n_0) \). A solution \( \{X(n)\} \) is called nonoscillatory if each component is either eventually positive or eventually negative and oscillatory otherwise.
Recently, there has been an increasing interest in the study of oscillatory and asymptotic behavior of the system (5.1.1), see, for example [1, 3, 10, 21, 43, 45] and the references cited therein, using the so called linearized oscillatory theory.

The scalar case was investigated in [33] where it was shown that the condition \( \sum_{n=1}^{\infty} Q(n) < \infty \) implies the existence of nonoscillatory solution of system (5.1.1) for \( p \neq -1 \). It is known that if \( p = -1, m = 1 \) and \( \sum_{n=1}^{\infty} Q(n) = \infty \), then all solutions of system (5.1.1) oscillate, see [22, 23]. Even in the case where the last summation is finite, one can have the oscillation of all solutions for \( p = -1 \), see [42].

In general, all the above mentioned results are valid only for some values of the parameter \( p \) and all those results are obtained for neutral equations of order one or two. Here we obtain the global result for the corresponding system in the variable coefficient case. Further our result gives sufficient conditions for the existence of a solution with a specific asymptotic behavior for all values of \( p \neq -1 \). In Section 5.2, we consider the case where \( p \) is scalar and in Section 5.3 we handle the case where \( p \) is matrix. Examples are provided to illustrate the results.
5.2 Existence of Nonoscillatory Solution when $p$ is scalar

In this section, we obtain sufficient conditions for the existence of nonoscillatory solutions for the system (5.1.1) when $p$ is a scalar constant.

**Theorem 5.2.1.** With respect to the system (5.1.1) assume that

$$\sum_{n=n_0}^{\infty} \|Q(n)\| < \infty$$  \hspace{1cm} (5.2.1)

where $p \neq -1$ and $\|\|$ is any norm in $\mathbb{R}^m$. Then system (5.1.1) has a nonoscillatory solution

**Proof.** The proof of the theorem is divided into five cases depending on the different ranges of the parameter $p$. Let $e$ be a vector such that $\|e\| = 1$.

**Case 1** $0 \leq p < 1$. Choose $n_1 \in N(n_0)$ sufficiently large so that $n_1 \geq n_0 + \theta$ and

$$\sum_{n=n_1}^{\infty} \|Q(n)\| \leq \frac{1 - p(1 + M_2) - M_1}{M_2}$$  \hspace{1cm} (5.2.2)

hold, where $M_1 < 1$, $M_2 > M_1$ are positive constants such that

$$M_1 + M_2 < 2 \text{ and } 1 - \frac{(M_1 + M_2)}{2} \leq p < \frac{1 - M_1}{1 + M_2}.$$  \hspace{1cm} (5.2.3)

Consider the Banach space $B(n_0)$ of all bounded vector functions
$X(n)$ for $n \geq n_0$ with supnorm $\|X\| = \sup_{n \geq n_0, 1 \leq i \leq m} |x^i(n)|$. We define a subset $S$ of $B(n_0)$ as

$$S = \{X \in B(n_0) : M_1 \leq \|X(n)\| \leq M_2, n \geq n_0\}.$$  

Clearly $S$ is a closed, bounded and convex subset of $B(n_0)$. Define a mapping $T : S \to B(n_0)$ as follows:

$$(TX)(n) = \begin{cases} (1 - p)e - pX(n - k) + \sum_{s = n}^{\infty} Q(s)X(s - \ell), n \geq n_1 \\ (TX)(n_1) & , n_0 \leq n \leq n_1. \end{cases}$$

Clearly $T$ is continuous. For every $X \in S$ and $n \geq n_1$, using (5.2.2) and (5.2.3), we obtain

$$\|(TX)(n)\| \leq \|(1 - p)e\| + \|pX(n - k)\| + \left|\sum_{s = n}^{\infty} Q(s)X(s - \ell)\right|$$

$$\leq (1 - p) + pM_2 + M_2 \sum_{s = n_1}^{\infty} \|Q(s)\|$$

$$\leq (1 - p) + pM_2 + M_2 \left(\frac{1 - p(1 + M_2) - M_1}{M_2}\right)$$

$$= 2(1 - p) - M_1$$

$$\leq M_2.$$  

Again, using (5.2.2), we have

$$\|(TX)(n)\| \geq \|(1 - p)e\| - \|pX(n - k)\| - \left|\sum_{s = n}^{\infty} Q(s)X(s - \ell)\right|$$
\[ \| (TX)(n) \| \geq 1 - p - pM_2 - M_2 \sum_{s=n_1}^{\infty} \| Q(s) \| \]
\[ \geq 1 - p - pM_2 - M_2 \left( \frac{1 - p(1 + M_2) - M_1}{M_2} \right) = M_1. \]

Thus, we have \( TS \subseteq S \). Next we prove \( T \) is a contraction mapping on \( S \). For every \( X, Y \in S \) and \( n \geq n_1 \), we have

\[ \| (TX)(n) - (TY)(n) \| \leq \| -p(X(n-k) - Y(n-k)) \| 
+ \left\| \sum_{s=n}^{\infty} Q(s) (X(s-\ell) - Y(s-\ell)) \right\| 
\leq p \| X - Y \| + \| X - Y \| \sum_{s=n_1}^{\infty} \| Q(s) \| 
= q_1 \| X - Y \|. \]

Hence, we have from (5.2.2) and (5.2.3)

\[ \| TX - TY \| \leq q_1 \| X - Y \| \]

where

\[ q_1 = p + \sum_{s=n_1}^{\infty} \| Q(s) \| = p + \frac{1 - p(1 + M_2) - M_1}{M_2} = \frac{1 - p - M_1}{M_2} < 1. \]

This proves that \( T \) is a contraction mapping and hence \( T \) has the fixed point \( X \) with \( \| X \| > 0 \) for all \( n \geq n_1 \), which is a nonoscillatory solution of system (5.1.1). This completes the proof for the Case 1.

**Case 2:** \( 1 < p < \infty \). Choose \( n_1 \in \mathbb{N}(n_0) \) sufficiently large so that

\[ n_1 + k \geq n_0 + \ell \quad (5.2.4) \]
and
\[ \sum_{n=n_1+k}^{\infty} \| Q(n) \| \leq \frac{p - 1 - pM_3 - M_4}{M_4} \] (5.2.5)

hold, where \( M_3 < 1, M_4 > M_3 \) are positive constants such that
\[ M_3 + M_4 < 2 \text{ and } \frac{1 + M_4}{1 - M_3} < p \leq \frac{2}{2 - M_3 - M_4}. \] (5.2.6)

Let \( B(n_0) \) be the Banach space considered in Case 1. Set
\[ S = \{ X \in B(n_0) : M_3 \leq \| X(n) \| \leq M_4, n \geq n_0 \}. \]

Clearly \( S \) is a closed, bounded and convex subset of \( B(n_0) \). Define a mapping \( T : S \rightarrow B(n_0) \) as follows:
\[
(TX)(n) = \begin{cases} 
\left(1 - \frac{1}{p}\right) e - \frac{1}{p} X(n + k) \\
+ \frac{1}{p} \sum_{s=n+k}^{\infty} Q(s) X(s - \ell), & n \geq n_1 \\
(TX)(n_1), & n_0 \leq n \leq n_1.
\end{cases}
\]

Clearly \( T \) is continuous. For any \( X \in S \) and \( n \geq n_1 \), using (5.2.5) and (5.2.6), we obtain
\[
\| (TX)(n) \| = \left\| \left(1 - \frac{1}{p}\right) e - \frac{1}{p} X(n + k) + \frac{1}{p} \sum_{s=n+k}^{\infty} Q(s) X(s - \ell) \right\|
\leq 1 - \frac{1}{p} + \frac{M_4}{p} + \frac{M_4}{p} \sum_{s=n_1}^{\infty} \| Q(s) \|.
\]
\[ \| (TX)(n) \| \leq 1 - \frac{1}{p} + \frac{M_4}{p} + \frac{M_4}{p} \left( \frac{p - 1 - pM_3 - M_4}{M_4} \right) \]
\[ = 2 \left( 1 - \frac{1}{p} \right) - M_3 \leq M_4. \]

Further
\[ \| (TX)(n) \| \geq \left( 1 - \frac{1}{p} \right) - \frac{M_4}{p} - \frac{M_4}{p} \left( \frac{p - 1 - pM_3 - M_4}{M_4} \right) = M_3. \]

Thus, we have \( TS \subseteq S \). Next we prove \( T \) is a contraction mapping on \( S \). For \( X, Y \in S \) and \( n \geq n_1 \), we have
\[ \| (TX)(n) - (TY)(n) \| = \left\| \frac{1}{p} (X(n+k) - Y(n+k)) \right\| \]
\[ + \frac{1}{p} \sum_{s=n+k}^{\infty} Q(s) (X(s-\ell) - Y(s-\ell)) \]
\[ \leq \frac{1}{p} \| X - Y \| + \frac{1}{p} \| X - Y \| \sum_{s=n_1}^{\infty} \| Q(s) \| \]
\[ = q_2 \| X - Y \|. \]

Hence, we have
\[ \| TX - TY \| \leq q_2 \| X - Y \| \]

where
\[ q_2 = \frac{1}{p} + \frac{1}{p} \sum_{s=n_1}^{\infty} \| Q(s) \| \leq \frac{1}{p} + \frac{p - 1 - pM_3 - M_4}{pM_4} = \frac{p - 1 - pM_3}{pM_4}. \]

It follows from (5.2.5) and (5.2.6) that \( q_2 < 1 \) which proves that \( T \) is a contraction mapping. Hence \( T \) has the fixed point \( X \) with \( \| X \| > 0 \).
for all \( n \geq n_1 \), which is a nonoscillatory solution of system (5.1.1).

This completes the proof for the Case 2.

**Case 3:** \( p = 1 \). Choose \( n_1 \in \mathbb{N}(n_0) \) sufficiently large so that (5.2.4) holds and

\[
\sum_{i=0}^{\infty} \sum_{s=n_1+(2i-1)k}^{n+2ik} \|Q(s)\| \leq \frac{\|p\| - p_1}{p_2}
\]

(5.2.7)

hold, where \( p \) is a nonzero constant vector and \( p_1 < p_2 \) are positive constants such that

\[
p_1 < \|p\| \leq \frac{p_1 + p_2}{2}.
\]

(5.2.8)

Let \( B(n_0) \) be the Banach space considered in Case 1. Set

\[
S = \{X \in B(n_0) : p_1 \leq \|X(n)\| \leq p_2, n \geq n_0\}.
\]

Clearly \( S \) is a closed, bounded and convex subset of \( B(n_0) \). Define a mapping \( T : S \rightarrow B(n_0) \) as follows:

\[
(TX)(n) = \begin{cases} 
\left\{ P + \sum_{i=0}^{\infty} \sum_{s=n+(2i-1)k}^{n+2ik} Q(s)X(s - \ell), n \geq n_1 \\
(TX)(n_1) \end{cases} \quad , n_0 \leq n \leq n_1.
\]

Clearly \( T \) is continuous and for every \( X \in S \) and \( n \geq n_1 \), using (5.2.7) and (5.2.8), we obtain

\[
\|(TX)(n)\| = \left\| P + \sum_{i=0}^{\infty} \sum_{s=n+(2i-1)k}^{n+2ik} Q(s)X(s - \ell) \right\|
\]
\[
\|(TX) (n)\| \leq \|P\| + p_2 \sum_{i=0}^{\infty} \sum_{s=n+(2i-1)k}^{n+2ik} \|Q(s)\| \\
\leq \|P\| + p_2 \frac{\|P\| - p_1}{p_2} = 2 \|P\| - p_1 \leq p_2.
\]

Further
\[
\|(TX) (n)\| \geq \|P\| - p_2 \sum_{i=0}^{\infty} \sum_{s=n+(2i-1)k}^{n+2ik} \|Q(s)\| \\
\geq \|P\| - p_2 \frac{\|P\| - p_1}{p_2} = p_1
\]

Thus, we have \(TS \subseteq S\). Next we prove \(T\) is a contraction mapping on \(S\). For \(X, Y \in S\), and \(n \geq n_1\), we have
\[
\|(TX) (n) - (TY) (n)\|
\leq \|X - Y\| \sum_{i=0}^{\infty} \sum_{s=n+(2i-1)k}^{n+2ik} \|Q(s)\|
\leq q_3 \|X - Y\|.
\]

Thus,
\[
\|TX - TY\| \leq q_3 \|X - Y\|
\]

where
\[
q_3 = \sum_{i=0}^{\infty} \sum_{s=n+(2i-1)k}^{n+2ik} \|Q(s)\| \leq \frac{\|P\| - p_1}{p_2}.
\]
It follows from (5.2.7) and (5.2.8) that $q_3 < 1$ which proves that $T$ is a contraction mapping. Hence $T$ has the fixed point $X$ with $\|X\| > 0$ for all $n \geq n_1$, which is a nonoscillatory solution of system (5.1.1).

This completes the proof of the Case 3.

**Case 4:** $-1 < p < 0$. Choose $n_1 \in \mathbb{N}(n_0)$ sufficiently large so that $n_1 \geq n_0 + \theta$ and

$$
\sum_{n=n_1}^{\infty} \|Q(n)\| \leq \frac{1 + p(1 + L_2) - L_1}{L_2} \quad (5.2.9)
$$

hold, where $L_1 < 1$ and $L_2$ are positive constants such that

$$
2(1 + p) < L_1 + L_2 < 2 \quad \text{and} \quad \frac{L_1 - 1}{1 + L_2} < p \leq \frac{L_1 + L_2}{2} - 1. \quad (5.2.10)
$$

Let $B(n_0)$ be the Banach space considered in Case 1. Set

$$
S = \{X \in B(n_0) : L_1 \leq \|X(n)\| \leq L_2, n \geq n_0\}.
$$

Clearly $S$ is a closed, bounded and convex subset of $B(n_0)$. Define the mapping $T : S \to B(n_0)$ as follows:

$$
(TX)(n) = \begin{cases} (1 + p)e - pX(n - k) + \sum_{s=n}^{\infty} Q(s)X(s - \ell), & n \geq n_1 \\ (TX)(n_1), & n_0 \leq n \leq n_1. \end{cases}
$$

Clearly $T$ is continuous and for every $X$ and $n \geq n_1$, using (5.2.9) and (5.2.10), we see that $TS \subseteq S$. Further, we can easily see that for every
$X, Y \in S$, and $n \geq n_1$,

$$\|TX - TY\| \leq q_4 \|X - Y\|$$

where

$$q_4 = -p + \sum_{s=n_1}^{\infty} \|Q(s)\| \leq -p + \frac{1 + p + pL_2 - L_1}{L_2} = \frac{1 + p - L_1}{L_2}.$$ 

It follows from (5.2.9) and (5.2.10) that $q_4 < 1$ which proves that $T$ is a contraction mapping. Thus $T$ has the fixed point $X$ with $\|X\| > 0$ for all $n \geq n_1$, which is a nonoscillatory solution of system (5.1.1). This completes the proof of the Case 4.

**Case 5:** $-\infty < p < -1$. Choose $n_1 \in \mathbb{N}(n_0)$ sufficiently large such that (5.2.4) holds, and

$$\sum_{s=n_1}^{\infty} \|Q(s)\| \leq \frac{pL_3 - 1 - pL_4}{L_4} \quad (5.2.11)$$

also holds, where $L_3$ and $L_4$ are positive constants such that

$$L_3 < L_4 < 1, L_3 + L_4 > 1 \text{ and } \frac{2}{L_3 + L_4 - 2} < p < \frac{1 + L_4}{L_3 - 1}. \quad (5.2.12)$$

Let $B(n_0)$ be as in Case 1. Set

$$S = \{X \in B(n_0) : L_3 \leq \|X(n)\| \leq L_4, n \geq n_0\}.$$ 

Clearly $S$ is a closed, bounded and convex subset of $B(n_0)$. Define
the mapping \( T : S \rightarrow B(n_0) \) as follows:

\[
(TX)(n) = \begin{cases} 
(1 + \frac{1}{p}) e - \frac{1}{p}X(n + k) \\
+ \frac{1}{p} \sum_{s=n+k}^{\infty} \mathcal{Q}(s)X(s - \ell), & n \geq n_1 \\
(TX)(n_1) & , n_0 \leq n \leq n_1.
\end{cases}
\]

Clearly \( T \) is continuous and for every \( X \in S \) and \( n \geq n_1 \), using (5.2.11) and (5.2.12), we see that \( TS \subseteq S \).

Further, for every \( X, Y \in S \), and \( n \geq n_1 \), we have

\[
\|TX - TY\| \leq q_5 \|X - Y\|
\]

where

\[
q_5 = \frac{1}{p} + \sum_{s=n_1}^{\infty} \|\mathcal{Q}(s)\| < \frac{1 + p - pL_3}{pL_4}.
\]

It follows from (5.2.12) that \( q_5 < 1 \) which proves that \( T \) is a contraction mapping. Hence \( T \) has the fixed point \( X \) with \( \|X\| > 0 \) for all \( n \geq n_1 \), which is a nonoscillatory solution of system (5.1.1). This completes the proof for the Case 5. The proof of the theorem is now complete.

Next we cite an example to illustrate Theorem 5.2.1.

**Example 5.2.1.** Consider the difference system

\[
\Delta \left[ X(n) + 2^{-k}X(n - k) \right] + \mathcal{Q}(n)X(n - \ell) = 0 \quad (5.2.13)
\]
where \( Q(n) = \begin{bmatrix} q_1(n) & q_2(n) \\ q_3(n) & q_4(n) \end{bmatrix} \) with \( q_1(n) + q_2(n) = q_3(n) + q_4(n) = 1 \), \( a \in \mathbb{R} \). Thus by Theorem 5.2.1, the system (5.2.13) has a nonoscillatory solution. In fact
\[
\{ X(n) \} = \left\{ \begin{bmatrix} a + 2^{-n} \\ a + 2^{-n} \end{bmatrix} \right\}, \quad n \geq n_0
\]
is one such solution.

**Remark 5.2.1.** If we allow \( \| e \| > 0 \) be arbitrary, instead of \( \| e \| = 1 \), then the solution of system (5.1.1) has much wider range for asymptotic behavior. In this case, the conditions (5.2.3), (5.2.6), (5.2.10) and (5.2.12) should be changed accordingly. For example, the condition (5.2.3) takes the form
\[
M_1 + M_2 < 2 \| e \| \quad \text{and} \quad 1 - \frac{M_1 + M_2}{2 \| e \|} \leq p < \frac{\| e \| - M_1}{\| e \| + M_2}.
\]

**Remark 5.2.2.** Theorem 5.2.1 can be extended to the system of nonlinear neutral delay equations
\[
\Delta \left( X(n) + p X(n - k) + f(n, X(n - \ell_1), \ldots, X(n - \ell_r)) \right) = 0
\]
where the function \( f \) satisfies the condition
\[
\| f(n, X(n - \ell_1), \ldots, X(n - \ell_r)) \| \leq \sum_{i=1}^{r} \| Q_i(n) X(n - \ell_i) \|
\]
\[(5.2.14)\]
where \( Q_i(n), i = 1, 2, \ldots, r \) are \( m \times m \) matrices for \( n \in \mathbb{N}(n_0) \) and \( i = 1, 2, \ldots, r \) are nonnegative integers. In this case condition (5.2.1) becomes

\[
\sum_{i=1}^{\infty} \| Q_i(n) \| < \infty \quad \text{for} \quad i = 1, 2, \ldots, r.
\]

In particular, the following interesting equation

\[
\Delta \left( X(n) + pX(n - k) \right) + Q_1(n)X(n - \ell_1) - Q_2(n)X(n - \ell_2) = 0
\]

is investigated in [6, 27, 44] for \( m = 1 \) and in [38] for \( m = 2 \).

### 5.3 Existence of Nonoscillatory Solution when \( p \) is matrix

Consider the neutral delay difference system with matrix coefficient of the form

\[
\Delta \left( X(n) + B X(n - k) \right) + Q(n)X(n - \ell) = 0 \quad (5.3.1)
\]

where \( B \) is a nonsingular \( m \times m \) matrix, \( X \in \mathbb{R}^m \) and \( k \) and \( \ell \) are nonnegative integers, \( Q \) is a continuous \( m \times m \) matrix for \( n \in \mathbb{N}(n_0) \). Assume that \( \| B \| = p \).

**Theorem 5.3.1.** With respect to the difference system (5.3.1) assume condition (5.2.1) holds. Then system (5.3.1) has a nonoscillatory solution.
Proof. **Case 1:** $0 \leq p < 1$. Choose $n_1 \in \mathbb{N}(n_0)$ sufficiently large so that $n_1 \geq n_0 + \theta$ and
\[
\sum_{n=n_1}^{\infty} \|Q(n)\| < \frac{1 - p(1 + M_2) - M_1}{M_2}
\]
(5.3.2)
hold, where $M_1 < 1$ and $M_2$ are positive constants such that
\[
1 - \frac{M_1 + M_2}{2} < p < \frac{1 - M_1}{1 + M_2} \quad \text{and} \quad 1 - p < M_1 + M_2 < 2.
\]
(5.3.3)

Let $B(n_0)$ be as in Theorem 5.2.1. Set
\[
S = \{X \in B(n_0) : M_1 \leq \|X(n)\| \leq M_2, n \geq n_0\}.
\]

Clearly $S$ is a closed, bounded and convex subset of $B(n_0)$. Define a mapping $T : S \to B(n_0)$ as follows:
\[
(TX)(n) = \begin{cases} 
    b - BX(n - k) + \sum_{s=n}^{\infty} Q(s)X(s - \ell), & n \geq n_1 \\
    (TX)(n_1), & n_0 \leq n \leq n_1.
\end{cases}
\]

where $b$ is a vector such that $\|b\| = 1 - p$. Clearly $T$ is continuous.

For every $X \in S$ and $n \geq n_1$, using (5.3.2) and (5.3.3), we obtain
\[
\|TX(n)\| = \left\| b - BX(n - k) + \sum_{s=n}^{\infty} Q(s)X(s - \ell) \right\|
\]
\[
\leq 1 - p + pM_2 + M_2 \sum_{s=n_1}^{\infty} \|Q(s)\|
\]
\[
\leq 1 - p + pM_2 + M_2 \left( \frac{1 - p(1 + M_2) - M_1}{M_2} \right)
\]
\[
= 2(1 - p) - M_1 \leq M_2.
\]
Further
\[
\| (TX) (n) \| \geq \| b \| - \| B X(n - k) \| - \left\| \sum_{s=n}^{\infty} Q(s) X(s - \ell) \right\| \\
\geq 1 - p - p M_2 - M_2 \left( \frac{1 - p (1 + M_2) - M_1}{M_2} \right) \\
= M_1.
\]

Thus, we have proved that \( TS \subseteq S \). Next we prove that \( T \) is a contraction mapping on \( S \). For every \( X, Y \in S \) and \( n \geq n_1 \), we have
\[
\| (TX)(n) - (TY)(n) \| = \| -B (X(n - k) - Y(n - k)) \\
+ \sum_{s=n}^{\infty} Q(s) (X(s - \ell) - Y(s - \ell)) \| \\
\leq \| B \| \| X(n - k) - Y(n - k) \| \\
+ \sum_{s=n}^{\infty} \| Q(s) (X(s - \ell) - Y(s - \ell)) \| \\
\leq p \| X - Y \| + \| X - Y \| \sum_{s=n_1}^{\infty} \| Q(s) \| \\
= r_1 \| X - Y \|.
\]

This implies that,
\[
\| TX - TY \| \leq r_1 \| X - Y \|
\]

where
\[
r_1 = p + \sum_{s=n_1}^{\infty} \| Q(s) \| \leq p + \frac{1 - p - p M_2 - M_1}{M_2} = \frac{1 - p - M_1}{M_2}.
\]
It follows from (5.3.3) that \( r_1 < 1 \) which proves that \( T \) is a contraction mapping. Hence \( T \) has the fixed point \( X \) with \( \|X\| > 0 \) for all \( n \geq n_1 \), which is a nonoscillatory solution of system (5.3.1). This completes the proof for the Case 1.

**Case 2:** \( 1 < p < \infty \). Choose \( n_1 \in N(n_0) \) sufficiently large so that \( n_1 + k \geq n_0 + \ell \) and

\[
\sum_{s=n_1+k}^{\infty} \|Q(s)\| \leq \frac{p - 1 - pM_3 - M_4}{M_4} \quad (5.3.4)
\]

hold, where \( M_3 > 1 \), and \( M_4 \) are positive constants such that

\[
M_3 + M_4 \leq 2 \quad \text{and} \quad \frac{1 + M_3}{1 - M_4} < p < \frac{2}{2 - M_3 - M_4}. \quad (5.3.5)
\]

Let \( B(n_0) \) be as in Theorem 5.2.1. Set

\[
S = \{X \in B(n_0) : \ M_3 \leq \|X(n)\| \leq M_4, n \geq n_0 \}.
\]

Clearly \( S \) is a closed, bounded and convex subset of \( B(n_0) \). Define the mapping \( T : S \rightarrow B(n_0) \) as follows:

\[
(TX)(n) = \begin{cases} 
  c - B^{-1}X(n - k) + B^{-1}\sum_{s=n}^{\infty} Q(s)X(s - \ell), & n \geq n_1 \\
  (TX)(n_1), & n_0 \leq n \leq n_1.
\end{cases}
\]

where \( c \) is a vector such that \( \|c\| = 1 - \frac{1}{p} \). Clearly \( T \) is continuous and for any \( X \in S \) and \( n \geq n_1 \), using (5.3.4) and (5.3.5), we have
$TS \subseteq S$. Further for $X, Y \in S$ and $n \geq n_1$, we have

$$\|TX - TY\| \leq r_2 \|X - Y\|$$

where

$$r_2 = \frac{1}{p} \left( 1 + \sum_{s=n_1}^{\infty} \|Q(s)\| \right)$$

$$\leq \frac{1}{p} \left( 1 + \frac{p - 1 - pM_3 - M_4}{M_4} \right)$$

$$= \frac{p - 1 - pM_3}{pM_4}.$$ 

It follows from (5.3.5) that $r_2 < 1$ which proves that $T$ is a contraction mapping. Hence $T$ has the fixed point $X$ with $\|X\| > 0$ for all $n \geq n_1$, which is a nonoscillatory solution of system (5.3.1). This completes the proof for the Case 2.

**Case 3:** $p = 1$. The proof is similar to that of in Theorem 5.2.1 and hence the details are omitted. The proof of the theorem is now complete.

**Remark 5.3.1.** If we allow arbitrary values for $\|b\| > 0$ and $\|c\| > 0$, instead of $\|b\| = 1 - p$ and $\|c\| = 1 - \frac{1}{p}$, then the obtained solutions of system (5.3.1) has much wider range of asymptotic behavior.
Remark 5.3.2. As in Remark 5.2.2, we can extend the results of Theorem 5.3.1 to more general nonlinear system

$$\Delta \left( X(n) + BX(n-k) \right) + f(n, X(n-\ell_1), \cdots, X(n-\ell_r)) = 0$$

where the function $f$ satisfies the condition (5.2.14) and $Q_i$, 
$(i = 1, 2, 3, \cdots, r)$ satisfies the same conditions as in Remark 5.2.2.

We conclude this chapter by illustrating our result with simple example of a system that has the solution which is asymptotically constant.

Example 5.3.1. Consider the difference system

$$\Delta \left( X(n) + \left[ \begin{array}{cc} \alpha & -\alpha \\ -\beta & \beta \end{array} \right] X(n-k) \right)$$

$$+ \left[ \begin{array}{cc} \frac{1}{an(n+1)} & \frac{1}{an(n+1)(an-al+1)} \\ 0 & \frac{n-\ell}{n(n+1)(an-al+1)} \end{array} \right] X(n-\ell) = 0$$

for some $\alpha, \beta, a \in \mathbb{R}$ and $k$ and $\ell$ are nonnegative integers. This system has a nonoscillatory solution by Theorem 5.3.1, and in fact one such solution is $\{X(n)\} = \left\{ \begin{bmatrix} a + \frac{1}{n} \\ a + \frac{1}{n} \end{bmatrix} \right\}$ for $n \geq \theta + 1$. 