Chapter 1

Introduction, Preliminaries and Review

In this chapter a brief review of the applications of graph theory in various fields and a collection of some basic definitions which are needed for the subsequent chapters are clearly brought out. A brief review of the origin and development of cycle multiplicity and tulgeity of a graph in the past five decades is dealt with.

1.1 Introduction

Graph theory is rapidly moving into the mainstream of mathematics. The prospects of further development in algebraic graph theory and the important link with computational theory indicate the possibility of the subject quickly emerging at the forefront of mathematics. Its scientific and engineering applications, especially to computer science, system theory and biology have already been accorded a place of pride in applied mathematics. Graphs serve as mathematical models to analyze successfully many concrete real-world problems. Certain problems in
physics, chemistry, communication science, computer technology, genetics, psychology, sociology and linguistics can be formulated as problems in graph theory. Also many branches of mathematics such as game theory, group theory, matrix theory, probability and topology, have interactions with graph theory. Some puzzles and various problems of a practical nature have been instrumental in the development of various topics in graph theory. The theory of acyclic graphs was developed for solving problems of electrical networks, and the study of trees was developed for enumerating isomers of organic compounds. The well-known four colour problem is the very basis for the development of planarity in graph theory and combinatorial topology. Problems of linear programming in operations research (such as maritime traffic problems) can be tackled by the theory of flows in networks.

In this sequence, enumerating or counting cycles is widely applied in constructing reliable network in communication systems. In this thesis a study of the counting of vertex disjoint and edge disjoint cycles in line, middle and total graph of some families of graphs is undertaken.

The concept of line graph of a given graph has been independently discovered by many authors. Each gave it a different name. Ore[58] calls it as ‘interchange graph’, Sabidussi[62] ‘derivative’, Beineke[5] ‘derived graph’, Seshu and Reed[67] ‘edge-to-vertex dual’, Kasteleyn[47] ‘covering number’ and Menon[55] ‘adjoin’. But both Whitney[79] and Krausz[50] used the construction of line graph before this. Various characterizations of line graphs have been developed by many authors. Total graph
was first studied by Behzad[4], which has surprisingly been discovered only once thus far, and has no other names. Middle graph was defined in succeeding years. The relationships among line graph, middle graph and total graph are studied by many authors and have widely applied in science and engineering. D.B. Dix [28], applied iterated line graphs in Biomolecular Conformation.

1.2 Basic definitions in graph theory

We present the definitions of basic terms and establish some of the notations that will be employed throughout the thesis. For graph theoretic terminology, we refer to Harary [39], Bondy and Murty [11]. Some special definitions are given in the respective sections when they are used.

A graph is an ordered triple $G = (V(G), E(G), I_G)$ where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$ and $I_G$ is an incidence map that associates with each element of $E(G)$, an unordered pair of elements of $V(G)$. Elements of $V(G)$ are called the vertices of $G$, and elements of $E(G)$ are called the edges of $G$.

For convenience, we use $uv$ to denote an edge joining the vertices $u$ and $v$, where $u$ and $v$ are called the ends of the edge $uv$. Two edges are adjacent if they have a common vertex. An edge $uv$ and its end $u$ are said to be incident with each other. Vertex having no incident edge is called an isolated vertex.

An edge with same end vertices is called a loop. Two or more edges are said to be parallel edges if their end vertices are same. A graph that has neither self
loops nor parallel edges are called a **simple graph**.

Two graphs $G = (V(G), E(G), I_G)$ and $H = (V(H), E(H), I_H)$ are said to be **isomorphic** if there exists a pair $(\varphi, \theta)$ where $\varphi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijections with the property that $I_G(e) = \{u, v\}$ if and only if $I_H(\theta(e)) = \{\varphi(u), \varphi(v)\}$. Two graphs are **homeomorphic** if both can be obtained from the same graph by a sequence of subdivisions of lines.

The **degree** of a vertex $v$ in a graph $G$, denoted by $\deg_G v$ or simply $\deg v$, is the number of edges of $G$ incident with $v$. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of vertices of $G$ respectively. A graph is **regular** ($d$-regular) if all its vertices have the same degree $d$.

A graph $H$ is a **subgraph** of graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, $H$ is a **spanning subgraph** of $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. If $S \subseteq V(G)$ then the subgraph induced by $S$ is the graph $\langle S \rangle$ with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$.

An $(v_0, v_r)$ -**walk** of length $r$ in a graph $G$ is a sequence of vertices of the form $v_0, v_1, v_2, \ldots, v_r$ where $v_i v_{i+1} \in E(G)$ for $i = 0, 1, 2, \ldots, r - 1$. This walk is denoted by $v_0v_1v_2\ldots v_r$ and $v_0$ is called the starting vertex and $v_r$ the ending vertex.

A **trail** is a walk in which all edges are distinct. A **path** is a trail in which all vertices are distinct. The first and last vertices of a path are its endpoints. The path with end points $u$ and $v$ is called $(u, v)$ path. The number of vertices in a path is called length of the path. The path of length $n$ is denoted by $P_n$. The length of
the shortest \((u, v)\) path is called the **distance** between the vertices \(u\) and \(v\). It is denoted by \(d(u, v)\).

A **cycle** is a trail in which all vertices are distinct except that the starting vertex is the same as the ending vertex. We denote a cycle on \(n\) vertices by \(C_n\). The number of vertices on a cycle is called the length of the cycle. The minimum of the length of the cycles in a graph \(G\) is called its **girth** and it is denoted by \(g(G)\). The maximum of the lengths of the cycles in a graph \(G\) is called its **circumference** and it is denoted by \(c(G)\).

A graph \(G\) is **connected** if there exists a \((u, v)\)-path for every pair of vertices \(u\) and \(v\) of \(G\). A **component** is a connected subgraph of \(G\) which is not properly contained in any other connected subgraph of \(G\). A graph \(G\) with \(E(G) = \emptyset\), empty set is called totally disconnected. A connected graph that has no cut vertices is called a **block**. A **block of a graph** is a subgraph that is a block and is maximal with respect to this property.

A simple graph in which every two distinct vertices are adjacent is a **complete graph**. The complete graph on \(n\) vertices is denoted by \(K_n\). A maximal complete subgraph of a graph \(G\) is called a **clique**. \(K_1\) is called **singleton graph**.

A **bipartite graph** is a graph whose vertex set can be partitioned into two sets \(A\) and \(B\) such that each edge joins a vertex of \(A\) to a vertex of \(B\).

A **complete bipartite graph** is a bipartite graph in which each vertex in the first set is adjacent to every vertex in the second set. If these two sets contain \(m\)
and $n$ vertices, respectively, then the complete bipartite graph is denoted by $K_{m,n}$.

A complete bipartite graph of the form $K_{1,n}$ is called a **star**.

An $r$-**partite graph** is a graph whose vertex set can be partitioned into $r$ subsets so that no edge has both ends in any one subset.

A complete $r$-**partite graph** is an $r$-partite graph in which each vertex is joined to every vertex that is not in the same subset.

**Line graph** $L(G)$ of a graph $G$ is defined with the vertex set $E(G)$, in which two vertices are adjacent if and only if the corresponding edges are adjacent in $G$.

**Middle graph** $M(G)$ of a graph $G$ is defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if either both are adjacent edges in $G$ or one of the elements is a vertex and the other one is an edge incident to the vertex in $G$.

**Total graph** $T(G)$ of a graph $G$ defined with the vertex set $V(G) \cup E(G)$, in which two elements are adjacent if and only if one of the following holds true (i) both are adjacent edges or vertices in $G$ (ii) one is a vertex and other is an edge incident to it in $G$.

We consider finite, simple, undirected graph $G$ with the vertex set $V(G)$ and edge set $E(G)$. $p$ and $q$ denote the number of vertices and edges of the graph $G$ respectively. For any real number $r$, $[r]$ and $\lceil r \rceil$ denote the largest integer not exceeding $r$ and the least integer not less than $r$, respectively. If $\mathcal{C}$ is the collection of cycles of a graph then $V(\mathcal{C})$ denotes the set of vertices belonging to the cycles of
\( \mathbb{C} \). \( \langle S \rangle \) denotes the subgraph of \( G \) induced by the set of vertices \( S \subseteq V(G) \). If \( C \) is a cycle induced by the three vertices \( v_1, v_2 \) and \( v_3 \) then we denote \( C \) by \( v_1v_2v_3 \). \( \emptyset \) denotes the empty set. The other notations and terminology used in this paper can be found in [39].

1.3 Cycle multiplicity and Tulgeity of graphs

Connection is a significant feature in communication systems. Thus they can be modeled and described as networks when dealing with some problems. Graph theories and algorithms are useful tools in the study of communication. Cycle is one of the distinct features to characterize a graph. The analysis of cycles in networks have different applications in the design and development of communication systems, such as the investigation of topological features[61], realiability, fault tolerance[54], etc. There are various problems related to the analysis of cycles in networks. Several problems of finding shortest path, shortest spanning trees, least cost Hamiltonian cycles, etc., of a graph have been studied. Enumerating or counting all cycles of given length, cycle decomposition, counting maximum number of vertex disjoint and edge disjoint cycles play a vital role in the design of reliable network. This motivates to study cycle multiplicity and tulgeity of certain family of graphs.

Graphs can be characterized by a type of configuration or subgraph they possess. Gary Chartrand et al.,[16], characterized several classes of graphs by defining them with the property \( P_n \). A graph \( G \) has the property \( P_n \), where \( n \) is a positive integer, if \( G \) contains no subgraph which is homeomorphic from the complete graph...
\( K_{n+1} \) or the complete bigraph \( K \left( \left\lfloor \frac{n+2}{2} \right\rfloor \left\lceil \frac{n+2}{2} \right\rceil \right) \). Trivially, a graph \( G \) has the property \( P_2 \), if \( G \) contains no cycles. They introduced the concept of point partition number and line partition number of a graph \( G \). The point partition number of a graph is defined as the minimum number of subsets into which the vertex set of \( G \) can be partitioned so that the subgraph induced by each subset has the property \( P_n \). The line partition number of a graph is defined as the minimum number of subsets into which the edge set of \( G \) can be partitioned so that the subgraph induced by each subset has the property \( P_n \). Dual to these concepts the dual point partition number of a graph is defined as the maximum number of subsets into which the vertex set of \( G \) can be partitioned so that the subgraph induced by each subset does not possesses the property \( P_n \) and dual line partition number of a graph is defined as the maximum number of subsets into which the edge set of \( G \) can be partitioned so that the subgraph induced by each subset does not possesses the property \( P_n \). The problems of Nordhaus - Gaddum type for the dual point partition number are investigated in [51].

In [16], Gary chartrand et al., defined the term cycle multiplicity of a graph as the dual line partition number for \( n = 2 \). (i.e) Cycle multiplicity is the maximum number of subsets into which the edge set of \( G \) can be partitioned so that the subgraph induced by each subset contains a cycle. Also they defined the term point cycle multiplicity or tulgeity of a graph as the dual point partition number for \( n = 2 \). (i.e) Tulgeity is the maximum number of subsets into which the vertex set of \( G \) can
be partitioned so that the subgraph induced by each subset contains a cycle.

Equivalently the cycle multiplicity of a graph is the maximum number of edge disjoint cycles in $G$ and it is denoted by $c(G)$, the tulgeity of a graph is the maximum number of vertex disjoint cycles in $G$ and it is denoted by $\tau(G)$. Since every cycle contains at least three vertices, an obvious upper bound for cycle multiplicity and tulgeity of any graph $G$ with $p$ vertices and $q$ edges are $c(G) \leq \lfloor q/3 \rfloor$ and $\tau(G) \leq \lfloor p/3 \rfloor$. $c(G) = \lceil q/3 \rceil$ and $\tau(G) = \lceil p/3 \rceil$ if $G$ is a complete graph.

In 1971, Gary Chartrand et al., [16], derived the formulas to find the cycle multiplicity of complete and complete bigraph. This was the first work carried out in cycle multiplicity. To derive these formulas they used the result of M.K. Fort et al., [32], that the edge set of any complete graph $K_n$, can be partitioned into subsets of three edges each such that every subset induces a triangle. Also they used the result of Guy [37] to prove these results. In 1972, Simões Pereira J.M.S[68] derived the formula to find the cycle multiplicity of line and total graph of complete graph. Also he derived a lower bound for the cycle multiplicity of line and total graph of any graph. In later decades, several results concerning the decomposition of vertex disjoint and edge disjoint cycles in a graph has been studied without considering the definition of cycle multiplicity and tulgeity. The definition of cycle multiplicity has been generalized by defining the cycle packing and its maximum cardinality. Cycle packing of a graph $G$ is defined as a collection of pairwise edge-disjoint cycles and several results were derived recently concerning the maximum cardinality of this
a new technique for packing pairwise edge-disjoint cycles of specified lengths in
complete graphs and used it to prove the existence of dense packings of the complete
graph with pairwise edge-disjoint cycles of arbitrary specified lengths and then used
this result to prove the existence of decompositions of the complete graph of odd
order into pairwise edge-disjoint cycles for a large family of lists of specified cycle
lengths. They also constructed new maximum packings of the complete graph with
pairwise edge-disjoint cycles of uniform length. In 2010, B.R. Smith [70] established
necessary and sufficient conditions for decomposing the complete multigraph $\lambda K_n$
into cycles of length $\lambda$.

Now we give a brief review on vertex disjoint cycles of a graph. In 1964,
bounds for maximum number of vertex disjoint cycles in a graph have been obtained
by K. Corrádi and A. Hajnal[22] and by G. Dirac and P. Erdős[27]. In 1968, Gary
chartrand et al., [17], derived a formula to find the tulgeity of complete multi partite
graph. In 1981, D. Sotteau [72] gave necessary and sufficient conditions in order that
$K_{m,n}$ admits a decomposition into $2k$-cycles. In 1996, H. Wang [74] showed that if $G$
is a bipartite graph $G = (V_1, V_2; E)$ with $|V_1| = |V_2| = n > 2k$, and minimum degree
at least $k + 1$, then $G$ contains $k$ vertex-disjoint cycles. In 1999,[75] he proved that
if $d(x) + d(y) \geq 4k - 1$ for every pair of non adjacent vertices $x$ and $y$ of a graph $G$
then $G$ contains $k$ vertex disjoint cycles. In the same year Chao-chih Chou et al.,
[18] showed that for any even $m, n$, $K_{m,n}$ can be decomposed into $p$ copies of $C_4$,
$q$ copies of $C_6$ and $r$ copies of $C_8$ for each triple $p, q, r$ of non negative integers such
that $4p + 6q + 8r = |E(G)|$ and proved the same for any $K_{n,n}$ minus 1-factor for odd $n$. Recently in 2005, Hajime Matsumura [53] derived the degree conditions for a bipartite graph to contain vertex disjoint 4-cycles and Dieter Rautenbach and Irene Stella [60] found the upper and lower bound for the maximum number of cycles in Hamiltonian graph. In 2004, Hung-Lin Fu, et al.,[45] proved that for any positive integer $m_1, m_2, \cdots, m_k$ not less than 3, the complete graph of order $2n + 1$ can be cyclically decomposed into $k(2n + 1)$ cycles such that, for each $i = 1, 2, \cdots, k$, the cycle of length $m_i$ occurs exactly $2n + 1$ times.

However, there exists no further research in cycle multiplicity and tulgeity of line, middle and total graph of a given graph in literature survey after the contribution of Simões Pereira J.M.S[68].