To find the distribution of $S_n$, the waiting time distribution, consider the Laplace transform of (2.5.1), which is obtained as

$$L_X(S) = \frac{\lambda_1 p}{\lambda_1 + S} + \frac{\lambda_2 (1 - p)}{\lambda_2 + S}. \quad (2.6.2)$$

Then the Laplace transform of $S_n$ is

$$L_{S_n}(S) = \left[ \frac{\lambda_1 p}{\lambda_1 + S} + \frac{\lambda_2 (1 - p)}{\lambda_2 + S} \right]^n, \quad (2.6.3)$$

which can be written as

$$L_{S_n}(S) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k} \left[ \frac{\lambda_1}{\lambda_1 + S} \right]^k \left[ \frac{\lambda_2}{\lambda_2 + S} \right]^{n-k}. \quad (2.6.4)$$

As Laplace transform uniquely determine the density the p.d.f of $S_n$ is given by

$$g(t) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k} \lambda_1^k \lambda_2^{n-k} t^{n-1} \times \phi_2(k, n - k, n - \lambda_1 t, -\lambda_2 t). \quad (2.6.5)$$

where $\phi_2(\cdot)$ is the hyper-geometric series with

$$\phi_2(\beta, \beta', \gamma, x, y) = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (\beta)_u (\beta')_v x^u y^v (\gamma)_{u+v} u! v!, \quad (2.6.6)$$

and

$$(\beta)_a = \beta (\beta + 1) \ldots (\beta + a - 1)$$

(see Erdelyi et al., 1954).

From the property of the function (2.6.6) it can be shown that the distribution function corresponding to (2.6.5) is

$$G(t) = \frac{1}{n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k} \lambda_1^k \lambda_2^{n-k} t^n \times \phi_2(k, n - k, n + 1 - \lambda_1 t, -\lambda_2 t). \quad (2.6.7)$$

Note that (2.6.5) is a generalization of mixture of two gamma densities. When $\lambda_1 = \lambda_2$ it reduces to the ordinary gamma density with parameters $\lambda_1$ and $n$. As

$$P(S_n \leq t) = P(N(t) \geq n) \quad (2.6.8)$$

(2.6.7) is the survival function of $N(t)$.
### 2.6.2 Renewal Function and Properties

An interesting concept in renewal theory is renewal function which is defined as

\[ M(t) = E(N(t)). \]  

(2.6.9)

In order to study \( N(t) \), the number of renewals by time \( t \), it is simplest to use the connection between \( N(t) \) and the random variable \( S_n \), the time of the \( n^{th} \) renewal. It is evident from the definition of the random variables \( N(t) \) and \( S_n \) that

\[ N(t) \leq n \text{ if and only if } S_n \geq t. \]

Equivalently

\[ N(t) < n \text{ if and only if } S_n > t. \]

Therefore

\[ P(N(t) < n) = P(S_n > t) = 1 - F_n(t). \]  

(2.6.10)

Hence

\[ P(N(t) = n) = F_n(t) - F_{n+1}(t). \]  

(2.6.11)

The renewal function \( M(t) \) can be, therefore, expressed in the following way:

\[
M(t) = \sum_{n=1}^{\infty} nP(N(t) = n) \\
= \sum_{n=1}^{\infty} n(P(N(t) \leq n) - P(N(t) \leq n + 1)) \\
= \sum_{n=1}^{\infty} n(F_n(t) - F_{n+1}(t)) \\
= \sum_{n=1}^{\infty} F_n(t), \text{ for } t \geq 0. \]  

(2.6.13)

For the model (2.5.1), the renewal function is in a closed form and we have the following theorem.
Theorem 2.6.1. The distribution of $X_n$ is (2.5.1) if and only if $M(t)$ is of the form

$$M(t) = \frac{\lambda_1 \lambda_2 t}{B} + C(1 - e^{-Bt}),$$ \hspace{1cm} (2.6.14)

where

$$B = (1 - p)\lambda_1 + p\lambda_2,$$ and

$$C = \frac{p (1 - p)(\lambda_1 - \lambda_2)^2}{B^2}.$$

Proof. Assume that (2.6.14) holds. Then we have

$$M(t) = \frac{\lambda_1 \lambda_2 t}{B} \left( t - \frac{1 - e^{-Bt}}{B} \right) + \frac{A}{B} (1 - e^{-Bt}),$$ \hspace{1cm} (2.6.15)

where

$$A = p\lambda_1 + (1 - p)\lambda_2.$$

Let $M^*(S)$ denote the Laplace transform of $M(t)$. Then, (2.6.15) gives

$$M^*(S) = \frac{\lambda_1 \lambda_2}{S^2(S + B)} + \frac{A}{S(S + B)},$$ \hspace{1cm} (2.6.16)

or

$$M^*(S) = \frac{p\lambda_1(\lambda_2 + S) + (1 - p)\lambda_2(\lambda_1 + S)}{S(S^2 + \lambda_1 S - p\lambda_1 S + p\lambda_2 S)}.$$ \hspace{1cm} (2.6.17)

From (2.6.8), $M(t) = \sum_{n=1}^{\infty} G(t)$ which implies

$$M^*(S) = \frac{f^*(S)}{S(1 - f^*(S))},$$ \hspace{1cm} (2.6.18)

or

$$f^*(S) = \frac{SM^*(S)}{1 + SM^*(S)},$$ \hspace{1cm} (2.6.19)

where $G(t)$ is the distribution function corresponding to $S_n$.

From (2.6.17) and (2.6.19), we get

$$f^*(S) = \frac{p\lambda_1(\lambda_2 + S) + (1 - p)\lambda_2(\lambda_1 + S)}{S^2 + \lambda_1 S + \lambda_2 S + \lambda_1 \lambda_2},$$ \hspace{1cm} (2.6.20)
which is (2.6.2). By the uniqueness property of Laplace transform, the distribution of $X_n$ is (2.5.1).

Conversely, when $X_n$ has the density (2.5.1), we have

$$M^*(S) = \frac{\lambda_1 \lambda_2 + AS}{S^2(S+B)},$$

(2.6.21)

which can be written as

$$M^*(S) = \frac{\lambda_1 \lambda_2}{B} \left[ \frac{1}{S^2} - \frac{1}{B} \left( \frac{1}{S} - \frac{1}{S+B} \right) \right] + \frac{A}{B} \left[ \frac{1}{S} - \frac{1}{S+B} \right].$$

(2.6.22)

Inverting the Laplace transform (2.6.22), we get $M(t)$ as given in (2.6.14).

This proves the theorem.

**Remark 2.6.1.** The renewal function $M(t)$ for the model (2.5.1) is obtained by Ross (2000), using transition probability approach. However, Theorem 2.6.1 provides a complete characterization for (2.5.1).

**Remark 2.6.2.** When $\lambda_1 = \lambda_2$, (2.6.14) is the well-known characterization of the Poisson process.

### 2.6.3 Renewal Density Function

Consider for any time $t$, the function $m(t)$, called renewal density

$$m(t) = \lim_{\Delta t \to 0} \frac{E(N(t,t+\Delta t))}{\Delta t},$$

which is the derivative $M'(t)$ of the renewal function. The renewal density specifies the mean number of renewals to be expected in a narrow interval near $t$. Its physical interpretation is that $m(t)\Delta t$ is asymptotically the chance of a renewal in the interval $(t, t+\Delta t)$. An alternative interpretation of $m(t)$ is that if we have a very large number $k$ of independent renewal processes in operation simultaneously, $km(t)\Delta t$ is the number of renewals in the time interval $(t, t+\Delta t)$. Mathematically, $m(t)$ is most easily calculated as the derivative of $M(t)$.

From (2.6.13), the renewal density $m(t)$ admits the form

$$m(t) = \sum_{n=1}^{\infty} f_n(t), \text{ for } t \geq 0,$$

(2.6.23)
where
\[ f_n(t) = \frac{d}{dt} F_n(t). \]
For the model (2.5.1), the renewal density is
\[ m(t) = \frac{\lambda_1 \lambda_2}{B} + CBe^{-Bt}, \tag{2.6.24} \]
where
\[ B = (1 - p) \lambda_1 + p \lambda_2, \]
and
\[ C = \frac{p (1 - p) (\lambda_1 - \lambda_2)^2}{B^2}. \]
As (2.6.24) is a decreasing function of \( t \), the renewal function \( M(t) \) is concave. The mixing weight \( p \) can be negative (see Titterington et al. (1985)). In that case \( M(t) \) will be a convex function. So the behavior of the renewal function \( M(t) \) depends on mixing weight.

### 2.6.4 Distribution of Age and Residual Life

In renewal theory, the random variables \( Y_1(t) = t - S_{N(t)} \) and \( Y_2(t) = S_{N(t)+1} - t \) are respectively called the age and the residual life at \( t \). Then the joint distribution are specified by the survival function
\[ P[Y_1(t) > x_1, Y_2(t) > x_2] = \int_0^{t-x_2} [1 - F(t + x_1 - y)] dU(y), \tag{2.6.25} \]
where \( U(y) = \sum_{n=0}^{\infty} F^{(n)}(y) \) and \( F^{(n)}(y) \) denotes the \( n \)-fold convolution of \( F \). As \( t \rightarrow \infty \) (2.6.25) tends to
\[ R(x_1, x_2) = \frac{1}{\mu} \int_{x_1+x_2}^{\infty} [1 - F(u)] du, \tag{2.6.26} \]
see Feller (1971), where \( R(x_1, x_2) \) is the joint survival function of \( Y_1 \) and \( Y_2 \) and \( \mu \) is the mean corresponding to \( F \). The marginal distributions of \( Y_1 \) and \( Y_2 \) are identical and is given by
\[ P[Y_i > x_i] = \frac{1}{\mu} \int_{x_i}^{\infty} [1 - F(u)] du, \tag{2.6.27} \]
which is known as the equilibrium distribution of the renewal process. From (2.6.26)

\[ R(x_1, x_2) \mu = \gamma(x_1 + x_2) F(x_1 + x_2), \]

(2.6.28)

where

\[ \gamma(t) = [1 - F(u)]^{-1} \int_t^\infty [1 - F(u)] du, \]

is the mean residual function corresponding to \( F \). From (2.6.28), we see that the underlying distribution of the renewal process can be determined uniquely by the joint distribution of \( Y_1 \) and \( Y_2 \). For the model (2.5.1), we have the following result.

**Theorem 2.6.2.** The distribution of \( X_n \) is (2.5.1) if and only if the joint distribution of \( Y_1 \) and \( Y_2 \) is given by the survival function

\[ R(x_1, x_2) = \frac{1}{\mu} \left[ \frac{pe^{-\lambda_1(x_1+x_2)}}{\lambda_1} + \frac{(1-p)e^{-\lambda_2(x_1+x_2)}}{\lambda_2} \right], \]

(2.6.29)

where \( t \) in (2.5.1) is equal to \( x_1 + x_2 \).

**Proof.** The proof follows directly from (2.6.26) and (2.6.28).

It is easy to see that the equilibrium distribution uniquely determine the underlying distribution of the renewal process (see also, Gupta and Kirmani, 1990). We characterize (2.5.1) using equilibrium distribution, which generalizes the result for the exponential distribution given in Gupta and Kirmani (1990).

**Theorem 2.6.3.** The distribution of \( X_n \) is (2.5.1) if and only if the equilibrium distribution of \( X_n \) is a mixture of two exponentials.

**Proof.** The proof follows easily from (2.6.26) and (2.6.27).

It is surprising to see that the marginal distribution of \( Y_i \) determine the joint density of \( (Y_1, Y_2) \). Notice that \( Y_i \)'s \( (i=1,2) \) are dependent.

### 2.6.5 Number of Renewals in a Random Time

Consider a random variable \( T \) so that \([0, T]\) is a random interval. Let \( N \) be the number of renewals in \([0, T]\) with \( G(z) \) as the probability generating function. Let
$G(t, z)$ be the probability generating function (p.g.f) of $N(t)$, where $N(t)$ is the number of renewals in a fixed interval $[0, t]$. If $q(t)$ is the p.d.f of $T$, then it is clear that

$$G(z) = \int_0^\infty G(t, z)q(t)dt. \quad (2.6.30)$$

Suppose that the p.d.f. of $T$ is (2.5.1) then

$$G(z) = \int_0^\infty G(t, z)[p\lambda_1 e^{-\lambda_1 t} + (1 - p)\lambda_2 e^{-\lambda_2 t}]dt, \quad (2.6.31)$$

which gives

$$G(z) = p\lambda_1 G^*(\lambda_1, z) + (1 - p)\lambda_2 G^*(\lambda_2, z), \quad (2.6.32)$$

where

$$G^*(\lambda_i, z) = Ee^{-\lambda_i G(t, z)}, \quad i = 1, 2.$$ 

From the definition of $G(t, z)$, we have

$$G(t, z) = \sum_{r=0}^\infty z^r P(N(t) = r)$$

$$= \sum_{r=0}^\infty z^r (F_r(t) - F_{r+1}(t))$$

$$= 1 + \sum_{r=1}^\infty z^{r-1}(z - 1)F_r(t), \quad (2.6.33)$$

where $F_r(t)$ is the distribution function of $S_r$. The equation (2.6.33) gives

$$G^*(\lambda_i, z) = \frac{1}{\lambda_i} + \frac{1}{\lambda_i} \sum_{r=1}^\infty z^{r-1}(z - 1)\left(f^*(\lambda_i)\right)^r, \quad i = 1, 2, \quad (2.6.34)$$

or

$$G^*(\lambda_i, z) = \frac{1 - f^*(\lambda_i)}{\lambda_i(1 - zf^*(\lambda_i))}, \quad i = 1, 2, \quad (2.6.35)$$

where $f^*(\cdot)$ is the Laplace transform of (2.5.1). Substituting (2.6.35) in (2.6.32). We get

$$G(z) = p\left(\frac{1 - f^*(\lambda_1)}{1 - zf^*(\lambda_1)}\right) + (1 - p)\left(\frac{1 - f^*(\lambda_2)}{1 - zf^*(\lambda_2)}\right), \quad (2.6.36)$$
which gives

\[ P(N = r) = p[f^*(\lambda_1)]^r[1 - f^*(\lambda_1)] + (1 - p)[f^*(\lambda_2)]^r[1 - f^*(\lambda_2)], \quad r = 0, 1, 2, \ldots \quad (2.6.37) \]

Thus, the distribution of number of renewals in a random time is a mixture of two geometric distributions.

**Corollary 2.6.1.** The distribution of number of renewals in a random time is geometric if the distribution of \( X_n \) is exponential.

### 2.7 Concluding Remarks

In this chapter, we discussed various properties of the finite mixture of exponential distribution in the context of renewal and reliability theory. We obtained a generalized mixture of gamma distributions in terms of the confluent hypergeometric function, as the waiting time distribution. We provided an expression for renewal function and renewal density in the mixture model. The distribution of age and residual life for the model are discussed. Finally, we proved that the distribution of number of renewals in a random time is a mixture of geometric if the interarrival distribution is mixture of exponential.
Chapter 3

Ageing Characteristics of the Finite Mixture of Geometric Distribution

3.1 Introduction

Geometric distribution plays a central role in analysis of lifetime or survival data, in part because of their convenient statistical theory, their important lack of memory property and their constant hazard rate. Mixture models, as discussed in chapter 1, provide an additional flexibility in modelling and analysis of lifetime from a heterogeneous population.

The more paradoxical case corresponds to finite mixture of geometric distribution that possess a decreasing failure rate (DFR), even though each component in the mixture has a constant failure rate. However, a systematic study towards the ageing characteristics of the finite mixture of geometric distribution is not yet carried out. Accordingly in the present chapter, we discuss various ageing characteristics of the finite mixture of geometric distribution.

The chapter is organized as follows. In section 3.1, we give the definition of the finite mixture of geometric distribution. Various properties of the mixture model are discussed in section 3.2. The properties, which are applicable in reliability theory, associated with the geometric mixture are included in section 3.3. In section 3.4, we investigate the discrete version of the finite mixture of exponential
distribution. A new explanation of decreasing failure rate of a mixture of geometric is given in section 3.5. Section 3.6 discusses the behavior of the failure rate of geometric mixture in the case of negative weights and also provides one relevant example to illustrate the property. Finally, concluding remarks are given in section 3.7.

**Definition 3.1.1.** Let $X$ be a discrete random variable in the set of non-negative integers $I^+$ with $E(X) < \infty$. Assume that the probability mass function (p.m.f) of $X$, $f(x)$ exists. A finite mixture of geometric distribution with $k$ components can be represented in the form

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + \ldots + p_k f_k(x),$$

where $p_i > 0$, $i = 1, 2, \ldots, k$; $\sum_{i=1}^{k} p_i = 1$ and

$$f_i(x) = q_i^x p_i, \; x = 0, 1, 2, \ldots, \quad q_i = 1 - p_i \; (i = 1, 2, \ldots, k). \quad (3.1.1)$$

As a special case of (3.1.1) with $k = 2$, we have,

$$f(x) = pq_1^x p_1 + (1 - p)q_2^x p_2 , \; x = 0, 1, 2, \ldots, \quad q_i = 1 - p_i \; (i = 1, 2). \quad (3.1.2)$$

For convenience we choose $p$ and $(1 - p)$ instead of $p_1$ and $p_2$. Throughout this chapter we consider a mixture of two geometric distributions. However, the results will be valid for any numbers of finite geometric mixtures.

### 3.2 Properties

Now, we present various properties associated with (3.1.2). The mean of the distribution (3.1.2) is obtained as

$$E(X) = \sum_{0}^{\infty} x f(x)$$

$$= \sum_{0}^{\infty} x[ pq_1^x p_1 + (1 - p)q_2^x p_2]$$

$$= \frac{q_1}{p_1} + (1 - p) \frac{q_2}{p_2}. \quad (3.2.1)$$
The variance of (3.1.2) is given as

\[
Var(X) = E(X^2) - [E(X)]^2 \\
= \sum_{x=0}^{\infty} x^2 f(x) - \left[ \sum_{x=0}^{\infty} x f(x) \right]^2 \\
= \sum_{x=0}^{\infty} x^2 [pq_1^x p_1 + (1-p)q_2^x p_2] - \left[ \sum_{x=0}^{\infty} x (pq_1^x p_1 + (1-p)q_2^x p_2) \right]^2 \\
= \frac{pq_1}{p_1^2} + \frac{(1-p)q_2}{p_2^2}.
\] (3.2.2)

Characteristic function of (3.1.2) is obtained as

\[
E(e^{itx}) = \sum_{x=0}^{\infty} e^{itx} f(x) \\
= \sum_{x=0}^{\infty} e^{itx} [pq_1^x p_1 + (1-p)q_2^x p_2] \\
= pp_1(1 - q_1 e^{it})^{-1} + (1-p) p_2(1 - q_2 e^{it})^{-1}.
\] (3.2.3)

Similarly moment generating function takes the form

\[
E(e^{tx}) = pp_1(1 - q_1 e^{t})^{-1} + (1-p) p_2(1 - q_2 e^{t})^{-1}.
\] (3.2.4)

### 3.3 Reliability Characteristics

The reliability function for the model (3.1.2) is obtained as

\[
R(x) = P(X \geq x) \\
= pq_1^x + (1-p)q_2^x.
\] (3.3.1)

\(R(x)\) gives the probability that the device will operate without failure for a time \(x\).

The hazard rate for the model (3.1.2) is given as

\[
h(x) = \frac{f(x)}{R(x)} \\
= \frac{pq_1^x p_1 + (1-p)q_2^x p_2}{pq_1^x + (1-p)q_2^x}.
\] (3.3.2)
Residual life after an elapsed time $x$ is denoted by $\{X - x/X > x\}$. The mean residual life function (M.R.L) for the model (3.1.2) is given by

$$r(x) = E[X - x/X > x],$$

$$= \frac{pq^{x+1} + (1-p)q^{x+1}p_1^{-1}}{pq^{x+1} + (1-p)q^{x+1}p_2^{-1}},$$

(3.3.3)

i.e., $r(x)$ expresses the expected additional lifetime given that a component has survived until time $x$ and is well defined when $E(X) < +\infty$.

Figures 3.1 to 3.6 show the behavior of hazard rate and the mean residual life function for different parameters of the model (3.1.2). From the figures, it follows that $h(x)$ is decreasing in $x$ while $r(x)$ is increasing in $x$. 
Figure 3.1: Failure rate for

\((p_1 = 0.4, p_2 = 0.5, p = 0.7)\)

![Graph 3.1]

Figure 3.2: Failure rate for

\((p_1 = 0.3, p_2 = 0.4, p = 0.3)\)

![Graph 3.2]

Figure 3.3: Failure rate for

\((p_1 = 0.5, p_2 = 0.6, p = 0.6)\)

![Graph 3.3]
Figure 3.4: Mean residual life function for

\((p_1 = 0.4, p_2 = 0.5, p = 0.7)\)

Figure 3.5: Mean residual life function for

\((p_1 = 0.3, p_2 = 0.4, p = 0.3)\)

Figure 3.6: Mean residual life function for

\((p_1 = 0.5, p_2 = 0.6, p = 0.6)\)
It is well known that $h(x) \; (r(x))$ uniquely determines the distribution. But in many situations $h(x) \; (r(x))$ is not in a simple closed form and hence cannot be used for characterizing the distributions. In such situations, one may explore simple relationship among $h(x)$ and $r(x)$ to characterize the distributions (see Nair and Sankaran, 1991).

**Theorem 3.3.1.** Let $X$ be a discrete random variable in the support of non-negative integers $I^+$ with $E(X) < \infty$. Then

$$r(x) = (p_1^{-1} + p_2^{-1}) - p_1^{-1} p_2^{-1} h(x + 1),$$

(3.3.4)

for all $x$ in $I^+$ if and only if $X$ has distribution with probability mass function

$$f(x) = pq_1^x p_1 + (1 - p)q_2^x p_2, \quad x = 0, 1, 2, \ldots, 0 < p_i < 1 \text{ and } q_i = 1 - p_i (i = 1, 2).$$

For the proof see Priya (2001).

**Remark 3.3.1.** The random variable $X$ follows geometric law with

$$f(x) = q^x p, \quad x = 0, 1, 2, \ldots, q = 1 - p$$

If and only if

$$r(x) = 2p^{-1} - p^{-2} h(x + 1).$$

(3.3.5)

The result follows by taking $p_1 = p_2 = p$ in theorem 3.3.1. This is a particular case of the result for the Ord family proved in Nair and Sankaran (1991). Further if one notes that $h(x)$ is a constant $p$ for the geometric distribution, we have Shanbhag’s (1970) characterization result.

### 3.4 Discrete Version of the Finite Mixture of Exponential Distribution

In this section, we prove that the finite mixture of exponential distribution with probability density function

$$f(x) = p\lambda_1 e^{-\lambda_1 x} + (1 - p)\lambda_2 e^{-\lambda_2 x}, \quad x > 0, \quad 0 < p < 1, \quad \lambda_1, \lambda_2 > 0,$$

(3.4.1)
is the continuous approximation of the geometric mixture (3.1.2). To prove this, we employ the slope-ordinate ratio method of Irwin (1975). This method consists in equating the ratio \( \frac{f_r - f_{r-1}}{\frac{1}{2}[f_r + f_{r-1}]} \) to the logarithmic derivative of \( f(x) \) evaluated at \( x = r - \frac{1}{2} \) and then solve the resulting differential equation. Notice that \( f_r \) is the frequency for \( r = 0, 1, 2, \ldots \) and \( f(x) \) is the corresponding continuous density.

Let \( f_{ir} \) be the frequency of \( i \)th component \( i = 0, 1, 2 \) and \( r = 0, 1, 2, \ldots \). Let \( f_i(x) \) be the corresponding continuous density.

For \( i = 1 \), (3.1.2) gives

\[ f_{1r} = N q_1^r p_1, \]

which gives

\[ f_{1r} - f_{1(r-1)} = N p_1 q_1^{r-1} (q_1 - 1) \]  \( (3.4.2) \)

and

\[ \frac{1}{2} \left[ f_{1r} + f_{1(r-1)} \right] = N p_1 q_1^{r-1} \left( \frac{q_1 + 1}{2} \right) \]  \( (3.4.3) \)

\[ \frac{1}{2} \left[ f_{1r} + f_{1(r-1)} \right] = \left( \frac{q_1 - 1}{q_1 + 1} \right)^{1/2}, \]  \( (3.4.4) \)

which is called the slope-ordinate ratio at \( x = r - \frac{1}{2} \).

Equating the ratio \( \frac{f_{1r} - f_{1(r-1)}}{\frac{1}{2}[f_{1r} + f_{1(r-1)}]} \) to the logarithmic derivate of \( f_1(x) \) evaluated at \( x = r - \frac{1}{2} \), we get a differential equation

\[ \frac{d}{dz} \log f_1(z) = \frac{(q_1 - 1)}{(q_1 + 1)}. \]  \( (3.4.5) \)

The solution of the differential equation is of the form \( f_1(z) = \lambda_1 e^{-\lambda_1 z} \) for some \( \lambda_1 > 0 \) which is the p.d.f of the first component of (3.4.1). The case for \( i = 2 \) is similar and thus (3.1.2) is the continuous approximation of (3.4.1).

### 3.5 A New Explanation of Decreasing Failure Rate of a Mixture of Geometric

The more paradoxical case corresponds to the Geometric distribution that shows a constant failure rate while Geometric mixtures belong to the decreasing failure
rate (DFR) class. Barlow (1985) gave a Bayes explanation for decreasing failure rate of a mixture of exponentials. Rodrigues and Wechsler (1993) used a similar Bayes explanation for the mixture of geometric distributions. Further a non-Bayes explanation is given by Jie Mi (1998) for the mixture of exponentials to have a decreasing failure rate. In this section, we are given a non-Bayes explanation for decreasing failure rate of a mixture of geometrics.

We can show that the proportion of strong subpopulation with small (large) failure rates in the mixture increases (decreases) as time passes. Based on this fact, a non-Bayes explanation is given for the mixture of geometric to have a decreasing failure rate.

To prove this we have the following notations

\( p_i \) failure rate

\( F(x) \) cdf of \( x \)

\( f(x) \) pdf of \( x \)

\( \bar{F}(x) \) survival function of \( x \)

\( h(x) \) failure rate for \( F(x) \)

\( g(p) \) weight function for the mixing measure

\( \{p_k\} \) Support set for \( g(p) \) when \( g(p) \) is a discrete function; \( k \geq 1 \)

\( n_i(x) \) expected number of survivors of type \( i \) device at time \( x \), \( x \geq 0 \)

\( n(x) \) \( \sum_i n_i(x) \) total number of survivors

\( M_x(\cdot) \) dynamic mixing probability measure.

Let there be \( n_i \) devices of type \( i \), \( i = 1, 2 \). These devices of different types perform the same function and are not distinguishable in operation. Let the lifetime of type \( i \) products follow geometric distribution with failure rate \( p_i \); and let \( p_1 < p_2 \). Mix these devices; there are \( n = n_1 + n_2 \) devices in total.
Define \( g(\cdot) \) on the set \( \{p_1, p_2\} \) by

\[
g(p_1) = p = \frac{n_1}{n},
\]

\[
g(p_2) = 1 - p = \frac{n_2}{n}; \quad 0 < p < 1.
\] (3.5.1)

If a device is randomly selected from the lot, then its lifetime distribution is the mixture of two geometric distributions with the mixing function \( g(p) \) that has probability masses at two points \( p_1 \) and \( p_2 \). The survival function and failure rate of the mixture are:

\[
F(x) = pq_1^x + (1 - p)q_2^x
\]

\[
h(x) = \frac{pq_1^x p_1 + (1 - p)q_2^x p_2}{pq_1^x + (1 - p)q_2^x}
\]

\[
p(x) = \frac{pq_1^x}{pq_1^x + (1 - p)q_2^x}
\]

\[
q(x) = \frac{(1 - p)q_2^x}{pq_1^x + (1 - p)q_2^x}
\]

\[
n(x) = n_1(x) + n_2(x).
\] (3.5.2)

\[
n_i(x) = n_i q_i^x, \text{ because the number of survivors is binomially distributed.}
\]

\[
\frac{n_1(x)}{n(x)} = \frac{n_1 q_1^x}{pq_1^x + n_2 q_2^x}
\]

\[
= \frac{pq_1^x}{pq_1^x + (1 - p)q_2^x}
\]

\[
= p(x).
\] (3.5.3)

Similarly,

\[
\frac{n_2(x)}{n(x)} = q(x).
\] (3.5.4)

The conditional survival function is

\[
F_x(t) = \frac{F(x + t)}{F(x)}
\]

\[
= q_1^t \frac{pq_1^x}{pq_1^x + (1 - p)q_2^x} + q_2^t \frac{(1 - p)q_2^x}{pq_1^x + (1 - p)q_2^x}
\]

\[
= q_1^t M_x(p_1) + q_2^t M_x(p_2).
\] (3.5.5)
Thus (3.5.4) – (3.5.6) define:

\[ M_x(p_1) = p(x) \]  
(3.5.7)

\[ M_x(p_2) = q(x) \]  
(3.5.8)

\[ M_0(p_i) = g(p_i), \quad i = 1, 2. \]  
(3.5.9)

which are the proportions of type \( i \) products in working state at time \( x \).

Equation (3.5.2) shows that \( h(x) \) of the mixture is the mixture of \( p_1 \) and \( p_2 \) according to \( M_x(p) \).

The \( M_x(p) \) evolves with \( x \) as shown:

\[ p'(x) = \frac{p(1-p)q_1^x q_2^x \log(q_1 q_2)}{(pq_1^x + (1-p)q_2^x)^2} > 0 \]

Since \( p_1 < p_2 \) and \( q_1 = 1 - p_1 \). Name the subpopulation consisting of:

- type 1 products as the strong one,
- type 2 products as the weak one,

because \( p_1 < p_2 \).

With this interpretation, (3.5.8) implies that:

- the proportion of the strong subpopulation strictly increases while that of the weak subpopulation strictly decreases;

\[ \lim_{x \to \infty} [p(x)] = 1, \]

i.e., the strong subpopulation will eventually dominate the performance of the mixture.

Based on these facts, the \((p(x))(q(x))\) of \( p_1(p_2) \) in (3.5.2) strictly increases (decreases), i.e., the strong (weak) subpopulation with a small (large) failure rate \( p_1(p_2) \) will contribute more (less) to \( h(x) \) of the mixture.

Of course, algebraically it is evident that if \( x_1 < x_2 \), then

\[ h(x_1) - h(x_2) = [p_1p(x_1) + p_2q(x_1)] - [p_1p(x_2) + p_2q(x_2)] \]
\[ = p_1[p(x_1) - p(x_2)] + p_2 [(q(x_1) - q(x_2))] \]
\[ = (p_1 - p_2) [p(x_1) - p(x_2)] < 0; \]
Thus it is clear that \( h(x) \) is strictly decreasing function of \( x \).

### 3.6 Behavior of the Failure Rate of Geometric Mixture using Negative Weights

Mixing weights are usually constrained to be non-negative. In general, this is not necessary, since (3.1.1) can be a probability mass function even when some of the \( p_i \)'s are less than zero, but the constraint \( \sum_{i=1}^{k} p_i = 1 \) is necessary (see Titterington et al., 1985 and Jorge Novarro and Pedrao Hernandez, 2006). However, if the non-negative constraints are to be relaxed, it is important to be able to guarantee that the mass function given in equation (3.1.1) is greater than or equal to zero everywhere. We can consider (3.1.1) with \( \sum_{i=1}^{k} p_i = 1 \), by allowing some of the \( p_i \)'s are less than zero.

Suppose \( X_1 \) and \( X_2 \) represents the lifetime of two components of a series system. The probability mass functions of \( X_1 \) and \( X_2 \) are given by

\[
f_1(x_1) = pq_1^{x_1} p_1 + (1 - p)q_2^{x_1} p_2 \tag{3.6.1}
\]

and

\[
f_2(x_2) = pq_3^{x_2} p_3 + (1 - p)q_4^{x_2} p_4. \tag{3.6.2}
\]

Thus the system lifetime is given by the p.m.f

\[
f_3(y) = p^2 (p_1 p_3)(q_1 q_3)^y + (1 - p)^2 (p_2 p_4) (q_2 q_4)^y + p(1 - p)(p_1 p_4)(q_1 q_4)^y + p(1 - p)(p_2 p_3)(q_2 q_3)^y, \tag{3.6.3}
\]

which is again a finite mixture of geometrics. In fact it is a mixture of four geometrics with mixing weights \( p^2 \), \( (1 - p)^2 \) and \( p(1 - p) \). Note that mixing weights in (3.6.1) and (3.6.2) are the same. In such situations, this property can be extended for any number of finite geometric mixtures.

For the model (3.1.2), the hazard (failure) rate is given by

\[
h(x) = \frac{pq_1^{x} p_1 + (1 - p)q_2^{x} p_2}{pq_1^{x} + (1 - p)q_2^{x}}. \tag{3.6.4}
\]
For $0 < p < 1$, $h(x)$ is decreasing in $x$. Hence the distribution (3.1.2) has decreasing failure rate (DFR) property. When one of the mixing weights is negative, (3.1.2) has the property of increasing failure rate. For example, consider the model with p.m.f

$$f(x) = 2q_1^x p_1 - q_2^x p_2, \quad x = 0, 1, 2, \ldots, \quad 2p_1 > p_2. \quad (3.6.5)$$

Then the failure rate for (3.6.5) is obtained as

$$h(x) = \frac{2q_1^x p_1 - q_2^x p_2}{2q_1^x - q_2^x p_2}, \quad (3.6.6)$$

and the derivative of $h(x)$

$$h'(x) = \frac{2(1 - p_1)p_2(q_1q_2)^x \log(q_1)}{(2q_1^x - q_2^x p_2)^2}. \quad (3.6.7)$$

Obviously, (3.6.7) is always non-negative and thus (3.6.5) has increasing failure rate (IFR) property. So the behavior of $h(x)$ depends on mixing weight.

### 3.7 Concluding Remarks

In the present chapter, we addressed various properties of the finite mixture of geometric distribution in the context of reliability theory. Various ageing phenomena of the model are discussed. A non-Bayes explanation is given for the mixture of geometrics to have a decreasing failure rate. It is well known that each component in the mixture has a constant failure rate; the failure rate of the mixture is a decreasing function. When one of the mixing weights is negative, the mixture has the property of increasing failure rate. Thus, mixture models can be employed for modelling the data of IFR nature, even when the individual component possess the DFR property.