There has recently been considerable interest in the nonequilibrium superconductivity. The nonequilibrium properties of a superconductor are interesting in themselves from a physical point of view, since many interesting phenomena occur which do not show up in the equilibrium state. The proper understanding of these nonequilibrium phenomena is of interest.

The nonequilibrium state can be created by external perturbations such as microwave and laser irradiation, phonon injection, quasiparticle injection, time dependent external currents etc. The resulting nonequilibrium state of the superconductor depends not only on how far the system is driven out of equilibrium but also upon the specific nature of the external driving source. A strongly driven superconductor, has all the three components, quasiparticles, phonons and pairfield, significantly altered from their respective equilibrium values.

In this chapter we introduce nonequilibrium...
superconductivity and discuss its various aspects. We introduce elaborate the concepts which are necessary for proper understanding of the nonequilibrium phenomena in superconductors.

2. SUPERCONDUCTING GROUND STATE AND EXCITATIONS

The equilibrium superconducting state is characterised by the transition temperature \( T_0 \), the temperature \( T \) and other quantities relevant to the metal in the normal state. However, as the superconductor is driven out of equilibrium the situation becomes complex and several other parameters are required. The superconductor must now be characterised by the distribution function \( f(E_k) \) for the quasiparticle excitations and occupation numbers of the phonons \( n_k \); \( k \) being the wave vector index. Furthermore, we must now know the complex order parameter \( \psi \) of the ground state i.e., both the phase \( \Theta \) and the superfluid density \( n_s \). While the phonon spectrum is not changed to any detectable extent when a metal becomes superconducting, the electrical excitations develop the excitations spectrum.

The ground state wave function \( \psi_g \) for a homogeneous superconductor is a linear combination of the
low lying normal configurations which gives a lower energy than the normal phase when there is a net attractive interaction between electrons. In metals the attraction arises from exchange of photons. The normal configurations which enter the sum are those in which states are occupied in pairs of opposite spin and momentum \((k\uparrow, -k\downarrow)\) such that if in a given configuration one of the two states is occupied, the other is also occupied. In terms of electron creation and destruction operators \(c_{k\uparrow}, c_{k\downarrow}\) for normal phase quasiparticle, we can write:

\[
\psi_{s} = \prod_{k} \left( u_{k} + v_{k} c_{k}^{\dagger} c_{k} \right) \psi_{0} = \sum_{N} a_{N} e^{ixN} \psi_{N}
\]

... (1)

where \(\psi_{0}\) is the vacuum state, and \(\psi_{N}\) is the many-particle wave function obtained by summing all terms in the product which have exactly \(N\) pairs. The phase \(\chi\) is a conjugate variable to the particle number \(N\), and these satisfy the uncertainty relation \(\Delta \chi \Delta N \approx 1\).

The coefficients \(u_{k}\) and \(v_{k}\) are normalised so that

\[|u_{k}|^2 + |v_{k}|^2 = 1,\]

and are called coherence factors. Here \(|u_{k}|^2\) represents the probability that the pair \(k\) is occupied and is given by:

...
where $\epsilon_k$ represents the energy of a quasiparticle in the normal state with wave vector $k$ and $E_k$ is the energy of the corresponding state in the superconducting phase.

$$\varepsilon_k = \sqrt{\epsilon_k^2 + \Delta^2} \quad \cdots (3)$$

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} - E_F - \mu_s \quad \cdots (4)$$

$$\mu_s = -\varepsilon_k - \mu_s$$

$\mu_s$ is the chemical potential of the electrons in the superconducting ground state and is related to the time derivative of the phase by Josephson relation:

$$\mu_s = \frac{\hbar}{2} \frac{d\Theta}{dt} \quad \cdots (5)$$

The energies are usually measured with respect to Fermi energy of the electrons in the normal state

$$E_F = \frac{\hbar^2 k_F^2}{2m}$$
\( \xi_k \) is the energy of electrons in the Bloch state with momentum \( k \). In thermal equilibrium \( \mu = 0 \).

The density of states \( N_e(E_k) \) of the excitations in the superconductor is proportional to the single electron density of states in the normal state taken at the Fermi surface, \( N(0) \), and is given by:

\[
N_e(E_k) = (u_k^2 - v_k^2)^{-1} N(0) \quad \ldots \ (6)
\]

The energy gap \( \Delta \) in the superconductor is given by the self-consistent BCS gap equation:

\[
\frac{\Delta}{V} = \Delta \sum_k \frac{1 - 2f_0(E_k)}{2E_k} \left[ \frac{\Delta}{\hbar \Omega_D} \int_{\Delta} \left( 1 - 2f_0(E) \right) dE \right] \quad \ldots \ (7)
\]

In this equation \( V \) is a model potential for the attractive electron-electron interaction assumed to be constant between electrons having energy \( \hbar \Omega_D \) (Debye energy) around Fermi sphere. The sum over \( k \) states has been transformed to an integral over the quasiparticle energy. The distribution function \( f_0(E) \) is the Fermi-Dirac distribution function

\[
f_0(E) = \left( 1 + \exp \frac{E}{k_B T} \right)^{-1} \quad \ldots \ (8)
\]
The solution of Eq. (7) $\Delta(T)$ is the energy gap at temperature $T$, and has maximum value at absolute zero. As the temperature is increased $f_{\text{eq}}(E)$ increases (i.e. more and more quasiparticle states become occupied) and thereby $\Delta$ decreases up to the critical temperature $T = T_c$ above which the only real solution of Eq. (7) is $\Delta = 0$. $\Delta_0$ and $T_c$ are related by:

$$\Delta(0) = \Delta_0 = 1.764 \, k_B T_c = 2 \frac{\hbar}{\sqrt{m^*}} e^{-1/N(0)V} \quad (9)$$

When distribution function $f$ departs from that corresponding to thermal equilibrium, the time required for the quasiparticles to come to equilibrium is generally much longer than that required for the pairs to come into equilibrium with $f(E_k)$ so that Eq. (7) remains valid even in nonequilibrium situations. This has been called the 'quasiparticle bottleneck'.

The minimum energy of a quasiparticle is $\Delta$. This is typical of the superconducting state contrary to the normal state of a metal. In practice a minimum energy of $2\Delta$ is required to break up a Cooper pair.

A quasiparticle is an elementary excitation of the superconducting ground state. The creation operator
of a quasiparticle is a linear combination of electron creation and destruction operators:

\[
\begin{align*}
\gamma_{k\uparrow}^+ &= u_k c_{k\uparrow}^+ - v_k c_{-k\downarrow}^+ \\
\gamma_{k\downarrow}^+ &= v_k c_{-k\downarrow}^+ + v_k c_{k\uparrow}^+
\end{align*}
\]

This transformation causes the charge of a quasiparticle \( q_k \) to be different from the electronic charge

\[
q_k = -\frac{e(\xi_k - \mu_s)}{E_k} = -e(u_k^2 - v_k^2) \quad (11)
\]

If \( \xi_k - \mu_s \) is positive then the charge of the quasiparticle is negative and is called an electron-like.

If \( \xi_k - \mu_s \) is negative charge of quasiparticle is positive and is called a hole-like. For \( |\xi_k - \mu_s| \gg \Delta \) the absolute value of the charge of quasiparticle approaches the electronic charge.

The total charge of the electronic system in a superconductor is equal to the total charge of the quasiparticles and the charge of the condensate. The charge \( q^* \) of the quasiparticles is given by:

\[
\ldots
\]
Fig. 1: Quasiparticle energy $E_K$, Coherence factors $U_K^2$ and $V_K^2$ and quasiparticle and $q_K$ as a function of the electron energy $\xi_K$ measured with respect to the pair chemical potential $\mu_s$. 

**Fig. 1:** Quasiparticle energy $E_K$, Coherence factors $U_K^2$ and $V_K^2$ and quasiparticle and $q_K$ as a function of the electron energy $\xi_K$ measured with respect to the pair chemical potential $\mu_s$. 

<table>
<thead>
<tr>
<th>$E_K/\Delta$</th>
<th>$q_K/e$</th>
<th>$\xi_K - \mu_s/\Delta$</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>-2</td>
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<td>2</td>
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where the sum is over $k$ states above and below $k_F$.

The factor 2 accounts for the summation over the spin states. In thermal equilibrium this charge is zero because the occupation probabilities of electronlike and holelike quasiparticles of equal energy and opposite charge are equal. The negative charge compensating the positive charge of the metal ions is contained in the condensate charge $Q_S$ of the superconducting state and is given by:

$$Q_S = -2 \cdot \sum_k v_k^2 \quad \ldots \quad (13)$$

In Fig.-(1) we have plotted the excitation spectrum, the coherence factors and the effective charge of the quasiparticles as a function of $\xi_k - \mu$. The situation for a normal metal for which $\Delta = 0$ is indicated by the dashed lines.

2.1 Nonequilibrium state

The Meissner state of type 1 superconductor in an external magnetic field has the lowest free energy and
is, therefore, a globally stable equilibrium. On the other hand the persistent current in a superconducting ring which sustains a trapped magnetic flux has higher free energy than the state with no current and no trapped flux and is therefore a metastable equilibrium state. For a bulk type I superconductor well below $T_c$, the life time of metastable state is $\sim 10^{10}$ years, exceeding even a true astronomical time scale. For a type II superconductor flux creep can occur at an observable rate. For a very thin superconductor $\sim 1/\mu m$ in diameter and 1mK below $T_c$, the characteristic decay time is of the order of a second. The simple equilibrium analysis must be supplemented by attention to thermally activated fluctuation effects to account for these dissipative phenomena, and thus one enters into the realm of nonequilibrium superconductivity.

The nonthermal equilibrium steady state is a dynamic equilibrium state in which a perturbing source is balanced by relaxation and diffusion. In this category are included the conversion of a normal current into supercurrent at an N-S interface with associated resistive voltage developed in the superconductor, and the stimulation (or weakening) of superconductivity by perturbations that effectively 'cool' (or 'heat') the electrons.
The most complex nonequilibrium state is the one in which both the magnitude and the phase of the order parameters $\Psi(\gamma, t) \sim |\Delta(\gamma, t)| e^{i\theta(\gamma, t)}$ vary in time as well as space. The current received much carrying filaments in this connection have received much attention. There is no general theoretical procedure which could provide rigorous treatment to this space and time dependent nonequilibrium regime and work in this area is particularly valuable. A superconducting nonequilibrium state can be achieved by external perturbations in the form of injection/ extraction of quasiparticles from a superconductor by passing a tunnelling current through a thin insulating barrier separating the superconductor from another metal which may be either normal or superconducting. The quasiparticles may further be influenced by electromagnetic radiation and if photon energy exceeds the gap value $2\Delta$, the ground state is also influenced by such radiations. The phonon injection can also be used to achieve states away from thermal equilibrium.

In general the nonequilibrium state is characterised by the resulting quasiparticle and phonon distributions which are dependent on the driving force, the thermal environment, and the magnitude of the energy gap $\Delta$. ...
Fig. 2:

(a) Dispersion curves excitation energies in normal and superconducting states, with schematic indication of occupation numbers in thermal equilibrium.

(b) Schematic indication of population with even mode excited.
In all these cases the resulting nonequilibrium state is characterised by actual distribution function of the quasiparticle excitations, which we write as:

\[ f(E_k) = f_0(E_k) + \delta f(E_k) \quad \ldots (14) \]

The equilibrium distribution function \( f_0(E_k) \) is an even function of the energy \( E_k \). The nonequilibrium part \( \delta f(E_k) \) can generally be written as a sum of two components which are even or odd in the energy \( E_k \). The even mode is called the longitudinal mode or energy mode, while the odd mode is known as transverse mode or branch imbalance or charge imbalance mode.

### 2.2 Longitudinal mode or Energy mode

In this nonequilibrium mode, the quasiparticle distribution function is even in the energy \( E_k \) of the electron states. The quasiparticles are distributed equally on both hole-like and electron-like branches so that the total charge \( Q^e \) of quasiparticles is zero as in equilibrium. The difference is only in energy distribution of the quasiparticles. Hence it is also called as the energy mode (see Fig.-2).

The longitudinal nonequilibrium state is excited by electrically neutral perturbations like photons or
photons. The most of the charged perturbations excite both longitudinal as well as transverse modes. The quasiparticles can be excited initially to energies much greater than $\Delta$ especially when optical phonons are used for excitation. However, these quasiparticles decay rapidly into lower energy states and emit phonons. These low energy phonons quasiparticles then recombine to form pairs. The recombination time is much larger than the time taken by the quasiparticles to cool down to lattice temperature. Thus quasiparticle population comes to thermal equilibrium with a lattice, but not with the condensate. The energy relaxation requires inelastic collisions, which can occur with phonons but not with ordinary impurities.

2.3 Transverse Mode or Charge Imbalance Mode

In this nonequilibrium mode the deviation of the quasiparticle distribution function from equilibrium is an odd function of $\varepsilon_k$, the energy of the electron states with wave vector $k$ measured with respect to chemical potential of the condensate. This type of nonequilibrium is generally created by charged perturbations disturbing the superconductor, e.g., particle injection from a normal metal into a superconductor through tunnel barrier or a conversion of supercurrent
Fig. 3 Each dot represents a quasiparticle in excess of the equilibrium distribution: (a) \(Q^* \neq 0, Q \neq 0\), (b) \(Q^* = Q = 0\), (c) \(Q^* \neq 0, Q = 0\), and (d) \(Q^* = 0, Q \neq 0\). In each case, the minimum value of \(E_k\) occurs at the pair chemical potential.
into normal currents. The resulting nonequilibrium distribution function has very often a longitudinal component also. The transverse mode is characterised by net quasiparticle charge imbalance mode in the electron-like and hole-like branches. We define a branch imbalance by:

\[
Q = \eta^+ - \eta^- = 2 N(0) \int_\Delta N_s(E) \left\{ f_k^+ - f_k^- \right\} dE
\]

... (15)

where \( N_s(E) = \frac{E}{(E^2 - \Delta^2)^{1/2}} \) is the normalised DOS density of states and \( f_k^+ \) and \( f_k^- \) are the quasiparticle distribution functions for the \( k > k_F \) and \( k < k_F \) branches of the quasiparticle spectrum. This quantity \( Q \) is the difference in populations of two branches regardless of the effective charge \( q_k \). Fig.- 3(a) represents the addition of one excitation to the equilibrium distribution. The quasiparticle charge \( \delta \) is non-zero but so is the branch imbalance \( Q \). In Fig.- 3(b) two excitations have been created on each branch in such a way that the net excitation population has been increased but \( \delta = Q = 0 \). This is the longitudinal mode of disequilibrium. In Fig.- 3(c) three quasiparticles have been added to each branch; thus \( \delta = 0 \), and \( Q = 0 \), because the
excitations have different $q_k$. However, this configuration of $Q$ can relax if the quasiparticles on the $k > k_F$ branch are scattered inelastically to lower energies. Fig.- 3(d) shows a branch imbalance $Q \neq 0$ but $Q^* = 0$. The charge of single $k > k_F$ quasiparticle (nearly unity) is sufficient to cancel exactly the charges of the three low lying $k < k_F$ quasiparticles that are each less than unity.

In the transverse mode, the quasiparticle charge $Q$ changes because in the integrand for $Q^*$ in Eq. (12), the distribution is multiplied by the charge $q(\xi)$ which is odd in $\xi_k$. The magnitude of the energy gap does not change because in gap Eq. (7) the distribution function is multiplied by a function even in $\xi_k$. Only phase of the order parameter changes in the transverse mode.

However, the total electronic charge of the superconductor remains unchanged. Any change in the quasiparticle charge is compensated for by an opposite change of the charge of the condensate. This implies that the chemical potential of the superfluid $\mu_s$ gets shifted from its equilibrium value ($\mu_s = 0$). With the change in $\mu_s$, the energy of quasiparticles $E_k$ with wave vector $k$ and coherence factors $u_k$ and $v_k$ also
Fig. 4. $E_0/\Delta$ and $v_1^2$ as a function of $\xi/\Delta$ in a situation of charge imbalance. The dashed lines give the equilibrium situation.
change, because $\xi_k$ is measured with respect to Fermi level of the ground state. The situation is shown in Fig. 4. The change in the chemical potential $\mu_s$ is directly proportional to the quasiparticle charge $Q^*$. The corresponding change in the condensate charge (reduction of number of pairs or Fermi level for pairs) which has been shown in Fig. 4 as shaded portion, can be easily got from Eq. (13) and is given by

$$\delta Q_s = -2 e \left\{ \sum_{k} \frac{\nu^2_k}{\mu_s \neq 0} - \sum_{k} \nu^2_k \right\}$$

$$= -2 e N(0) \delta \mu_s \quad \cdots (16)$$

Thus quasiparticle charge imbalance, which is opposite to the change in condensate charge can be written as

$$Q^* = -\delta Q_s - 2 e N(0) \delta \mu_s \quad \cdots (17)$$

The charge imbalance is characterised by the shifting of quasiparticles in one direction along the $\xi$ axis and of the chemical potential $\mu_s$ in the opposite direction. If the quasiparticles are in internal equilibrium, and are in disequilibrium with the condensate, the quasiparticle distribution function is the Fermi-Dirac function with the energy measured with respect to a

...
chemical potential $\mu_a$ of the quasiparticle which is different from $\mu_s$. Near $T_c$, the chemical potential is related to $\mu_s$ by the relation:

$$\mu_a = -\mu_s \frac{\pi \Delta}{k_B T_c} \quad \cdots (18)$$

The charge imbalance is, therefore, due to the difference in electrochemical potentials of the condensate and the quasiparticles. This potential difference has been measured and thus demonstrated the existence of the charge imbalance.

3. Relaxation Processes

A disequilibrium created by external perturbation will relax with characteristic relaxation time. The quasiparticle imbalance $\delta f$ which we have discussed in previous section will also relax. Here two kinds of the relaxation processes are to be distinguished. The elastic scattering of quasiparticle (e.g., impurity scattering) with a characteristic time $\tau_1$ tend to drive the quasiparticles towards an isotropic state in the momentum space. In these processes the momentum and not the energy is changed. On the other hand the inelastic scattering processes which are caused by electron-phonon interactions tend to establish the equilibrium distribution—
Fig. 5  Basic electron phonon processes: (a) scattering process and (b) recombination process or phonon pair breaking process.
function \( f_0(E) \). Further, the inelastic processes are of two types. The one in which absorption or emission of a phonon by a quasiparticle takes place so that quasiparticle energy is increased or decreased by \( \hbar \Omega \), where \( \Omega \) is the phonon frequency. The other is the recombination of two quasiparticles forming a Cooper pair, with an emission of phonon. Here the number of quasiparticles is reduced. Conversely the number of quasiparticles is increased by breaking up of a Cooper pair by absorption of a phonon of energy \( \hbar \Omega > 2 \Delta \) (see Fig. 5). These inelastic processes have their characteristics time constant and in general depend on the energy \( E \) of the quasiparticles involved.

3.1 Relaxation of Charge Imbalance

(a) Phonon Scattering

When an electron-like excitation is injected into the superconductor it creates a charge imbalance and also carries the energy of the quasiparticles well above the thermal energy. The relaxation process, therefore, involves cooling as well as charge imbalance relaxation. However, these are not necessarily independent process. A quasiparticle initially at a higher energy above \( \Delta(T) \) and \( k_B T \), may emit a phonon and scatter on the same branch to an energy not much greater than \( \Delta(T) \).
Four inelastic processes that contribute to the relaxation of the charge of a quasiparticle with energy $E$. (a) scattering to a state of lower charge on the $k>k_F$ branch; (b) scattering to a state of lower charge of opposite sign on the $k<k_F$ branch; (c) recombination with a quasiparticle near the gap edge on the $k>k_F$ branch; (d) recombination with a quasiparticle near the gap edge on the $k<k_F$ branch. Processes (a) and (b) are governed by the coherence factor $(uu'-vv')$, while (c) and (d) are governed by the coherence factor $(vu'+uv')$. 
The quasiparticle thus has cooled and has also given up some of its charge because \( q = q_e \) has been reduced from an initial value close to unity to a lower value.

In this subsection we will be concerned with charge imbalance in a homogeneous superconductor with an isotropic energy gap and with no magnetic impurities, so that relaxation takes place by inelastic scattering only.

Let there be excess quasiparticles on the \( k > k_F \) branch distributed in energy from \( \Delta \) to \( k_B T \). Consider first the scattering of a quasiparticle of energy \( \sim k_B f \). This may scatter with the emission or absorption of a phonon with energy \( \hbar \Omega \) to a new state \( E' \) on either \( k > k_F \) or the \( k < k_F \) branch. The maximum phonon energy available for absorption is \( \sim k_B T \). The coherence factor that governs these processes is given by

\[
(\nu \nu' - \nu \nu')^2 = \frac{1}{2} \left( 1 + \frac{\xi \xi' - \Delta^2}{E E'} \right)
\]

...(19)

We consider various situations illustrated in Fig. 6.

(1) Both the quasiparticle states are on the same branch. Since \( \xi, \xi' \) are both positive and \( E \sim k_B f \gg \Delta \) the coherence factor is substantial for all values of \( E' \) (Fig. 6(a)). Upward transitions with phonon absorption do not change \( q \) significantly, while downward transitions with phonon emission, change \( q \) significantly only when \( \Delta \leq E' \leq 2 \Delta \). Thus of all possible transitions of
this nature, only a fraction \( \sim \Delta/2 \) can relax \( \zeta^* \) significantly.

(ii) The scattering onto the \( k < k_y \) branch (see Fig. 6b). Given that \( E \approx \xi \gg \Delta \), we see that if \( E' \) so that \( \xi' \approx -E' \), the coherence factor is very small. Thus scattering to energies \( E' \) higher than \( E \) is rare. Moreover scattering to energies lower than \( E \) is likely only if \( E' \) is reasonably close to \( \Delta \) (say between \( \Delta \) and \( 2 \Delta \)). At \( E' = 2 \Delta \) the coherence factor is

\[
\approx 1/2 (1 - \frac{\xi}{\Delta}) \approx 0.07.
\]

Thus again, only a fraction \( \sim \Delta/2 \) of the transitions relax \( \zeta^* \).

(iii) We now consider the relaxation of charge imbalance by recombination process for which the coherence factor is

\[
(v u' + u v')^2 = \frac{1}{2} (1 + \frac{\Delta^2 - \xi \xi'}{E E'}) \quad (20)
\]

where \( E \) is the energy of the quasiparticle under consideration and \( E' \) is the energy of the quasiparticle with which it combines.

First consider the recombination when \( E \) and \( E' \) refer to same branch (Fig. 6c). If \( E, E' \) are both large compared with \( \Delta \), the coherence factor becomes very small. However, if one of the energies is not greater than \( 2 \Delta \), the coherence factor is relatively large and recombination can make a substantial contribution to charge relaxation. A fraction \( \sim \Delta/k_B T \) of quasiparticles can relax \( \zeta^* \) in this way...
(iv) Now consider recombination with $E$ and $E'$ on different branches (Fig. 6d). Since the product is now positive, the coherence factor is always at least $1/2$. However, these recombination processes have only small effect on $q^*$ for energies where $|q|$ is reasonably close to unity. When one quasiparticle lies between $\Delta$ and $-2\Delta$ and other is at higher energies, $q^*$ does relax sufficiently. Such processes are a fraction $\Delta / k_B T$ of the total.

Thus we see that of all the inelastic processes involving recombination or creation of quasiparticles only a fraction $\Delta / k_B T$ participate in quasiparticle charge imbalance relaxation. This leads to the conclusion that the charge imbalance relaxation time $\tau_\xi^*$ will be a factor of order $k_B T / \Delta$ larger than the effective inelastic scattering time $\tau_k$. There are several calculations of charge relaxation time, all leading to the same result:

$$\tau_\xi^* = \frac{4 k_B T}{\pi \Delta (1)} \tau_k \quad \cdots (21)$$

As $T$ approaches $T_c$, the value of $\tau_\xi^*$ diverges as $(1 - T/T_c)^{-1/2}$ and it becomes increasingly difficult to transfer charge from the quasiparticles to the condensate. The charge imbalance mode represents an extra degree of freedom for a superconductor and is not a feature of normal metal.
Fig. 7. Principle of charge relaxation by elastic scattering in the presence of gap anisotropy. The two excitation spectrum refer to two different regions of the Fermi surface where the energy gaps are different.
We have considered so far an isotropic superconductor with nonmagnetic impurities. The elastic scattering from nonmagnetic impurities can relax $Q^*$ only if a quasiparticle changes its branch. This process in isotropic superconductor is forbidden, by coherence factor Eq. (19). This can be seen immediately by realizing that if $E$ and $E'$ refer to different branches then $u' = v$ and $v' = u$.

In the next section we discuss the elastic scattering processes in an anisotropic superconductor and from magnetic impurities.

(b) Anisotropic Energy Gap

When the energy gap is anisotropic, the coherence factor $(u u' - v v')^2$ does not vanish and even elastic scattering can transfer quasiparticles from one branch to other. Fig. 7 shows the excitation spectra at two regions of the Fermi surface having different energy gap values. Since the values of $|\xi|$ and $|\xi'|$ are different for initial and final states, the coherence factor is nonzero even though $E$ and $E'$ are equal. This situation was first pointed out by Tinkham and later discussed quantitatively by Chi and Clarke. The elastic scattering rate $\frac{C_{el}}{d \xi}$ is given by...
\[ \tau^{-1} \xi = \frac{1}{2} \left( \rho (E) \left[ 1 - \frac{\Delta - \epsilon}{E^2} - \left( 1 - \frac{\Delta^2}{E^2} \right)^{1/2} \left( 1 - \frac{\Delta^2 - \epsilon^2}{E^2} \right)^{1/2} \right] \right) \]  

\[ \times \left[ \xi \left( 1 - \xi' \right) - \xi' \left( 1 - \xi \right) \right] \quad \ldots (22) \]

Here \( \tau^{-1} \) is the elastic scattering rate of an electron when the metal is in normal state, \( \rho (E) \) is the final density of states, one half times the first square bracket is the coherence factor and the two terms in the second square bracket are the occupation factors for elastic scattering from \( \xi \) to \( \xi' \) and the reverse process, respectively. The symbol \( \langle \rangle \) indicates an angular average over the gap anisotropy distribution.

To express the quantities in \( \langle \rangle \) in terms of the mean square anisotropy, the coherence factor is to be averaged. This is done by setting \( \Delta = \bar{\Delta} + \delta \) and \( \Delta' = \bar{\Delta} + \delta' \), where \( \bar{\Delta} \) is the average energy gap

\[ \langle \delta^2 \rangle = \langle (\Delta - \bar{\Delta})^2 \rangle \quad \ldots (23) \]

Using \( \rho (E) = \rho (E) \left( \frac{E^2 - \bar{\Delta}^2}{E^2 - \bar{\Delta}^2} \right)^{1/2} \) and the fact that \( \langle \delta \rangle_{\Delta} = \langle \delta' \rangle_{\Delta'} = 0 \), we find

\[ \langle (uu' - vv')^2 \rangle_{\bar{\Delta}, \bar{\Delta}} = \frac{\langle \delta^2 \rangle_{\bar{\Delta}, \bar{\Delta}}}{2 \left( E^2 - \bar{\Delta}^2 \right)} \quad \ldots (24) \]

This leads to

\[ \langle \rho (E) (uu' - vv')^2 \rangle_{\bar{\Delta}, \bar{\Delta}} = \frac{\langle \delta^2 \rangle_{\bar{\Delta}, \bar{\Delta}}}{2 \left( E^2 - \bar{\Delta}^2 \right)^{3/2}} \quad \ldots (25) \]

for \( E > \bar{\Delta} + \langle \delta^2 \rangle_{\bar{\Delta}, \bar{\Delta}}^{1/2} / 2 \).
we define the normalized mean-square gap anisotropy $^{12}$

$$\langle a^2 \rangle = \frac{\langle \delta^2 \rangle}{\Delta^2} \quad \ldots (26)$$

For clean bulk superconductor ($\Rightarrow h v_F / \Delta), \langle a^2 \rangle$ is a constant independent of temperature. However, if the mean free path is shortened by impurities or surface scattering $\langle a^2 \rangle$ is reduced from its clean limit value $^{13}$ $\langle a^2 \rangle_c$ to a value given by:

$$\langle a^2 \rangle = \frac{\langle a^2 \rangle_c}{1 + \left( \frac{\hbar}{2 \tau_1 \Delta} \right)^2} \quad \ldots (27)$$

Combining Eqs. (22), (25), (26) and (27), we have final expression for the elastic scattering rate

$$\tau^{-1}_e \tau^{-1}_S \left( \tau^{-1}_S \right) \frac{\Delta^2}{1 + \left( \frac{\hbar}{2 \tau_1 \Delta} \right)^2} \quad \left( \frac{\Delta^2}{2 \tau_1 \Delta} \right)^{3/2} \left( \frac{\Delta^2}{2 \tau_1 \Delta} \right)^{3/2} \quad \ldots (28)$$

In this equation, we have set $\xi' = \xi$ which is a good approximation for the values of anisotropy encountered in real metals.

$$\tau^{-1}_e \tau^{-1}_S \left( \tau^{-1}_S \right) \text{ appears as a function of } \tau^{-1}_S \text{. In the clean limit case in which } \left( \frac{\hbar}{2 \tau_1 \Delta} \right)^2 \ll 1 \text{ i.e. } \ell \gg \frac{\hbar v_F}{2 \Delta},$$

if we increase $\tau^{-1}_S$ by adding impurities $\tau^{-1}_S$ increases proportionally. In this limit shortening the mean free path increases the scattering rate much more than it decreases the mean square anisotropy.
In the dirty limit \( \left( \frac{\hbar}{2 \tau \Delta} \right)^2 \gg 1 \), the \( \tau^{-1}_{el, \xi} \) is proportional to \( \tau_1^{-1} \). Thus as we increase \( \tau_1^{-1} \) the rate \( \tau^{-1}_{el, \xi} \) decreases. In this limit the mean square anisotropy is reduced more rapidly than increase in the scattering rate.

From Eq. (28) it is seen that at a given temperature the scattering rate is maximum when \( \tau^{-1}_1 = \frac{\hbar}{2 \Delta l} \) or \( l = \frac{\hbar}{2 \Delta v_F} \). Thus sufficiently close to \( T_c \), the sample must be in the limit in which \( \frac{\hbar}{2 \tau_1 \Delta} \gg 1 \), and \( \tau^{-1}_{el, \xi} \) will be proportional to \( \frac{\tau_1 \Delta^4}{(k_B T_c)^2} \) for typical quasiparticle energies \( \sim k_B T_c \). Thus we expect the elastic charge relaxation rate to fall off much more rapidly than the inelastic rate as \( T \to T_c \), so that the inelastic rate eventually dominates above a temperature that depends on the relative magnitudes of \( \tau_1^{-1} \) and \( \tau^{-1}_e \). For this reason, while discussing phase slip centres which are formed close to \( T_c \) in current carrying filaments in Chapter III, we consider the relaxation of charge imbalance by inelastic processes only.

To account quantitatively for charge relaxation by elastic scattering Chi and Clarke have used \( \tau^{-1}_{el, \xi} \) in the Boltzmann equation and solved it numerically in the limit \( \left( \frac{\hbar}{2 \Delta \tau_1} \right)^2 \gg 1 \) for Aluminium films and have shown that Eq. (28) can be written as

\[
\tau^{-1}_{el, \xi} - \tau^{-1}_1 \left( 1 - \frac{\Delta}{k_B T_c} \right)^2 \frac{E}{(E^2 - \Delta^2)^{3/2}} \left( \frac{E}{\Delta} - \frac{\Delta}{E} \right)
\]

111 (29)
Computed distribution for a 100nm-thick film at $0.9T_c$ with $\Delta=100\text{meV}$ for electrons injected from a normal film through a $1\Omega$ 3x3mm tunnel junction biased at $10\Delta/e$.

Curves a and b are for $\tau_{Q*e1}^{-1}(0)\tau_E=0$ and 0.93, respectively.
where

\[ \frac{-1}{Q^* e} (0) = 2 \frac{1}{Q^* e} (k_B T_e)^2 \frac{\langle a^2 \rangle}{t^2} \quad \ldots \quad (30) \]

is the characteristic elastic charge relaxation rate. Fig. 8 shows the results of Chi and Clarke for

\[ \frac{-1}{Q^* e} (0) \approx 0, \quad \text{and for} \quad \frac{-1}{Q^* e} (0) \approx 0.95. \]

We see that the addition of an elastic relaxation process reduces the charge imbalance and shifts the peak in the distribution to higher energies. This shift illustrates the fact that the elastic relaxation is much faster at low energies than at high energies because of the factor \((\Delta^2 - A^2)^{-3/2}\) in Eq. (29).

(e) **Magnetic Impurities**

The magnetic impurities, unlike nonmagnetic impurities, relax the quasiparticle charge even if the energy gap is isotropic. The addition of magnetic impurities to a superconductor destroys the degeneracy between time-reversed electron states through the exchange interaction between the conduction electrons and the impurities. This gives the Cooper pairs a finite lifetime the inverse of which is called the pair breaking rate \(^{14}\). The pair breaking rate is the elastic spin-flip scattering rate \(\xi^{-1}\) for electrons in the normal metal. Impurities also smear out the peak in the BCS density of states over an energy range \(-\hbar / \xi\) (so that the energy gap and order parameter are no longer equal), depresses the transition temperature by \(\frac{\hbar}{4 k_B \xi} \ldots \ldots \)
and alter the temperature dependence of the order parameter from the usual BCS form.

In addition, the impurities have a significant effect on the charge relaxation rate, as was first pointed out by Schmid and Schönh and later studied by Pethick and Smith, Entin-Khofman and Orbach and Lemberger and Clarke. This can be understood by noting that the coherence factor for elastic spin-flip scattering from one quasiparticle branch to the other in an isotropic superconductor is not zero, but is of the form

$$(u' u + v' v)^2 = 4 u^2 v^2 = \frac{\Delta^2}{E^2}$$ \quad (31)

This factor approaches unity as $E \to \Delta$, therefore we expect spin-flip scattering to have an appreciable effect on $\bar{\tau}_{Q^*}$ when $\frac{\zeta_s}{\tau_s^*} > \frac{\Delta}{E}$. Schmid and Schönh have derived an expression for the charge imbalance relation rate, by using microscopic technique which includes the spin-flip scattering from magnetic impurities and is given by:

$$\bar{\tau}^{-1}_{Q^*} = \frac{\pi \Delta}{4 k_B T_C \zeta_E} \left( 1 + \frac{2 \zeta_E}{\zeta_s} \right)^{1/2} \left( 1 + \frac{\hbar^2 T}{\Delta^2 \zeta_E} \right)^{1/2}$$

for $\bar{\tau}^{-1}_{Q^*} \ll \bar{\tau}^{-1}_{E}$ \quad (32)
where

\[ \Gamma = (2 \tau_E)^{-1} + \tau_S^{-1} \]

The factor \((1 + \frac{\hbar \Gamma}{\Delta^2 \tau_E})^{1/2}\) for the present case is very close to unity and takes account of the smearing in the density of states.

For the case of \(\tau_S^{-1} \ll \tau_E^{-1}\) in which the spin-flip scattering is a weak perturbation on the inelastic scattering and to a first approximation does not effect the quasiparticle distribution created by the inelastic processes, Eq. (32) can be expanded to give

\[ \frac{1}{\tau} = \frac{\pi \Delta}{4 k_BT} \left( \frac{1}{\tau_E} + \frac{1}{\tau_S} \right) \text{ for } (\tau_S^{-1} \ll \tau_E^{-1}, \tau_Q^{-1} \ll \tau_E^{-1}) \]

... (33)

The spin-flip term \(\frac{\pi \Delta}{4 k_BT} \tau_S\) has this form because coherence factor is substantial for quasiparticles in the range from \(\Delta\) to \(\sim 2 \Delta\) so that only a fraction \(\sim \Delta / k_B T\) of the excess quasiparticles can relax through this process.

In the limit \(\tau_S^{-1} \gg \tau_E^{-1}\) Eq. (32) reduces to

\[ \frac{1}{\tau} = \frac{\pi \Delta}{4 k_BT} \left( \frac{2}{\tau_E \tau_S} \right)^{1/2} \text{ for } \frac{1}{\tau} \gg \frac{1}{\tau_E}, \tau_S^{-1} \ll \tau_E^{-1} \]

... (34)
In this limit, the spin-flip scattering modifies the quasiparticle distribution substantially. This is because the lower energy excess quasiparticles undergo spin-flip scattering to the other branch more rapidly than higher energy quasiparticles can cool to replace them. As a result, the energy below which spin-flip charge relaxation is important is increased from $-2\Delta$ to an energy $E^*$. This energy can be estimated by equating the cooling rate $\tau^{-1}$ with that of spin-flip branch crossing rate $\sim \frac{\Delta^2}{E^* \tau_5}$. We then find $E^* \sim \Delta \left( \frac{\tau_E}{\tau_5} \right)^{1/2} \gg \Delta$. Thus at temperature near $T_c$ of the quasiparticle scattered downwards by cooling process a fraction $\sim \frac{E^*}{k_B T_c}$ contributes significantly to charge relaxation. Thus the rate is $\sim \frac{E^*}{k_B T_c \tau_E}$.

\[ \sim \left( \frac{\Delta}{k_B T_c} \right) \left( \frac{\tau_E}{\tau_5} \right)^{-1/2} \]

This is in agreement with Eq. (34). It may be emphasized that $\tau_E^{-1}$ enters the result not because it contributes to the charge relaxation in this case because it determines the rate at which quasiparticles scatter downwards into the region from which they spin-flip scatter to the other branch.

The spin-flip scattering rate for a quasiparticle at energy $E$ is given by (for Boltzmann equation approach)

\[ \frac{1}{\tau_{sf}} = \frac{1}{\tau_E} \frac{\Delta^2}{E^2} \left| \frac{E}{E^*} \right| \left( f_f - f_s \right) \]  

(35)
where \( \zeta_s \) is the elastic spin-flip scattering rate, \( \Delta^2 / E^2 \) is the coherence factor, \( E / |\xi| \) is the final density of states and \( (f_\uparrow - f_\downarrow) \) is the usual occupation factor. In practice it is quite complicated to assess explicitly the role of magnetic impurity scattering because the scattering by magnetic impurities is never the only important mechanism, so it is necessary to consider the combined effects of magnetic impurities and phonons, etc.

(d) **Supercurrent**

In the presence of a supercurrent, non-magnetic impurities can relax charge imbalance even if the energy gap is isotropic. This is because the collision term now contains an energy dependent relaxation time similar to that for scattering by magnetic impurities or by non-magnetic impurities when the gap is anisotropic.

When a supercurrent flows in a superconductor the quasi-particle energies are raised by an amount \( P_k \cdot v_\uparrow \), where \( P_k \) is the momentum of quasiparticle in state \( k \) and \( v_\uparrow \) is the superfluid velocity\(^{17, 18} \). Thus quasiparticle at the Fermi energy are raised (lowered) by \( P_\uparrow \cdot v_\uparrow \) where \( v_\uparrow \) and \( P_\uparrow \) are in same (opposite) direction. This current induced anisotropy causes elastic scattering between \( k \) and \( k' \) states as an intrinsic gap anisotropy does (Fig. 7). Maki has shown that an appropriate electron relaxation rate for thin films in the dirty limit \(( 1 \ll \xi_0 \) ) is:

\[
\frac{1}{\zeta_s} = \frac{1}{6} \left( \frac{P_\uparrow}{h} \right)^2 \quad \cdots \quad (36)
\]
where $\xi_0 = k \nu_p / \pi \Delta (0)$ and $p_s$ is the momentum of the Cooper pairs. We can use Eq. (36) in Eq. (32) or Eq. (35) to predict the charge relaxation rate as a function of $p_s$.

The expression for the detector voltage (which is proportional to charge imbalance) at a given temperature and injection current in presence of supercurrent $I_s$ can be written as

$$v_d (I_s) = \frac{v_d (0)}{(1 + 2 \frac{\gamma_v}{\gamma_s})^{1/2}} - \frac{v_d (0)}{(1 + b_{ss} I_s^2)^{1/2}}$$

where

$$b_{ss} = \frac{2 \frac{\gamma_v}{\gamma_s} I_s}{2 \frac{\gamma_v}{\gamma_s}} - \frac{\gamma_v}{3 \frac{\gamma_v}{\gamma_s}} \frac{\nu_p p_s^2}{I_s^2}$$

For uniform currents much less than the critical current, we can use the relations $p_s = 2wV_s$, $J_s = n_s e v_s$, $n_s = mc^2 / 4 \pi e^2 \lambda^2$, and in dirty limit $\lambda (0) \approx \lambda_L (0) (\frac{\xi_0}{\xi_0})_0$. We thus find

$$b_{ss} = \left( \frac{8 \pi e}{\hbar c^2} \right)^2 \frac{\gamma_v \nu_p \xi_0^2 \lambda_L (0)}{3 \lambda d^2 v^2} \frac{\lambda (0)}{\lambda (0)} \frac{\lambda (0)}{\lambda (0)}$$

where $w$ is film width, $\lambda_L (0)$ is London penetration depth and $\frac{\lambda (0)}{\lambda (0)}$ is a well known function. Thus Eqs. (30) and (33) represent the Schmid and Schön prediction for the effect of a supercurrent on the value of $\xi_0$ at fixed temperature and
Fig. 9. Typical experimental plot of $V_d$ vs. $I_s$ for fixed $I_i$. The points are a fit to a function of the form

$$V_d(I_s) = V_d(0)/(1 + b I_s^2)^{1/2},$$

with $b$ as the fitting parameter.

Fig. 10. Data points are measured values of $b$, solid and dashed lines are fits of $b^{ss}$ and $b^{num}$ to the data.
injection current. (See Fig. 9).

Lemberger and Clarke (1984) solved the Boltzmann equation in presence of a supercurrent by replacing $1/\tau_5$ in Eq. (35) by Eq. (36) to obtain

$$\tau_{s,5} = \frac{\varepsilon}{6} \left( \frac{p_s}{k} \right)^2 \frac{\Delta^2}{k |\xi|} (f_+ - f_-)$$

for fixed temperature and injection voltage

$$v_d(I_s) = \frac{v_d(0)}{(1/\text{num}^2)}$$

where $\text{num}$ is the computed number. The experimentally determined values of $b$ are shown in Fig. 10 for two samples. The temperature dependence of $b^\text{meas}$ is in agreement with the data, while $b^\text{num}$ is in substantial disagreement. Thus Schmid and Schön theory fits the measured data accurately, even at values of $\Delta/k_B T$ much larger than one would expect, while computed solution to Boltzmann equation approach to charge relaxation does not fit the data. Thus the charge relaxation in the presence of a pair breaking mechanism is not yet well understood. The Boltzmann equation approach produces results that are not in agreement with the data, whereas the Schmid and Schön theory seems to be valid over a temperature range much wider than is expected.

...
3.2 Order parameter relaxation

If the quasiparticle distribution is known, the energy gap for a spatially homogeneous superconductor can be calculated using the BCS gap equation (7), with $f_0(E)$ being replaced by $f(E) = f_0(E) + \delta f(E)$. Close to $T_c$, this equation can be approximated by

$$\left[ \alpha + b \left( \frac{\Delta}{k_B T_c} \right)^2 - \xi^2(0) \nabla^2 - \chi \right] \Delta = 0$$

where $\alpha = \frac{T - T_c}{T_c} = t - 1$, $b = \frac{7 \xi(3)}{8 \pi^2} \approx 0.105$

$$\xi(T) = \xi(0) \left( 1 - \frac{T}{T_c} \right)^{-1/2}$$

and $\chi = \int \frac{\delta f(E)}{\left( E^2 - \Delta^2 \right)^{1/2}} \, dE$

The nonequilibrium quasiparticle distribution which gives rise to a nonzero $\chi$ in the Eq. (42) leads to the change in the gap. For this reason $\chi$ is called the gap-control function. Any increase in the quasiparticle distribution function decreases the energy gap, while any decrease increases the gap. The gap can also be changed if the quasiparticles are shifted in energy. This is caused by the denominator in the integrand of the gap-control function, which is a decreasing function of energy $E$. Thus quasiparticles near the gap edge make larger contribution to the gap control function than quasiparticles of higher energy. Shifting quasiparticles to higher energy while keeping their total number constant thus increases the energy gap. In
order to calculate the gap \( \Delta \), the function \( \delta f(E) \) has to be known, which in the simplest way can be obtained by using a Boltzmann equation with a relaxation time approximation

\[
\frac{\partial f(E)}{\partial t} = \frac{f(E) - f_0(E, T)}{\tau_E} + I_{qp}(E) \quad \ldots (43)
\]

The term \( I_{qp}(E) \) is the driving term of the external disturbance causing the nonequilibrium state. The effective relaxation time \( \tau_E \) for the quasiparticles is assumed to be independent of the quasiparticle energy. It is also assumed that the lattice temperature \( T \) is not altered by the nonequilibrium state. When external disturbance, which brings a superconductor out of equilibrium, is switched off the superconductor will relax to the thermal equilibrium state. This process is described by the Boltzmann equation (43) for the quasiparticle relaxation and the gap equation (42) for the variation of \( \Delta \). An interesting consequence of the coupling of these two equations is that near the critical temperature \( T_c \) the order parameter relaxation time \( \tau_\Delta \) is much longer than the quasiparticle relaxation time \( \tau_E \). This is due to the fact that when the gap \( \Delta \) changes with time the energy of a quasiparticle with wave vector \( k \) also changes because of the relaxation \( E_k - (\frac{\hbar^2 k^2}{2m} + \Delta^2)^{1/2} \) being the actual time-dependent gap value. This means that the energy \( E_k \) in the Fermi function \( f_0 \) in Eq. (43) is time dependent. The Boltzmann Eq. (43) describes the relaxation of the distribution function to the equilibrium function \( f_0(E_k) \), which is itself a function of time because the changing distribution
function results in a time-varying gap value. This causes the relaxation of to take longer time than the time constant $\tau_E$.

(a) **Slow variation of order parameter**

In order to illustrate this process we consider the order parameter varying on a time scale which is long compared with $\tau_E$ so that the quasiparticle population lags behind by the relaxation time $\tau_E$. We can then write for temperature near $T_c$,

$$f(E_k) \approx f_0 \left( \frac{E_k}{T} \right) - \tau_E \frac{\partial f_0 \left( \frac{E_k}{T} \right)}{\partial t} \ldots \ (44)$$

Then

$$\delta f(E_k) = - \tau_E \frac{\partial f_0 \left( \frac{E_k}{T} \right)}{\partial t} \ldots \ (45)$$

where

$$\frac{\partial f_0 (E_k)}{\partial t} = \frac{\partial f_0 (E_k)}{\partial E_k} \frac{\partial E_k}{\partial \Delta} \frac{\partial \Delta}{\partial t} \ldots \ (46)$$

Restricting attention to $E_k \approx \Delta \ll T \approx T_c$, as is appropriate in the GL regime, $\frac{\partial f_0}{\partial E} = -\frac{1}{4 k_B T_c}$ so that Eq. (45) and (46) lead to

$$\delta f(E_k) = -\frac{\tau_E}{4 T_c} \Delta \frac{\partial \Delta}{\partial t} \ldots \ (47)$$

Using this in gap control function, we obtain
\[
\chi = \int \frac{\delta f}{(E^2 - \Delta^2)^{1/2}} \, dE - \frac{\pi E}{E_B T_0} \Delta \frac{\partial \Delta}{\partial t} \int \frac{dE}{E (E^2 - \Delta^2)^{1/2}} \approx \frac{\pi E}{4 k_B T_0} \frac{\partial \Delta}{\partial t} \quad \text{... (48)}
\]

The gap equation now becomes

\[
\left[ \frac{E - T_0}{T} + \beta \frac{\Delta^2}{k_B T_0^2} - \xi^2(\phi) \nabla^2 \right] \Delta = - \frac{\pi T}{4 k_B T_0} \xi E \Delta \frac{\partial \Delta}{\partial t} \quad \text{... (49)}
\]

It is convenient to rewrite this equation in terms of a normalized order parameter \( f = \Delta / \Delta_0(T) \)

\[
\left[ 1 + (t - 1)^{-1} \beta \frac{\Delta^2}{(k_B T_0^2)} - (t - 1)^{-1} \xi^2(\phi) \nabla^2 \right] \Delta = - \frac{\pi T}{4 k_B T_0} \xi E \Delta \frac{\partial \Delta}{\partial t} (t - 1)^{-1} \quad \text{(t - 1)}^{-1}
\]

Using \( \Delta_0(T) = \beta^{-1/2} k_B T_0 (1 - t)^{1/2} \), and

\[ \xi(\phi) = \xi(T) (1 - t)^{1/2} \]

we have

\[
\left[ 1 - f^2 + \xi^2(T) \nabla^2 \right] f = 2 \xi \Delta \frac{\partial f}{\partial t} \quad \text{... (50)}
\]

where

\[
\xi = \frac{\xi(T)}{R} = \frac{\pi E}{8 \left[ \xi(1 - t) \right]^{1/2}} = \frac{1/2 \xi E}{(1 - t)^{1/2}} \quad \text{... (51)}
\]
To check the appropriateness of this definition of $\zeta_\Delta$, consider a spatially uniform fluctuation $\delta f$ about solution $f = 1$ of the static Ginsburg-Landau equation. By linearisation of Eq. (50), one finds that $\delta f$ is given by

$$f_0 + \delta f - (f_0^2 + \delta^2 f + 2 f_0 \delta f)(f_0 + \delta f)$$

$$+ \xi^2(1) \nabla^2 (f_0 + \delta f) - 2 \zeta_\Delta (f_0 + \delta f) \frac{\partial}{\partial t} (f_0 + \delta f)$$

In linearisation only linear terms are to be retained. Thus

$$f_0 + \delta f - f_0^3 - 2 f_0^2 \delta f - f_0^2 \delta f$$

$$- 2 \zeta_\Delta (f_0 + \delta f) \frac{\partial}{\partial t} (f_0 + \delta f)$$

for $f_0 = 1$

$$\frac{\partial (\delta f)}{\partial t} = - \zeta_\Delta \delta f$$

$$\ldots (52)$$

Thus indeed this $\zeta_\Delta$ gives the exponential damping time for small fluctuations from uniform solution.

(b) Rapid variation of order parameter

If the gap variation is too rapid so that $\zeta_\Delta \gg \zeta_E$, no longer holds, the solution of the Boltzmann Eq. (43) can no longer be approximated by Eq. (45). One can write the solution of Boltzmann equation in the relaxation time approximation as

$$\delta f(E, t) = \int_{-\infty}^{+\infty} dt' \frac{\partial f_0(E, t')}{\partial t'} e^{-(t - t')/\zeta_E} \ldots (53)$$
Using Eq. (46) and evaluating \( \frac{\partial f_0}{\partial E} \) near \( T_c \) as before, we can write Eq. (53) as

\[
\delta f(E, t) = \frac{1}{4 k_B T_c} \int_{-\infty}^{t} \frac{\Delta'_2}{(E^2 + \Delta'^2)^2} \frac{\partial \Delta}{\partial t'} e^{-(t-t')/\tau_E} \quad \text{... (54)}
\]

Inserting this in the gap-control function \( \chi \), we have

\[
\chi(t) = \frac{1}{4 k_B T_c} \int \int \frac{\Delta'}{(E^2 + \Delta'^2)^2 (E^2 + \Delta^2)^2} e^{\frac{\partial \Delta}{\partial t'} - \frac{e^{-(t-t')/\tau_E} \int \int dE dt' \quad \text{... (55)}}
\]

If \( \Delta \propto \Delta' \) the integral is elementary and has the value \( \pi/\Delta \). Its exact value can be written in terms of elliptic integrals and is well approximated (to a few percent) by the symmetric interpolation formula \( \pi \frac{(\Delta \Delta')^{1/2}}{(\Delta' + \Delta)} \) so long as \( \Delta, \Delta' \) differ by no more than a factor of \( \sim 2 \). (Schmid and Schön)\(^9\). This allows us to write Eq. (55) as

\[
\chi(t) = \frac{\pi}{4 k_B T_c} \sqrt{\Delta(t)} \int_{-\infty}^{t} \frac{1}{\sqrt{\Delta(t')}} \frac{\partial \Delta}{\partial t'} e^{-(t-t')/\tau_E} \quad \text{... (56)}
\]

If \( \Delta \) and \( \frac{\partial \Delta}{\partial t} \) vary slowly on the scale of \( \tau_E \) this equation reduces to earlier result (Eq. 48). For example, a small sudden drop of the gap at \( t = 0 \) from \( \Delta \) to \( \Delta - \delta \Delta \) can be
described by \( \frac{\partial \Delta}{\partial t} = - (8\Delta) \delta(t) \). According to Eq. (48) this would cause an infinitely large instantaneous \( \delta \) - function spike in \( \chi = \frac{\pi T_c}{4k_BT_c} (8\Delta) \delta(t) \). But with Eq. (56), the same change in \( \frac{k_B T_c}{\Delta} \) produces a step change \( \chi = \frac{4k_B T_c}{\Delta} \), which would then relax with time constant \( \tau_E \). This limiting form of the behaviour reflects the fact that no matter how rapidly the change in \( \Delta \) occurs the maximum departure from the equilibrium population for a given \( 8\Delta \) is that which exists right after the change in gap and before any relaxation of the previously existing \( f(E) \) towards the new equilibrium has occurred.

It is worthwhile to obtain a more transparent version of Eq. (56) by making a further approximation that \( \Delta(t) \approx \Delta(t') \) for \( t - t' \leq \tau_E \), even when \( \partial \Delta / \partial t \) may change rapidly on this time scale. Thus Eq. (56) becomes

\[
\chi(t) = \frac{\pi T}{4k_B T_c} \int_{-\infty}^{t} \frac{3\Delta}{\partial t'} e^{-(t-t')/\tau_E} dt'. \tag{57}
\]

Inserting the generalised expression (57) into the DL Eq. (42) and again expressing the result in terms of \( f = \Delta / \Delta_o \), we obtain an integro-differential equation

\[
[1 - f^2 + \frac{\pi^2}{8} \nabla^2] f
- a f \int_{-\infty}^{t} \frac{\partial f}{\partial t'} e^{-(t-t')/\tau_E} dt'. \tag{58}
\]

where \( a = \frac{2 \tau_A}{\tau_E} = \frac{\pi}{4} g^{-1/2} (1-g)^{-1/2} = 2.41(1-t)^{-1/2} \).
The parameter $a$ is typically of order 10. As a check we note that if $\partial f/ \partial t$ varies slowly on the scale of $\tau_E$, it can be taken out of the integral and we recover the conventional partial differential equation approximation Eq. 50.

4. ENHANCEMENT OF SUPERCONDUCTIVITY

It is known that thermally excited quasiparticles suppress superconductivity since they restrict the phase space available to Cooper pairs. Less known, however, is the fact that excitations near the gap edge are more destructively blocking the pairs than those which are off the edge. Recognizing this, Elishberg$^{20}$, and Ivlev et al.$^{21}$ pointed out that electromagnetic radiation tends to remove, under certain conditions, excitations from the gap edge. As a consequence there results an enhancement of the order parameter and stimulation of superconductivity above $T_c$. Radiation of say, frequency $\omega$ induces transition of the quasiparticles from states of energy $E$ to states of energy $E + \hbar \omega$. Initially the gap edge is heavily populated with excitations on account of its high density of states, and consequently the net rate of transitions is away from the gap edge. In Boltzmann equation these transitions are represented by a finite source term which is proportional to the applied radiation power.

4.1 Enhancement by Microwave Irradiation

The microwave field of frequency $\omega$ is represented by a vector potential...
\[ A = A (-i \omega t) \]  

which is defined in the London gauge; \( \Delta \) is assumed to be real. The power of the RF field is expressed as

\[ \alpha = \frac{2 e^2 D}{\hbar c^2} \Delta^2 \]  

where

\[ D = \frac{1}{3} v_F l, \]  

is the diffusion coefficient.

The driving term can be obtained with a Golden Rule calculation in which the electron creation and destruction operators are replaced by the corresponding linear combinations of quasiparticle operators according to the Bogoliubov transformation. The result \( ^{21, 22} \) is

\[ I_{qp}(E) = 2 \Delta^{-1} \alpha \left[ \frac{E - t \omega}{[(E - t \omega)^2 - \Delta^2]^{1/2}} \left( 1 + \frac{\Delta^2}{E(E - t \omega)} \right) \right] \]

\[ \times \left( f(E - t \omega) - f(E) \right) \Theta(E - t \omega - \Delta) \]

\[ - \frac{E + t \omega}{[(E + t \omega)^2 - \Delta^2]^{1/2}} \left( 1 + \frac{\Delta^2}{E(E + t \omega)} \right) \left( f(E) - f(E + t \omega) \right) \]

\[ + \frac{t \omega - E}{[(E - t \omega)^2 - \Delta^2]^{1/2}} \left( 1 + \frac{\Delta^2}{E(t \omega - E)} \right) \Theta(t \omega - E - \Delta) \]

\[ \times \left( 1 - f(E) - f(t \omega - E) \right) \Theta(t \omega - E - \Delta) \]  

... (61)

where \( \Theta \) is the Heaviside step function. The various terms in eq. (61) can easily be identified. The first term is the net
flux of quasiparticles between energy levels $E - \hbar \omega$ and $E$. It consists of particles going from $E - \hbar \omega$ to $E$ by absorption of a microwave photon or from $E$ to $E - \hbar \omega$ by emission of a photon. This process of course is only possible if $E - \hbar \omega > \Delta$.

The second term describes the same process between the levels $E$ and $E - \hbar \omega$.

The third term describes the recombination of quasiparticles into Cooper pairs or the breaking up of Cooper pairs with respectively the emission or the absorption of a photon with energy $\hbar \omega > E + \Delta$. For microwave radiation with $\hbar \omega < 2\Delta$ this pair breaking term does not exist. The transition rates for each of the above mentioned processes in Eq. (61) is the product of the coupling strength with the field, the density of states $E / (E^2 - \Delta^2)^{1/2}$ of each of the levels involved, the coherence factor, the difference in occupation probabilities of the levels and the suitable step functions $\Theta$ that take into account that levels with $E < \Delta$ are not present.

For stationary irradiation the nonequilibrium part of the distribution function is obtained from Boltzmann equation (45) and is given by

$$\delta f(E) = \zeta E I_{qp}(E) \quad \ldots (62)$$

For low microwave intensities $\alpha \zeta \ll 1$, the perturbation $\delta f(E)$ is small and $I_{qp}(E)$ can be calculated by taking $f(E) = f_0(E)$ (the equilibrium function) in Eq. (61).
\[
\frac{\hbar \omega}{k_B T_c} \ll 1, \text{ then to the first order in } \frac{\hbar \omega}{k_B T_c},
\]

we have

\[
f(E) - f(E + \hbar \omega) = \frac{\hbar \omega}{4 k_B T_c}
\]

With this approximation and using Eq. (62) the resulting \( \Delta(\alpha) \) as a function of the microwave power parameter \( \alpha \) is

\[
\frac{\Delta^2(\alpha)}{\Delta^2(0)} = 1 + 2 \alpha^2 \frac{\pi}{E} \left( \frac{\hbar \omega}{\Delta(0)} \right)^2
\]

\[
\left[ -1 + \frac{1}{\hbar \omega T_B} \frac{k_B T_c}{\hbar \omega} + \frac{1}{2 \pi \hbar \omega} g \left( \frac{\Delta(\alpha)}{\hbar \omega} \right) \right]
\]

\[\ldots (63)\]

with \( \Delta(0) \) as the gap value without microwaves at the temperature \( T \). The function \( g(u) \) is due to the gap control function and is given by

\[
g(u) = \begin{cases} 
2 \pi u (1 - u^2)^{-1/2} & ; \quad u < \frac{1}{2} \\
\frac{2}{u + 1/2} \left\{ k \left( \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \right) + 4 u^2 \left[ \Pi \left( \frac{1}{(2u + 1)^2}, \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \right) \right] \\
- k \left( \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \right) \right\} & , \quad u \geq 1/2 \end{cases}
\]

\[\ldots (64)\]

where \( k \) and \( \Pi \) are the complete elliptic integrals of the first and the third kinds, respectively. This \( g \) function has two useful limits

\[\ldots\]
\[ \varepsilon = \begin{cases} 2 \pi u & \text{for } u \to 0 \\ 2 \ln (2 \cdot \zeta u) / u & \text{for } u \to \infty \end{cases} \quad \cdots (65) \]

for \( u = 1/2; \ \varepsilon (1/2) = 3.63. \)

The nonequilibrium term, proportional to \( \propto E \) in Eq. (63) contains three contributions: (i) the first one is negative and tends to reduce the gap. It is due to excess quasiparticles at energies \( E \gg \Delta \); (ii) The second one is also negative and is due to the pair breaking caused by the vector potential; and (iii) The third term is positive. It is caused by the decrease of the quasiparticle population just at and above \( E = \Delta \) where the quasiparticles are removed by absorption of microwave photons.

There is a critical microwave frequency \( \omega_c \) above which the sum of the three contributions is positive and the gap is enhanced under the influence of microwave irradiation. It is also found that with microwaves the critical temperature is enhanced. There is a temperature region above \( T_c \) where there are two non-zero solutions to the gap Eq. (63). One solution \( \Delta_1 > \frac{T_c \omega_c}{2} \) is unstable. At a new critical temperature \( T_c (\propto) \), which is higher than \( T_c \), both solutions collapse. Above this temperature there is no longer a non-zero solution for \( \Delta (\propto) \).

(a) Effect of phonons

So far we have considered the nonequilibrium quasiparticle
Fig. 11: The dependence of the energy gap upon the microwave power parameter $A$, which corresponds to $\sim 4 \propto \omega_B$.
Curves a-d correspond to $t_{\omega} = \Delta_0 / 2$, curves e and f to $t_{\omega} = \Delta_0$ and curves g and h to $t_{\omega} = 3 \Delta_0 / 2$. The phonon escape time $\tau_{es} = 0.2 \tau_B$ for curve a, $2 \tau_B$ for b, $8 \tau_B$ for curves c, e, and g and $16 \tau_B$ for curves d, f, and h ($\tau_B = \text{pair breaking time at } T = 0$). [Ref 24]
driving term $I_{qp}(E)$ only and have taken $I_{ph}(-\Omega) = 0$. The calculated excess quasiparticle distribution shows the characteristic feature that for low-lying states ($\Delta < E < \Delta + \hbar \omega^*$) one finds $\delta f(E) < 0$ and for states with $E > \Delta + \hbar \omega^*$ one has $\delta f(E) > 0$. The absorption of photons shifts the quasiparticles to higher energies. If the recombination rate of the quasiparticles were independent of energy the total number of quasiparticles would remain constant when they are shifted to higher energies. It has been argued in previous section that such a shift in quasiparticle distribution leads to an enhanced energy gap. However, the recombination rate is energy dependent. The higher energy quasiparticles states have larger recombination rate $\gamma_R^{-1}$ than the lower states because of increasing phonon density of states for the recombination phonons. The total number of quasiparticles will, therefore, be less than that in a state of equilibrium. This is a second cause of the gap enhancement. Calculated values of the gap as a function of the microwave power are shown in Fig. (11). For curves a and b the enhancement caused by the decrease in the number of quasiparticle accounts for about 70% of total enhancement while the remaining 30% is caused by the direct influence of the redistribution on the gap equation.

Chang and McElpinco\textsuperscript{24} considered a set of Boltzmann-like kinetic equations for both the quasiparticles and the phonons. In the specific case of microwave stimulation the source of nonequilibrium is the interaction with the field, which is introduced with an expression similar to Eq. (61). The gap is calculated with the gap equation and follows from the quasiparticle distribution. In the numerical calculations, the
Fig. 12:
(a) Quasiparticle distribution, and (b) Phonon spectrum. Solid lines show the equilibrium values at $T = 0.9 T_c$, $A_{cs} = 0$. Dotted lines represent the nonequilibrium values for microwave irradiation at $T = 0.9 T_c$, $\hbar \omega = \Delta_0 / 2$, $A_{cs} = 0.45$ and $\zeta_{es} = 2 \zeta_B$. [Ref 24].
Fig. 13: (a) Quasiparticle density $N_{qp}$; and (b) energy gap $\Delta$ as a function of $A_{cs}$. $T = 0.9 T_c$, $C_{cs}/C_B = 8$, $\hbar \omega = 0.5, 1, 1.5$ times $\Delta(0)$. [Ref. 24].
whole scheme is run until a stationary, self consistent solution is obtained. An important parameter is the coupling of the phonons to the helium bath and the substrate. They use the pa parameter $\tau_{es}$, the 'escape time' to describe this coupling. This $\tau_{es}$ is measured in units of $\tau_B$, the phonon Cooper pair breaking time at zero temperature. A relaxation time approximation is used for the phonons, where the number of phonons leaving the metal is proportional to $\left\{ n(\Omega) - n^0(\Omega) \right\} / \tau_{es}$. As a whole there are four input parameters: temperature, $\tau_{es}$, microwave frequency and microwave intensity. The microwave intensity is expressed as $A_{cs}$ which when compared with Eliashberg parameter $\lambda$ in its scattering effect on the quasiparticles corresponds to

$$A_{cs} = 92 \propto \tau_E \quad \ldots \quad (66)$$

In Chang and Scalapino calculations the interaction of quasiparticles at energy $E$ is considered with phonons of energy $\Omega$ without the assumption of effective relaxation time. In Fig. 12 the quasiparticle distribution function and phonon spectrum are given for equilibrium and for microwave irradiation. In Fig. 13 we give both the density of quasiparticle and the gap as a function of $A_{cs}$, at $T = 0.9 \, T_c$ with $\tau_{es} = 3 \, \tau_B$ and for $\frac{\omega}{\Delta(T)} = 1/4; \, 1/2; \, 1$ and $3/2$, which roughly correspond to $1/2; \, 1; \, 2$ and $3$ times the equilibrium $\Delta(T)$. At the highest frequency pair breaking dominates, $N_{qp}$ increases and hence no enhancement occurs. Other curves show initial enhancement. At higher microwave powers more and more phonons are created with sufficient
energy to break pairs. A maximum is found for $\Delta$ as a function of $A_{cs}$.

b) Effect of Supercurrent

We now discuss the more general case of enhancement by including the supercurrent density, which is important also for the microwave enhancement of the critical current density. Assuming phonon equilibrium and validity of Boltzmann equation approach with constant scattering time $\tau_E$ (Eq. 43), and linearization of the equations for low microwave power levels, the gap equation is

$$\frac{\Delta^2(\alpha, j_s)}{\Delta^2(o)} - \frac{4}{27} \left( \frac{j_s}{j_c(o)} \right)^2 \frac{\Delta^4(o)}{\Delta^4(\alpha, j_s)}$$

$$= 1 + \alpha \tau_E \frac{\pi}{2} \left( \frac{\tau o}{\Delta(o)} \right)^2$$

$$+ \left[ -1 - \frac{1}{2 \omega^2 \tau_E} \frac{k_B T_s}{\hbar \omega} + \frac{1}{2 \pi \beta} \frac{k_B T_s}{\hbar \omega} \frac{\Delta(\alpha, j_s)}{\hbar \omega} \right]$$

$$\ldots (67)$$

This is valid for $T$ near $T_c$, and microwave frequency $\omega$ high compared to $\tau^{-1}$ but small compared with $\tau^{-1}$. For this range of value of $\omega$, the e.m. field can be treated without complications. The extra term in Eq. (67) compared to Eq. (63) represents the pair-breaking effect of the supercurrent density $j_s$. The critical current density $j_c(0)$ for $\alpha = 0$ is given in the dirty limit by$^{25}$

$$\ldots$$
The function \( g \) in Eq. (67) is due to integration of the external driving term \( I_{qp} (E) \). Since the density of states for a current-carrying superconductor is different from the noncurrent-carrying state, \( I_{qp} (E) \) has to be changed accordingly. By using the same function \( g \) as given in Eq. (64) with current-dependent gap \( \Delta (\alpha, J_g) \) a good approximation can be obtained.

From Eq. (67) the critical current density as a function of the microwave power \( J_c (\alpha) \) can be calculated as the largest possible value of \( J_g \) such that a solution for \( \Delta (\alpha, J_g) \neq 0 \) exists. The result shows a critical current enhancement \( \delta J_c (\alpha) = J_c (\alpha) - J_c (0) \), increasing linearly with \( \alpha \leq E \).

It is interesting to note\(^{26}\) that the gap enhancement
\[
\delta \Delta (\alpha) = \Delta (\alpha, J_g) - \Delta (0, J_g)
\]
is itself a function of the transport current, it increases with increasing current up to \( J_c (0) \). This indicates that near transition temperature the superconductor is more susceptible to microwave irradiation.

4.2 Enhancement by Phonons

Superconductivity can also be enhanced by phonon irradiation. The interaction of quasiparticles with photons...
(microwaves) and with phonons have similar consequences. The
same mechanism of redistribution of quasiparticles to higher
energies is possible and enhancement of the gap can be obtained.
However, the interaction with longitudinal phonons is characteri-
sed by different coherence factors in comparison with those
that characterizes the interaction with a e.m. vector field.
The coherence factor for scattering of quasiparticles between
E and $E'$ for longitudinal phonons is $\left(1 - \frac{\Delta^2}{E E'}\right) / 2$. This
is to be compared with $\left(1 + \frac{\Delta^2}{E E'}\right) / 2$ for microwaves.
For scattering near the gap edge the phonon coherence factor
is small. The pair breaking processes have large coherence
factor. When phonon energy $\hbar \Omega$ exceeds $2 \Delta$ even stronger
effects are expected than with microwaves. The theory can be
developed in analogy with the microwave case and the result is
equivalent to Eq. (63) with a different $g(u)$ function which
is discontinuous at $u = 1/2$.

$$G(u) = -2 \pi u (1 - u^2)^{1/2} \quad \text{for } u < 1/2$$
and
$$G(u) = g(u) \quad \text{for } u \geq 1/2$$

Here $\omega$ is to be replaced by the phonon frequency $\Omega$, and
and $\alpha$ measuring the phonon power.

In this case there is no contribution analogous to the
pair breaking caused by the static part of the microwaves.
Further, the fact that pair breaking process has a large coherence
factor shows up in the $G(u)$ function. It is negative for $u < 1/2$. 

...
Fig. 14: Relative increase of critical current of a microbridge at two phonon power levels as a function of temperature. The fully drawn lines result from Eliashberg's theory. [Ref 27]
This means that there can be no enhancement effect for 
\[ \Omega > 2\Delta \] and thus no critical temperature enhancement.

Fredwell and Jacobson\textsuperscript{27} have experimentally investigated the effect of phonons in Aluminium film. Phonons of 10 GHz frequency were directed to the film. The superconducting properties of the Aluminium film were probed by measuring the critical current of either a tin point contact to the film or a microbridge fabricated in the film. It was observed that the critical current increased under the influence of phonon radiation and this increase was interpreted as being caused by the increase of the energy gap in Aluminium film. Fig. 14 shows the relative increase of the critical current for two fixed phonon power densities as a function of temperature. The measurements relate to a microbridge for which critical current is proportional to \((1 - I / I_c)^{3/2}\). Good agreement with Eliashberg's theory was found. The enhancement was observed only below the temperatures where injected phonon energy \(\Omega\) equals gap \(2\Delta(T)\). There is no indication of critical temperature enhancement.

More recently preliminary results have been obtained by Saligson and Clarke\textsuperscript{28} who observed gap enhancement of Aluminium film by 9 GHz phonons using a tunnel junction. It is of great interest to obtain more experimental data and compare it with the theoretical expectations.

4.3 Enhancement by Quasiparticle Tunneling

Early in 1961 Farrenber\textsuperscript{29} suggested that the gap may be
enhanced by extraction of quasiparticles resulting in a reduced number of quasiparticles. This effect can be realized with a tunnel current between two superconductors having different gap values. The enhancement by tunnelling is also possible in s-symmetric SIS junctions\textsuperscript{30}. These mechanisms were later discussed by Kirichenko, Peskovatskii and Seminshanko\textsuperscript{31}. These effects can also be discussed in analogy with the microwave and phonon irradiation in terms of gap control function. The result is again an equation like (45), with yet another rather complicated $g$-function\textsuperscript{23}.

The tunnelling enhancement offers an advantage over microwaves. The injection rate can be precisely measured with an ammeter and the voltage $V_1$ (equivalent to the microwave frequency for experiments with equal gaps) can be varied easily from zero to $2 \Delta / e$. Measurements by Gray\textsuperscript{32} confirm the general picture of gap enhancement and of reduction in the quasiparticle density $N$. The requirement of identical gap is not easily met and additional enhancement was found if $eV_1$ is approximately equal to the difference in the gaps. With unequal gaps, there is a net extraction (injection) of quasiparticles in the film with smaller (larger) gap. Enhancement due to extraction processes has been studied in detail by Shi and Clarke\textsuperscript{33}. They find enhancements of up to $40\%$. However, at higher voltages, they propose that nonequilibrium phonons transmitted between films, limit the size of the enhancement.
5. SUPPRESSION OF SUPERCONDUCTIVITY

The optical illumination with laser light and injection of quasiparticles by means of tunnelling can create large deviations in the quasiparticle population and tend to decrease the energy gap. We here first discuss briefly two models, $\mu^*$ model\textsuperscript{34} and $T^*$ model\textsuperscript{35}, which have been used to explain the experiments on the gap suppression.

5.1 The $\mu^*$ and $T^*$ models

The energy distribution of the quasiparticles in the nonequilibrium state under an external dynamic pair breaking influence can be described by the $\mu^*$ model of Owen and Scalapino\textsuperscript{34}. In this model the quasiparticles are assumed to be in thermal equilibrium with the phonons, but not in equilibrium with the pairs. This model is applicable only for systems in which quasiparticles recombination time is long compared to the time for the quasiparticle to thermalize with the bath (or lattice). Under external pair breaking conditions the quasiparticle distribution function will be described by Fermi function, with lattice temperature $T$ and a chemical potential of quasiparticles $\mu_{qp} (= \mu^*)$ different from the chemical potential of the condensate $\mu_s$ which is taken to be zero. Therefore, we have

$$f(E) = \left[ 1 + e^{(E - \mu^*) / k_B T} \right]^{-1} \quad \ldots \quad (69)$$

Substituting this distribution function in the gap equation,
\( \Delta(n) \) is obtained as a solution of the algebraic equation, in the zero temperature limit

\[
\left( \frac{\Delta}{\Delta_0} \right)^2 = \left\{ \left( \frac{\Delta}{\Delta_0} \right)^2 + n^2 \right\}^{1/2} - n \right\}^2 \quad ... (70)
\]

where \( n \) is the excess quasiparticle concentration normalized to \( 4N(0)\Delta_0 \), and

\[
n = \frac{N_{qp}(\mu^*) - N_{qp}(0)}{4N(0)\Delta_0} \quad ... (71)
\]

The resulting \( \Delta \) is double valued. Only the solution which starts at \( \Delta_0 \) for \( n = 0 \) is stable. Increasing \( n \) decreases \( \Delta \) until at \( n = 0.15 \), where \( \Delta = 0.62 \Delta_0 \), a first order phase transition to the normal state occurs. At finite temperature similar behaviour is found. For small values of \( n \) the critical temperature \( T_c \) at which the first-order phase transition occurs decreases linearly with \( n \)

\[
T_c(n) \approx (1 - 4n)T_c(0), \quad n \leq 0.1 \quad ... (72)
\]

A characteristic feature of the \( \mu^* \) model is the prediction of this first order phase transition to the normal state. The model also predicts a spatial instability of the energy gap in the superconductor.\(^{36}\)

Another model for describing a superconductor under external pair-breaking perturbations was developed by Langer\(^{35}\), called the \( \tau \) model. In this model the quasiparticles are
assumed to be in thermal equilibrium with phonons of energy greater than \( 2 \Delta \) at an elevated temperature \( T^* \), as well as in chemical equilibrium with the pairs \( (\mu = 0) \). The phonons with \( \hbar \Omega < 2 \Delta \) remain undisturbed at both temperature \( T \).

This model is applicable to the experimental situation where the quasiparticle thermalization time is long as compared to the recombination time. The elevated temperature \( T^* \) turns out to be considerably greater than that resulting from simple heating. For \( T = 0 \) and \( n \lesssim 0.1 \), the result for \( \Delta (n) \) are almost identical with the results of the \( T^* \) model. For larger values of \( n \) the \( T^* \) model differs markedly from the \( \Delta^* \) model.

Specifically it does not predict a first-order phase transition but a continuous second-order phase transition at a value of \( n \approx 0.4 \).

The quasiparticle distribution function of \( T^* \) model is in reasonable agreement with that calculated numerically from kinetic equations\(^{37}\). However, the phonon distribution function obtained from \( T^* \) model deviates appreciably from the results of the numerical calculations. Chang et al.\(^{37}\), therefore, introduced an extended \( T^* \) model. Using the approximation for the quasiparticle distribution from the \( T^* \) model they obtained from the kinetic equations for the phonon distribution an analytical expression for the nonequilibrium phonon distribution. This was in better agreement with the numerical solution of coupled kinetic equations than the distribution function obtained with the \( T^* \) model.
5.2 Optical Illumination

The response of a superconductor to laser illumination was first determined by Testardi\textsuperscript{38} by measuring the voltage across the current-carrying films.

Under the same illumination conditions he observed a transition in the thin films (27 nm) to a normal resistive state down to temperatures as far as $3k$ below $T_c$, whereas the thick film (200 nm) showed the transition to the resistive state at a temperature $T_c = 0.45\,\text{k}$ as expected. Only films with thickness comparable to the optical penetration depth and superconducting coherence length showed this anomaly of the photon-induced resistive state. From the delay time between the laser pulse and the onset of the resistive state Testardi estimated a temperature rise of $3 - 10\,\text{k}$ in the electron gas. It was on the basis of this observation that Owen and Scalapino developed their $\mu^*$ model.

A beautiful experiment to investigate how the $\mu^*$ model and $\Delta^*$ model agree with experiments was carried out by Smith, Skeoopol and Tinkham\textsuperscript{39}. They determined the quasiparticle distribution function under laser illumination. The distribution function allows a much more severe check on the validity of the different models than quantities like energy gap, the d.c. resistance or the transmission coefficients at microwave frequencies which contain only an integral over the distribution function. The I-V characteristics of Aluminium - Nb Bi tunnel junction under laser illumination were measured and analysed to...
obtain the quasiparticle distribution function. The results were in good agreement with the predictions of the $F^*$ model for the distribution function, but in disagreement with the $\mu^*$ model.

5.3 Quasiparticle Injection

Injection of quasiparticle or phonons can also disturb the equilibrium between quasiparticles, the condensate and the phonons in a superconductor. Quasiparticle injection is usually made by means of tunnel junctions. In many experiments the well known three-layer geometry is used. The first two films serve as the injection junction. The change in the energy gap of the middle films caused by the injection of quasiparticles is detected by means of the tunnel junction formed between the second and the third film.

In an Sn-Sn-Pb tunnelling structure Fuchs et al., measured the energy gap reduction of tin as a function of injection current. The results followed the $\mu^*$ model and showed significant deviation from $F^*$ model. They observed a first order transition predicted by the $\mu^*$ model for gap reduction larger than 40%. In contrast Rajeev Kumar et al., found in a similar experiment on a Sn-Sn-Sn double tunnel junction that the decrease of the energy gap as a function of the quasiparticle injection current for not too high values of this current could be described by the $F^*$ model and not by the $\mu^*$ model, or by simple heating model. At high injection level the energy gap decreased discontinuously to zero, in disagreement with the $F^*$ model. At high quasiparticle injection rates ...
splitting of the energy gap into two distinct gaps has been observed in some experiments. Dynes et al.,42 measured the gap instability in an Al-Al-Al tunnel junction. They attributed the development of a second energy gap to an intrinsic property of the superconducting state. They rule out critical current and beating effects.

The observation of a multiple gap structure was repeated by Iguchi et al.,43. The injection voltages were chosen well above the gap edge for a uniform quasiparticle injection over the tunnelling region. The results were interpreted in terms of the $\mu^*$ model, but first order phase transition, which is an inherent feature of this model was not observed.

It appears from these experiments that the question as to whether a first-order phase transition really occurs at high quasiparticle injection rates is still unsettled. It is also not clear, why some experiments agree satisfactorily with the $\mu^*$ model and others with $T^*$ model. In case of laser illumination it is difficult to ensure that illumination of tunnel junction is really homogeneous. More precise knowledge of experimental conditions is needed in respect of tunnel barrier and films to compare the data with the expectations of theoretical models.
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