Chapter IV

A Stronger Form of \((\omega)\)paracompactness

In this chapter, we define a stronger form of \((\omega)\)paracompactness and obtain some characterizations of this notion involving closure preserving property.

4.1 Introduction

The previous chapter contains characterizations of \((\omega)\)paracompactness along the lines of the characterizations of paracompactness obtained by Michael [28].

In this chapter, we introduce a stronger form of \((\omega)\)paracompactness, which we call \((\omega^s)\)paracompactness. We give an example to show that this notion is stronger than \((\omega)\)paracompactness. Using the notion of hereditarily closure preserving collection introduced by Bruke, Engelking and Lutzer [6], we obtain a set of characterizations of an \((\omega^s)\)paracompact space in terms of closure preserving covers, as done by Michael [29] for paracompact spaces.

\[\text{The results discussed in this chapter, are from our published paper: ‘Int. J. Pure Appl. Math.}\ 59\ (2010), \text{ no. 4, 429-434’ [4].}\]
We require the following known definitions and results.

**Definition 4.1.1.** (Michael [29]) A collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{J})$ is said to be $(\mathcal{J})$closure preserving if for every subcollection $\mathcal{B}$ of $\mathcal{A}$,

$$
\bigcup\{(\mathcal{J})clB \mid B \in \mathcal{B}\} = (\mathcal{J})cl(\bigcup\{B \mid B \in \mathcal{B}\}).
$$

**Definition 4.1.2.** (Bruke, Engelking and Lutzer [6]) A collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{J})$ is hereditarily $(\mathcal{J})$closure preserving if, whenever a subset $B_A$ of $\mathcal{A}$ is chosen for each $A \in \mathcal{A}$, the resulting collection $\mathcal{B} = \{B_A \mid A \in \mathcal{A}\}$ is $(\mathcal{J})$closure preserving.

**Definition 4.1.3.** (Michael [29]) A collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{J})$ is said to be $(\mathcal{J})$discrete if every $x \in X$ has a $(\mathcal{J})$open neighbourhood intersecting at most one $A \in \mathcal{A}$.

**Definition 4.1.4.** (Bruke, Engelking and Lutzer [6], Michael [29]) A collection $\mathcal{A}$ of subsets of $(X, \mathcal{J})$ is said to be $\sigma$−hereditarily $(\mathcal{J})$closure preserving (resp. $\sigma$−$(\mathcal{J})$discrete) if $\mathcal{A}$ can be written as a countable union of hereditarily $(\mathcal{J})$closure preserving (resp. $(\mathcal{J})$discrete) subcollections.

**Lemma 4.1.1.** (Michael [29]) If $\{C_\alpha \mid \alpha \in \mathcal{A}\}$ is a $(\mathcal{J})$locally finite collection of subsets of a topological space $(X, \mathcal{J})$ and if, for every $\alpha$, $\mathcal{B}_\alpha$ is a $(\mathcal{J})$closure preserving collection of subsets of $C_\alpha$, then the collection $\bigcup\{\mathcal{B}_\alpha \mid \alpha \in \mathcal{A}\}$ is also $(\mathcal{J})$closure preserving.
Lemma 4.1.2. (Dowker [10]) If $\{V_\lambda \mid \lambda \in \mathcal{C}\}$ be a disjoint collection of $(\mathcal{J})$ open subsets of a normal space $(X, \mathcal{J})$, if $D_\lambda \subset V_\lambda$ for each $\lambda$, and if $\bigcup_\lambda D_\lambda$ is $(\mathcal{J})$ closed, then there exist a $(\mathcal{J})$ discrete family $\{W_\lambda \mid \lambda \in \mathcal{C}\}$ of $(\mathcal{J})$ open subsets of $X$ such that $D_\lambda \subset W_\lambda \subset V_\lambda$, for all $\lambda$.

4.2 $(\omega^s)$paracompactness

We now introduce the notion of $(\omega^s)$paracompactness as follows:

Definition 4.2.1. An $(\omega)$topological space $X$ is said to be $(\omega^s)$paracompact if for every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a $(\mathcal{J}_m)$ locally finite $(\mathcal{J}_m)$ open refinement.

Obviously an $(\omega^s)$paracompact space is $(\omega)$paracompact. However, the converse is not true as seen below.

Example 4.2.1. Taking $X = [0, 1]$, let us define an $(\omega)$topological space $(X, \{\mathcal{J}_n\})$ as follows:

Let $I_n = [0, 1 - \frac{1}{n+1})$ and $\mathcal{T}_n = \mathcal{U} \mid I_n$, where $\mathcal{U}$ denotes the usual topology on $\mathbb{R}$. Now for any $n \in \mathbb{N}$, define $\mathcal{J}_n$ to be the topology generated by the base $\mathcal{T}_n \cup \mathcal{S}$, where $\mathcal{S}$ is the collection defined in Example 2.4.2. This space is $(\omega)$paracompact but not $(\omega^s)$paracompact:

Let $\mathcal{U}$ be any $(\omega)$open cover of $X$. Obviously $\mathcal{U}$ must contain
an interval \((a, 1]\) having 1 as the right end point. We choose \(n\) such that \(1 - \frac{1}{n+1} > a\). Let \(\mathcal{U}_1\) be a subclass of \(\mathcal{U}\) covering \([0, a]\). As \([0, a]\) is \((\omega)\)-compact, \(\mathcal{U}_1\) must contain a finite subcollection say \(\mathcal{V}\), which covers \([0, a]\). The family \(\mathcal{V} \cup \{(a, 1]\}\) forms a finite subcover of \(\mathcal{U}\). Hence \(X\) is \((\omega)\)-compact and so \((\omega)\)-paracompact.

Next we show \(X\) is not \((\omega^s)\)-paracompact. Let \(\mathcal{D}\) be an \((\omega)\)-open cover of \(X\). If possible, suppose \(X\) is \((\omega^s)\)-paracompact, then there exists an \(m \in \mathbb{N}\) having the requisite properties. We choose a \((\mathcal{J}_n)\)-open refinement \(\mathcal{V}\) of \(\mathcal{D}\) with \(n > m\) and containing sets of the form \((a, 1 - \frac{1}{n+1})\), \((b, 1]\), such that

\[
a < 1 - \frac{1}{n} < b < 1 - \frac{1}{n+1}
\]

(This is possible from the construction of the space \((X, \{\mathcal{J}_n}\})\). Then \(\mathcal{V}\) must have a \((\mathcal{J}_m)\)-locally finite \((\mathcal{J}_m)\)-open refinement. However, this is not possible. For, we see the portion \((1 - \frac{1}{n}, b)\) cannot be contained in a \((\mathcal{J}_m)\)-open refinement as \((1 - \frac{1}{n}, b)\) is a \((\mathcal{J}_n)\)-open set but it is not \((\mathcal{J}_m)\)-open for any \(m < n\). Hence \(X\) is not \((\omega^s)\)-paracompact.

The following theorems give us some characterizations of \((\omega^s)\)-paracompactness.

**Theorem 4.2.1.** Suppose, for each \(n \in \mathbb{N}\), \((X, \mathcal{J}_n)\) is a regular topological space. Then the following statements are equivalent:

(a) \(X\) is \((\omega^s)\)-paracompact.

(b) For every \((\omega)\)-open cover \(\mathcal{U}\) of \(X\), there exists an \(m\) such
(ω*)paracompactness 64

that for any \( n \), each \((\mathcal{J}_n)\)open refinement of \( \mathcal{U} \) has a \( \sigma-(\mathcal{J}_m)\)locally finite \((\mathcal{J}_m)\)open refinement.

(c) For every \((\omega)\)open cover \( \mathcal{U} \) of \( X \), there exists an \( m \) such that for any \( n \), each \((\mathcal{J}_n)\)open refinement of \( \mathcal{U} \) has a \((\mathcal{J}_m)\)open refinement \( \mathcal{V} \) and \( \mathcal{V} \) has a \((\mathcal{J}_m)\)locally finite refinement.

(d) For every \((\omega)\)open cover \( \mathcal{U} \) of \( X \), there exists an \( m \) such that for any \( n \), each \((\mathcal{J}_n)\)open refinement of \( \mathcal{U} \) has a \((\mathcal{J}_m)\)open refinement \( \mathcal{V} \) and \( \mathcal{V} \) has a \((\mathcal{J}_m)\)locally finite \((\mathcal{J}_m)\)closed refinement.

Proof. Similar to that of Theorem 3.3.1. \( \square \)

Lemma 4.2.2. Let \( X \) be an \((\omega)\)topological space in which for every \((\omega)\)open cover \( \mathcal{U} \) of \( X \), there exists an \( m \) such that for any \( n \), each \((\mathcal{J}_n)\)open refinement of \( \mathcal{U} \) has a \((\mathcal{J}_m)\)open refinement \( \mathcal{U}_0 \) and \( \mathcal{U}_0 \) has a \((\mathcal{J}_m)\)closure preserving \((\mathcal{J}_m)\)closed refinement. If \( \mathcal{U}_0 = \{U_\alpha|\alpha \in A\} \), then there exists a \((\mathcal{J}_m)\)closure preserving \((\mathcal{J}_m)\)closed refinement \( \{K_\alpha|\alpha \in A\} \) of \( \mathcal{U}_0 \) such that \( K_\alpha \subset U_\alpha, \alpha \in A \).

Proof. Let \( \mathcal{B} \) be a \((\mathcal{J}_m)\)closure preserving \((\mathcal{J}_m)\)closed refinement of \( \mathcal{U}_0 \). We write \( K_\alpha = \cup\{B \in \mathcal{B}|B \subset U_\alpha\} \) for all \( \alpha \). Then for any \( \alpha \), \( K_\alpha \subset U_\alpha \) and

\[
(\mathcal{J}_m)cl K_\alpha = (\mathcal{J}_m)cl (\cup\{B \in \mathcal{B}|B \subset U_\alpha\})
= \cup\{(\mathcal{J}_m)cl B | B \in \mathcal{B}, B \subset U_\alpha\}
= \cup\{B \in \mathcal{B}|B \subset U_\alpha\}
= K_\alpha.
\]
Therefore $K_\alpha$ is $(\mathcal{J}_m)$closed for each $\alpha \in A$. Since $\mathcal{B}$ is $(\mathcal{J}_m)$closure preserving, it follows that $\{K_\alpha | \alpha \in A\}$ is $(\mathcal{J}_m)$closure preserving.

We now prove the following theorem which provides us with a set of characterizations of $(\omega^*)$paracompactness in terms of closure preserving refinements.

**Theorem 4.2.3.** Suppose, for each $n \in \mathbb{N}$, $(X, \mathcal{J}_n)$ is a regular topological space. Then the following statements are equivalent:

(i) $X$ is $(\omega^*)$paracompact.

(ii) For every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a hereditarily $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$open refinement.

(iii) For every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a $\sigma$–hereditarily $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$open refinement.

(iv) For every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a $(\mathcal{J}_m)$open refinement $\mathcal{U}_0$ and $\mathcal{U}_0$ has a $(\mathcal{J}_m)$closure preserving refinement.

(v) For every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a $(\mathcal{J}_m)$open refinement $\mathcal{U}_0$ and $\mathcal{U}_0$ has a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed refinement.
(ω*) paracompactness

Proof. (i) ⇒ (ii): Follows from the fact that a \( (\mathcal{J}_m) \) locally finite collection of subsets of \( X \) is hereditarily \( (\mathcal{J}_m) \) closure preserving.

(ii) ⇒ (iii): The proof is trivial.

(iii) ⇒ (iv): Let \( \mathcal{U} \) be an \( (\omega) \) open cover of \( X \) and \( \mathcal{V} \) be a \( (\mathcal{J}_n) \) open refinement of \( \mathcal{U} \). By (iii), there exists a \( (\mathcal{J}_m) \) open refinement \( \mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i \) of \( \mathcal{V} \) such that each \( \mathcal{G}_i \) is hereditarily \( (\mathcal{J}_m) \) closure preserving. Let

\[
G_i = \bigcup \{ G \mid G \in \mathcal{G}_i \}, \\
K_1 = X, \\
K_i = X - \bigcup_{j=1}^{i-1} G_j, i = 2, 3, ...
\]

Then \( \{ K_i \mid i \in \mathbb{N} \} \) is \( (\mathcal{J}_m) \) locally finite. For each \( i \in \mathbb{N} \), let

\[
\mathcal{B}_i = \{ G \cap K_i \mid G \in \mathcal{G}_i \}.
\]

Since each \( \mathcal{G}_i \) is hereditarily \( (\mathcal{J}_m) \) closure preserving, each \( \mathcal{B}_i \) is \( (\mathcal{J}_m) \) closure preserving. Therefore by Lemma 4.1.1, it follows that \( \mathcal{B} = \bigcup \{ \mathcal{B}_i \mid i \in \mathbb{N} \} \) is \( (\mathcal{J}_m) \) closure preserving. Also \( \mathcal{B} \) is a refinement of \( \mathcal{G} \). Taking \( \mathcal{G} \) as \( \mathcal{U}_0 \), (iv) follows.

(iv) ⇒ (v): Let \( \mathcal{U} \) be an \( (\omega) \) open cover of \( X \) and \( \mathcal{U}_0 \) be a \( (\mathcal{J}_m) \) open refinement of a \( (\mathcal{J}_n) \) open refinement of \( \mathcal{U} \). Consider \( x \in X \) and \( U_x \in \mathcal{U}_0 \) such that \( x \in U_x \). By the \( (\mathcal{J}_m) \) regularity of \( X \), we get a \( V_x \in \mathcal{J}_m \) with \( x \in V_x \subset (\mathcal{J}_m) \text{cl} V_x \subset U_x \). Then \( \mathcal{V} = \{ V_x \mid x \in X \} \) is a \( (\mathcal{J}_m) \) open refinement of \( \mathcal{U} \). Taking \( n = m \), we may consider \( \mathcal{V} \) as a \( (\mathcal{J}_n) \) open refinement of \( \mathcal{U} \). By (iv), \( \mathcal{V} \) has a \( (\mathcal{J}_m) \) closure preserving refinement \( \mathcal{B} \). For \( B \in \mathcal{B} \), there
exists a $V_x$ such that $B \subset V_x$ and $(\mathcal{J}_m)clB \subset (\mathcal{J}_m)clV_x \subset U_x$. Therefore $\{(\mathcal{J}_m)clB \mid B \in \mathcal{B}\}$ is the required refinement of $U_0$.

$(v) \Rightarrow (i)$: Let $\mathcal{U}$ be an $(\omega)$open cover of $X$ and $U_0 = \{U_\alpha \mid \alpha \in A\}$ be a $(\mathcal{J}_m)$open refinement of some $(\mathcal{J}_n)$open refinement of $\mathcal{U}$. Let the index set $A$ be well-ordered.

For each positive integer $i$, we first construct a collection $\{K_{\alpha,i} \mid \alpha \in A\}$ of subsets of $X$ satisfying the following conditions:

1. $\{K_{\alpha,i} \mid \alpha \in A\}$ is a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed cover of $X$ with $K_{\alpha,i} \subset U_\alpha$.
2. $K_{\alpha,i+1} \cap K_{\beta,i} = \emptyset$ for all $\alpha > \beta$.

Using Lemma 4.2.2, we get the cover $\{K_{\alpha,1} \mid \alpha \in A\}$ satisfying (1) for $i = 1$. Suppose the covers $\{K_{\alpha,i} \mid \alpha \in A\}$ have been constructed for $i = 1, 2, ..., r$. Let

$$U_{\alpha,r+1} = U_\alpha - \bigcup_{\beta < \alpha} K_{\beta,r}, \text{ for all } \alpha \in A,$$

$U_{\alpha,r+1}$ is $(\mathcal{J}_m)$open. It follows that $\{U_{\alpha,r+1} \mid \alpha \in A\}$ is a $(\mathcal{J}_m)$open refinement of $U_0$ and hence of $\mathcal{U}$. By Lemma 4.2.2, we get a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed refinement $\{K_{\alpha,r+1} \mid \alpha \in A\}$ such that $K_{\alpha,r+1} \subset U_{\alpha,r+1}$ for all $\alpha$. The condition (1) for $i = r + 1$ and the condition (2) for $i = r$ are obviously satisfied.

Now we write

$$V_{\alpha,i} = X - \bigcup_{\beta \neq \alpha} K_{\beta,i}, \text{ for all } \alpha, i$$

and show that:
(a) \( \{ V_{\alpha,i} \mid \alpha \in \mathbb{A}, \ i \in \mathbb{N} \} \) is a \( (\mathcal{J}_m) \) open cover of \( X \) and \( V_{\alpha,i} \subset U_\alpha \) for all \( \alpha \) and \( i \).

(b) \( V_{\alpha,i} \cap V_{\beta,i} = \emptyset \) whenever \( \alpha \neq \beta \).

Since \( \{ K_{\beta,i} \mid \beta \in \mathbb{A} \} \) is \( (\mathcal{J}_m) \) closure preserving and each \( K_{\beta,i} \) is \( (\mathcal{J}_m) \) closed, the \( (\mathcal{J}_m) \) openness of each \( V_{\alpha,i} \) follows. Also since \( \{ K_{\alpha,i} \mid \alpha \in \mathbb{A} \} \) forms a cover of \( X \), we get

\[
V_{\alpha,i} \subset K_{\alpha,i} \subset U_\alpha, \text{ for all } \alpha \text{ and } i,
\]

and hence (b) follows. It remains to show that \( \{ V_{\alpha,i} \mid \alpha \in \mathbb{A}, \ i \in \mathbb{N} \} \) is a cover of \( X \).

Let \( x \in X \) be any point. If \( \alpha_l = \min \{ \alpha_i \mid i \in \mathbb{N} \} \) where, for all \( i \in \mathbb{N} \), \( \alpha_i = \min \{ \alpha \in \mathbb{A} \mid x \in K_{\alpha,i} \} \), then

\[
x \in V_{\alpha_l,l+1} = X - \left( \bigcup_{\beta \neq \alpha_l} K_{\beta,l+1} \right).
\]

In fact, if \( \alpha < \alpha_l \), then \( x \notin K_{\alpha,l+1} \) by the definition of \( \alpha_l \) and if \( \alpha > \alpha_l \), then \( x \notin K_{\alpha,l+1} \) by the relation (2) with \( i = l \) and \( \beta = \alpha_l \) and by the fact that \( x \in K_{\alpha_l,l} \). Thus \( \{ V_{\alpha,i} \mid \alpha \in \mathbb{A}, \ i \in \mathbb{N} \} \) forms a cover of \( X \). Therefore it is a \( (\mathcal{J}_m) \) open refinement of \( \mathcal{U}_0 \) and hence of \( \mathcal{U} \).

By Lemma 4.2.2, we get a \( (\mathcal{J}_m) \) closure preserving \( (\mathcal{J}_m) \) closed refinement \( \{ F_{\alpha,i} \mid \alpha \in \mathbb{A}, \ i \in \mathbb{N} \} \) of \( \{ V_{\alpha,i} \mid \alpha \in \mathbb{A}, \ i \in \mathbb{N} \} \) such that \( F_{\alpha,i} \subset V_{\alpha,i} \) for all \( \alpha \) and \( i \).

Again we see that the topological space \( (X, \mathcal{J}_m) \) is \( (\mathcal{J}_m) \) normal: Let \( A \) and \( B \) be two disjoint \( (\mathcal{J}_m) \) closed subsets of \( X \). Then \( \{ X - A, X - B \} \) is a \( (\mathcal{J}_m) \) open cover of \( X \).
Therefore there exists a \((J_m)\)closed cover \(\{F, K\}\) of \(X\) such that \(F \subset X - A\) and \(K \subset X - B\). Obviously \(A \subset X - F\), \(B \subset X - K\) and \((X - F) \cap (X - K) = \emptyset\). So \(X\) is \((J_m)\)normal.

Further, \(\{F_{\alpha,i} \mid \alpha \in A, i \in \mathbb{N}\}\) is a \((J_m)\)closure preserving \((J_m)\)closed refinement so \(\bigcup_{\alpha \in A} F_{\alpha,i}\) is \((J_m)\)closed. Using Lemma 4.1.2, we obtain for each \(i \in \mathbb{N}\), a \((J_m)\)discrete collection \(\{W_{\alpha,i} \mid \alpha \in A\}\) of \((J_m)\)open sets such that

\[
F_{\alpha,i} \subset W_{\alpha,i} \subset V_{\alpha,i}, \text{ for all } \alpha.
\]

The collection \(\{W_{\alpha,i} \mid \alpha \in A, i \in \mathbb{N}\}\) is a \(\sigma-(J_m)\)discrete and hence \(\sigma-(J_m)\)locally finite \((J_m)\)open refinement of \(U_0\). Therefore by Theorem 4.2.1, \(X\) is \((\omega^*)\)paracompact. \(\square\)