Chapter VIII

(ω)TOPOLOGICAL CONNECTEDNESS AND
HYPERCONNECTEDNESS

In this chapter, the notions of connectedness and hyperconnectedness in an (ω)topological space are introduced and studied. We also introduce (ω)semiopen sets as an analogue of semiopen sets introduced by Levine.

8.1 Introduction

The notion of (ω)connectedness is introduced in the next section. Some of the results obtained in this section are as follows:

An (ω)topological space $X$ is (ω)connected if the topology of all its $(\sigma\omega)$open sets is connected (Theorem 8.2.1).

A maximal (ω)connected space is $(\omega)T_0$ (Theorem 8.2.6).

An (ω)open (ω)connected subset of a maximal

The results of this chapter have been published in the paper: ‘Note Mat. 31 (2011), no. 2, 93-101’ [5].
\((\omega)\)connected space is also maximal \((\omega)\)connected (Theorem 8.2.7).

\((\omega)\)hyperconnected spaces are introduced in the third section of this chapter. Also we introduce the notion of \((\omega)\)semiopen sets as an analogue of semiopen sets [24]. We summarize below some of the results obtained in this section.

An \((\omega)\)topological space \(X\) is \((\omega)\)hyperconnected if and only if the topology of all its \((\sigma\omega)\)open sets is hyperconnected (Theorem 8.3.1).

An \((\omega)\)topological space is \((\omega)\)hyperconnected if and only if the collection of all non-empty \((\omega)\)semiopen sets forms a filter (Theorem 8.3.2).

In a maximal \((\omega)\)hyperconnected space the collection of all \((\omega)\)open sets is identical to the collection of all \((\omega)\)semiopen sets (Corollary 8.3.4).

If an \((\omega)\)topological space is maximal \((\omega)\)hyperconnected, then the collection of all non-empty \((\omega)\)semiopen sets forms an ultrafilter (Theorem 8.3.5).

An \((\omega)\)topological space is \((\omega)\)hyperconnected \((\omega)\)door if and only if the collection of all non-empty \((\omega)\)open sets forms an ultrafilter (Theorem 8.3.7).

An \((\omega)\)hyperconnected, \((\omega)\)door space is an maximal
(ω)hyperconnected and minimal (ω)door space. (Theorem 8.3.8).

An (ω)topological space is maximal (ω)hyperconnected if and only if it is (ω)submaximal and (ω)hyperconnected (Theorem 8.3.9).

We require the following known definitions.

**Definition 8.1.1.** (Levine [24]) A set $A$ in a topological space $(X, J)$ is said to be *semiopen* if there exists an open set $O$ such that $O \subset A \subset (J)\text{cl}O$.

**Definition 8.1.2.** (Levine [25]) Let $(X, J)$ be a topological space and $J \subset J^*$. Then $J^*$ is called a *simple extension* of $J$ if there exists an $A \notin J$ such that $J^* = \{O \cup (O' \cap A) | O, O' \in J\}$. We write $J^* = J(A)$.

**Definition 8.1.3.** (Kelley [19], p. 76) A topological space $(X, J)$ is said to be a *door space* if every subset of $X$ is either open or closed.

**Definition 8.1.4.** (Mathew [27]) A topological space is said to be *submaximal* if every dense subset is open.

We recapitulate the following notions discussed in Section 2.2 of Chapter II.

**Definition 8.1.5.** A non-empty set $X$ equipped with an increasing sequence $\{J_n\}$ of topologies is called an (ω)*topological space* and the sequence $\{J_n\}$ of topologies is called an (ω)*topology* on $X$. 
In an \((\omega)\)topological space the union and intersection of a finite number of \((\omega)\)open sets is \((\omega)\)open. However, the countable union of \((\omega)\)open sets may not be \((\omega)\)open. These sets are called \((\sigma\omega)\)open sets. Since the arbitrary union of \((\mathcal{J}_n)\)open sets is \((\mathcal{J}_n)\)open, the union of an arbitrary number of \((\omega)\)open sets is also \((\sigma\omega)\)open. Similarly, \((\delta\omega)\)closed sets are defined as the intersection of a countable number of \((\omega)\)closed sets. The intersection of all \((\omega)\)closed sets containing a set \(A\) is called the \((\omega)\)closure of \(A\) and is denoted by \((\omega)\text{cl}A\). Obviously, it is a \((\delta\omega)\)closed set.

It is clear that the class \(\mathcal{T}\) of all \((\sigma\omega)\)open sets in \(X\) forms a topology on \(X\).

**Definition 8.1.6.** A set \(A \subset X\) is said to be \((\omega)\)dense in \(X\) if for every non-empty \((\omega)\)open set \(G\), \(A \cap G \neq \emptyset\).

It is easy to see that a set \(A \subset X\) is \((\omega)\)dense if and only if \((\omega)\text{cl}E = X\).

### 8.2 \((\omega)\)connectedness

This section deals mainly with connectedness in an \((\omega)\)topological space, we call it \((\omega)\)connectedness. We also introduce the notion of a maximal (resp. minimal) \((\omega)\)topological space. These notions are used to obtain certain results.

We introduce the following definitions.
Definition 8.2.1. An \((\omega)\)topological space \(X\) is said to be \((\omega)\)connected, if \(X\) cannot be expressed as the union of two disjoint non-empty \((\omega)\)open sets.

Obviously, when \(J_n = J\) for all \(n\), the space \((X, \{J_n\})\) is an \((\omega)\)connected space if and only if the topological space \((X, J)\) is connected. Further, an \((\omega)\)topological space \((X, \{J_n\})\) is \((\omega)\)connected if and only if \((X, J_n)\) is \((\omega)\)connected for all \(n\).

Definition 8.2.2. A subset \(Y\) of an \((\omega)\)topological space \((X, \{J_n\})\) is said to be \((\omega)\)connected, if the \((\omega)\)topological space \((Y, \{J_n|Y\})\) is \((\omega)\)connected.

Theorem 8.2.1. If the space \((X, \mathcal{T})\) is connected, then the \((\omega)\)topological space \((X, \{J_n\})\) is \((\omega)\)connected.

Proof. Since every \((\omega)\)open set is \((\sigma\omega)\)open, the result follows.

We now give an example to show that the converse of the above theorem is not true.

Example 8.2.1. The \((\omega)\)topological space \((\mathbb{N}, \{J_n\})\) considered in Example 2.3.5 is \((\omega)\)connected. The union of no two disjoint \((\omega)\)open sets of this \((\omega)\)topological space can be \(\mathbb{N}\), as from the definition of the \((\omega)\)topological space it is clear that any two disjoint \((\omega)\)open sets must be finite.

However, the topology of all \((\sigma\omega)\)open sets of the above \((\omega)\)topology is not connected. The set of all even natural numbers and the set of all odd natural numbers are two disjoint
(σω)open sets whose union is $\mathbb{N}$.

If $X$ is not ($\omega$)connected, then there exist two disjoint non-empty ($\omega$)open sets $A$ and $B$ such that $X = A \cup B$. In this case, we write $X = A|B$. We call it an ($\omega$)separation of $X$. Since the two ($\omega$)open sets $A$ and $B$ belong to some $\mathcal{J}_n$, it is clear that if $X$ is not ($\omega$)connected then for some $n$, the topological space $(X, \mathcal{J}_n)$ is not connected. As a consequence we get the following theorem.

**Theorem 8.2.2.** If $C$ is an ($\omega$)connected subset of an ($\omega$)topological space $X$ which has the ($\omega$)separation $X = A|B$, then either $C \subset A$ or $C \subset B$.

**Proof.** Straightforward. $\square$

**Corollary 8.2.3.** The union of a family of ($\omega$)connected sets having non-empty intersection is ($\omega$)connected.

**Proof.** Straightforward $\square$

**Corollary 8.2.4.** If any two points of $Y \subset X$ are contained in some ($\omega$)connected subset of $Y$, then $Y$ is ($\omega$)connected.

**Proof.** Straightforward. $\square$

**Corollary 8.2.5.** If $C$ is an ($\omega$)connected set in $X$ and $C \subset E \subset (\omega)\text{cl}C$, then $E$ is ($\omega$)connected.

**Proof.** If $E$ is not ($\omega$)connected, then it has an ($\omega$)separation $E = A|B$. By Theorem 8.2.2, $C \subset A$ or $C \subset B$. Let us assume
\( C \subset A \). Suppose \( A, B \in \mathcal{J}_n|E \). Then
\[
B = B \cap (\omega)clC \\
\subset B \cap (\mathcal{J}_n|E)clA \\
= \emptyset \text{ (since } A \cap B = \emptyset\).
\]
This is a contradiction and so \( E \) is \((\omega)\)connected. \( \square \)

**Definition 8.2.3.** An \((\omega)\)topology \( \{\mathcal{J}'_n\} \) on \( X \) is said to be **stronger** (resp. **weaker**) than an \((\omega)\)topology \( \{\mathcal{J}_n\} \) on \( X \) if
\[
\bigcup_n \mathcal{J}_n \subset \bigcup_n \mathcal{J}'_n \quad \text{(resp. } \bigcup_n \mathcal{J}'_n \subset \bigcup_n \mathcal{J}_n \text{).}
\]
If, in addition, \( \bigcup_n \mathcal{J}_n \neq \bigcup_n \mathcal{J}'_n \), then \( \{\mathcal{J}'_n\} \) is said to be **strictly stronger** (resp. **strictly weaker**) than \( \{\mathcal{J}_n\} \).

**Definition 8.2.4.** An \((\omega)\)topological space \((X, \{\mathcal{J}_n\})\) with property \( P \) is said to be **maximal** (resp. **minimal**) if for any other \((\omega)\)topology \( \{\mathcal{J}'_n\} \) strictly stronger (resp. strictly weaker) than \( \{\mathcal{J}_n\} \), the space \((X, \{\mathcal{J}'_n\})\) cannot have this property.

**Theorem 8.2.6.** If \( X \) is maximal \((\omega)\)connected, then \( X \) is \((\omega)T_0\).

**Proof.** Suppose, if possible \( X \) is not \((\omega)T_0\), then there exist \( x, y \in X \), \( x \neq y \) such that \( x \in (\omega)cl\{y\} \) and \( y \in (\omega)cl\{x\} \). Let \( \mathcal{J}'_n \) be the topology generated by \( \mathcal{J}_n \cup \{y\} \). The \((\omega)\)topological space \((X, \{\mathcal{J}'_n\})\) is not \((\omega)\)connected. Let \( X = A|B \) be an \((\omega)\)separation of \((X, \{\mathcal{J}'_n\})\). By Theorem 8.2.2, either \( \{x, y\} \subset A \) or \( \{x, y\} \subset B \). Suppose \( \{x, y\} \subset A \). There exists a set \( U \in \mathcal{J}_n \) with \( x \in U \). But \( U \) also contains \( y \). Since for any point
$z \in A$ with $z \neq y$, there is a ($\mathcal{J}_n$)open neighbourhood $G \subset A$ of $z$, it follows that $A \in \mathcal{J}_n$. Clearly $B \in \mathcal{J}_n$. Thus $A|B$ is an ($\omega$)separation of $X$ which is a contradiction. \qed

Below we provide examples to show that a maximal ($\omega$)connected space may or may not be ($\omega$)$T_1$.

**Example 8.2.2.** Let us consider the ($\omega$)topological space defined in Example 2.3.5. Clearly $(\mathbb{N}, \{\mathcal{J}_n\})$ is an ($\omega$)$T_1$—space as for any two points $l, m \in \mathbb{N}$, the sets $\{l\}$ and $\{m\}$ are ($\omega$)open.

Also this space is ($\omega$)connected and can therefore be extended to a maximal ($\omega$)connected space.

**Example 8.2.3.** Let us consider the ($\omega$)topological space $(\mathbb{N}, \{\mathcal{J}_n\})$ where

$$\mathcal{J}_n = \{\emptyset\} \cup \{E \subset \mathbb{N} | 1 \in E\}$$

for all $n$.

We consider another ($\omega$)topological space $(\mathbb{N}, \{\mathcal{J}'_n\})$ which is stronger than $(\mathbb{N}, \{\mathcal{J}_n\})$. It must contain a set $A$ which is ($\mathcal{J}'_n$)open for some $n$, but not ($\mathcal{J}_n$)open for any $n$. Obviously $1 \notin A$. Therefore the sets $A$ and $\mathbb{N} - A$ form an ($\omega$)separation of $(\mathbb{N}, \{\mathcal{J}'_n\})$. So $(\mathbb{N}, \{\mathcal{J}'_n\})$ is not ($\omega$)connected. However, $(\mathbb{N}, \{\mathcal{J}_n\})$ is ($\omega$)connected as it does not contain any pair of disjoint non-empty ($\omega$)open sets. Thus $(\mathbb{N}, \{\mathcal{J}_n\})$ is a maximal ($\omega$)connected space.

However, $(\mathbb{N}, \{\mathcal{J}_n\})$ is not an ($\omega$)$T_1$—space as every non-empty ($\omega$)open set contains 1.
Theorem 8.2.7. Let \((X, \{J_n\})\) be maximal \((\omega)\)connected and \(G\) be an \((\omega)\)open \((\omega)\)connected subset of \(X\). Then \((G, \{J_n|G\})\) is maximal \((\omega)\)connected.

Proof. If possible, suppose \( (G, \{J_n|G\})\) is not maximal \((\omega)\)connected. Let \(\{L_n\}\) be an \((\omega)\)topology on \(G\) strictly stronger than \(\{J_n|G\}\) and \((G, \{L_n\})\) is \((\omega)\)connected. Let \(H \subset G\) be such that \(H \in L_{n_0} - J_{n_0}|G\) for some \(n_0\). If \(Q_n\) is a topology on \(G\) generated by \((J_n|G) \cup \{H\}\), then \(\{Q_n\}\) is an \((\omega)\)connected \((\omega)\)topology on \(G\). Also if \(K_n\) is the topology on \(X\) generated by \(J_n \cup \{H\}\), then the \((\omega)\)topology \(\{K_n\}\) on \(X\) is strictly stronger than \(\{J_n\}\) and so \((X, \{K_n\})\) is not \((\omega)\)connected. Let \(X = A | B\) be an \((\omega)\)separation of \((X, \{K_n\})\). Then either \(G \subset A\) or \(G \subset B\), since, otherwise, \((G \cap A)|(G \cap B)\) is an \((\omega)\)separation of \((G, \{Q_n\})\). Suppose \(G \subset A\). Since \(G \in \bigcup_n J_n\), it follows that \(A \in \bigcup_n J_n\). But obviously \(B \in \bigcup_n J_n\). Therefore \(X = A | B\) is an \((\omega)\)separation of \((X, \{J_n\})\) which is a contradiction. 

Definition 8.2.5. Let \(x \in X\). The component \(C(x)\) of \(x\) in \(X\) is the union of all \((\omega)\)connected subsets of \(X\) containing \(x\).

From Corollary 8.2.3, it follows that \(C(x)\) is \((\omega)\)connected.

Theorem 8.2.8. In an \((\omega)\)topological space \(X\),

(i) each component \(C(x)\) is a maximal \((\omega)\)connected set in \(X\),
(ii) the set of all distinct components in \(X\) forms a partition of \(X\).
(iii) each $C(x)$ is $(\delta \omega)$ closed in $X$.

Proof. Straightforward.

8.3 $(\omega)$ hyperconnectedness

In this section, we introduce the notions of $(\omega)$ hyperconnectedness and $(\omega)$ semiopen sets. We also define $(\omega)$ door spaces and $(\omega)$ submaximal spaces. Some results on the notions introduced are obtained.

Definition 8.3.1. $X$ is said to be $(\omega)$ hyperconnected if for any two non-empty $(\omega)$ open sets $U$ and $V$, $U \cap V \neq \emptyset$.

Therefore for any non-empty $(\omega)$ open set $U$, $(\omega)clU = X$, since otherwise $V_1 = X - (\omega)clU$ is a non-empty $(\sigma \omega)$ open set and $U \cap V_1 = \emptyset$ which implies that for any non-empty $(\omega)$ open set $V \subset V_1$, we have $U \cap V = \emptyset$.

Theorem 8.3.1. $X$ is $(\omega)$ hyperconnected if and only if the topological space $(X, T)$ is hyperconnected.

Proof. Suppose $X$ is $(\omega)$ hyperconnected. Let $A$ and $B$ be two non-empty $(T)$ open sets. Then $A = \bigcup_{i=1}^{\infty} A_i$, $B = \bigcup_{j=1}^{\infty} B_j$ where $A_i$ and $B_j$ are non-empty $(\omega)$ open sets. Now

$$A \cap B \supseteq A_i \cap B_j \neq \emptyset.$$ 

Thus $(X, T)$ is hyperconnected.
Conversely, since each (ω)open set is (σω)open set, the hyperconnectedness of the space \((X, T)\), implies that the (ω)topological space \(X\) is (ω)hyperconnected.

\[\text{Definition 8.3.2.}\] A set \(A \subset X\) is said to be (ω)semiopen if there exists an \(n\) such that for some \(U \in J_n\), we have \(U \subset A \subset (ω)cl U\).

Let \(SO_ω(X, \{J_n\})\) or, simply, \(SO_ω(X)\) denote the set of all (ω)semiopen sets. If some set \(A\) satisfies the above relation for some set \(U \in J_n\), we say that \(A\) is \((J_n − ω)\)semiopen. The set of all \((J_n − ω)\)semiopen sets is denoted by \((J_n)SO_ω(X)\). Thus

\[SO_ω(X) = \bigcup_n(J_n)SO_ω(X)\]

\[\text{Theorem 8.3.2.}\] X is (ω)hyperconnected if and only if \(SO_ω(X) − \{∅\}\) is a filter.

\[\text{Proof.}\] Suppose X is (ω)hyperconnected. Let \(A, B \in SO_ω(X) − \{∅\}\). There exists a \(k \in \mathbb{N}\) such that for some \(U\) and \(V\) with \(U, V \in J_k\), we have

\[U \subset A \subset (ω)cl U,\]
\[V \subset B \subset (ω)cl V.\]

Since \(X\) is (ω)hyperconnected, \(U \cap V \neq ∅\) and \((ω)cl(U \cap V) = X\). Therefore \(A \cap B \neq ∅\) and

\[U \cap V \subset A \cap B \subset (ω)cl(U \cap V).\]
Thus $A \cap B \in SO_\omega(X) - \{\emptyset\}$. Again if $B \supset A \in SO_\omega(X) - \{\emptyset\}$, there exists, for some $k$, a $U \in \mathcal{J}_k$ such that

$$U \subset A \subset (\omega)clU \text{ and so } U \subset B \subset (\omega)clU \text{ (since } (\omega)clU = X).$$

Hence $B \in SO_\omega(X) - \{\emptyset\}$. Therefore $SO_\omega(X) - \{\emptyset\}$ is a filter.

Since every $(\omega)$open set is $(\omega)$semiopen, the converse follows.

It is easy to see that the union of an arbitrary number of $(\mathcal{J}_n - \omega)$semiopen sets is $(\mathcal{J}_n - \omega)$semiopen. Also if $X$ is $(\omega)$hyperconnected, then the intersection of a finite number of $(\mathcal{J}_n - \omega)$semiopen sets is $(\mathcal{J}_n - \omega)$semiopen. Thus if $X$ is $(\omega)$hyperconnected, then the class $(\mathcal{J}_n)SO_\omega(X) = \mathcal{S}_n$ forms a topology on $X$ and $\mathcal{S}_n \subset \mathcal{S}_{n+1}$. Hence $\{\mathcal{S}_n\}$ is an $(\omega)$topology on $X$.

From Theorem 8.3.2, we get the following result.

**Theorem 8.3.3.** If $X$ is $(\omega)$hyperconnected, then so is $(X, \{\mathcal{S}_n\})$.

**Corollary 8.3.4.** If $X$ is maximal $(\omega)$hyperconnected, then $\bigcup_n \mathcal{J}_n = \bigcup_n \mathcal{S}_n$.

**Proof.** It is obvious that $\bigcup_n \mathcal{J}_n \subset \bigcup_n \mathcal{S}_n$. Further by Theorem 8.3.3 $(X, \{\mathcal{S}_n\})$ is $(\omega)$hyperconnected. Therefore by the maximality of $(\omega)$hyperconnectedness of $X$ it follows that $\bigcup_n \mathcal{S}_n \subset \bigcup_n \mathcal{J}_n$. Hence $\bigcup_n \mathcal{J}_n = \bigcup_n \mathcal{S}_n$. □
In an \((\omega)\)topological space \(X\), for any set \(A \notin \bigcup_n \mathcal{J}_n\), let \(\mathcal{J}_n(A)\) denote the simple extension of \(\mathcal{J}_n\). Then \((X, \{\mathcal{J}_n(A)\})\) forms an \((\omega)\)topology on \(X\) and \(\mathcal{J}_n \subset \mathcal{J}_n(A)\) for all \(n\). We call \(\{\mathcal{J}_n(A)\}\), a simple extension of \(\{\mathcal{J}_n\}\).

**Theorem 8.3.5.** If \(X\) is maximal \((\omega)\)hyperconnected, then \(SO_\omega(X) - \{\emptyset\}\) is an ultrafilter.

**Proof.** Suppose \(X\) is maximal \((\omega)\)hyperconnected. For \(E \subset X\), suppose \(E \notin SO_\omega(X) - \{\emptyset\}\). Then \(E \notin \bigcup_n \mathcal{J}_n\). Let us consider the simple extension \(\{\mathcal{J}_n(E)\}\) of \(\{\mathcal{J}_n\}\). Since \(X\) is maximal \((\omega)\)hyperconnected, \((X, \{\mathcal{J}_n(E)\})\) is not \((\omega)\)hyperconnected. Therefore for some \(n\), there exist two non-empty sets \(G, H \in \mathcal{J}_n(E)\) such that \(G \cap H = \emptyset\). Let

\[
G = G_1 \cup (G_2 \cap E) \quad \text{and} \quad H = H_1 \cup (H_2 \cap E)
\]

where \(G_1, G_2, H_1, H_2 \in \mathcal{J}_n\).

Then \(G_1 \cap H_1 = \emptyset\). Since \(X\) is \((\omega)\)hyperconnected, either \(G_1 = \emptyset\) or \(H_1 = \emptyset\). Suppose \(G_1 = \emptyset\). If \(H_1 = \emptyset\), then \(G_2 \neq \emptyset\) and \(H_2 \neq \emptyset\), since \(G \neq \emptyset\) and \(H \neq \emptyset\). Thus by \((\omega)\)hyperconnectedness of \(X\), \(G_2 \cap H_2 \neq \emptyset\). Again since \(G \cap H = \emptyset\), we have \(G_2 \cap H_2 \cap E = \emptyset\). Hence \(G_2 \cap H_2 \subset E^c\), and therefore by Theorem 8.3.2, \(E^c \in SO_\omega(X) - \{\emptyset\}\). Now consider the case \(H_1 \neq \emptyset\). Since \(G \neq \emptyset\), we have \(G_2 \neq \emptyset\). Therefore \(G_2 \cap H_1 \neq \emptyset\). From the relation \(G \cap H = \emptyset\), it follows that \((G_2 \cap E) \cap H_1 = \emptyset\). Hence \(G_2 \cap H_1 \subset E^c\), and so \(E^c \in SO_\omega(X) - \{\emptyset\}\). Thus \(SO_\omega(X) - \{\emptyset\}\) is an ultrafilter. \(\Box\)

Using Corollary 8.3.4, we get the following result.
Corollary 8.3.6. If $X$ is maximal $(\omega)$hyperconnected, then the class of all non-empty $(\omega)$open sets is an ultrafilter.

Definition 8.3.3. $X$ is said to be an $(\omega)$door space if for every subset $E$ of $X$, $E \in \mathcal{J}_n$ or $E^c \in \mathcal{J}_n$ for some $n$.

We now show that for an $(\omega)$door space $X$ the topological spaces $(X, \mathcal{J}_n)$ need not be door spaces.

Example 8.3.1. Let us define an $(\omega)$topological space $(\mathbb{N}, \{\mathcal{J}_n\})$ as follows:

$$\mathcal{J}_n = \{\emptyset, \mathbb{N}\} \cup \{E \subset \{1, 2, \ldots, n\} | 1 \in E\} \text{ for all } n < 10,$$

$$\mathcal{J}_n = \{\emptyset\} \cup \{E \subset \mathbb{N} | 1 \in E\} \text{ for all } n \geq 10.$$

$(\mathbb{N}, \{\mathcal{J}_n\})$ is an $(\omega)$door space as for any $E \subset \mathbb{N}$ either $E$ or $E^c$ contains 1 and belongs to $\mathcal{J}_{10}$.

But for any $n < 9$, the topological space $(\mathbb{N}, \mathcal{J}_n)$ is not a door space, as for example neither of the sets $\{2\}$ and $\mathbb{N} - \{2\}$ can be $(\mathcal{J}_n)$open for any $n < 9$.

Example 8.3.2. Taking $X = [0, 1)$, let us define an $(\omega)$topological space $(X, \{\mathcal{J}_n\})$ as follows:

$\mathcal{J}_n$ is the topology generated by the subbase

$$\{E \subset [0, 1 - \frac{1}{n+1}) | 0 \in E\} \cup \{\emptyset\} \cup \{E \subset X | 0 \in E \text{ and } 1 \text{ is a limit point of } E\}.$$

It is easy to see that $(X, \{\mathcal{J}_n\})$ forms an $(\omega)$door space. However, $(X, \mathcal{J}_n)$ is not a door space for any $n$. 
Definition 8.3.4. A property $P$ of an ($\omega$)topological space $X$ is said to be contractive (resp. expansive) if it is possessed by ($\omega$)topological spaces $(X, \{J'_n\})$ whenever it is possessed by $(X, \{J_n\})$, where the ($\omega$)topologies $\{J'_n\}$ are weaker (resp. stronger) than $\{J_n\}$.

It is clear that ($\omega$)connectedness and ($\omega$)hyperconnectedness are contractive properties while ($\omega$)doorness is an expansive property.

Theorem 8.3.7. $X$ is an ($\omega$)hyperconnected ($\omega$)door space if and only if $\mathcal{F} = (\bigcup_n J_n) - \{\emptyset\}$ is an ultrafilter.

Proof. Suppose $X$ is an ($\omega$)hyperconnected ($\omega$)door space. Then for $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$. Now let $B \supset A \in \mathcal{F}$. If $B = X$, then $B \in \mathcal{F}$. If $B \neq X$, then $B^c \notin \mathcal{F}$, since otherwise $A \cap B^c \neq \emptyset$. Therefore $B \in \mathcal{F}$. Hence $\mathcal{F}$ is a filter and so an ultrafilter.

The converse part is obvious.

Theorem 8.3.8. If $X$ is ($\omega$)hyperconnected and ($\omega$)door, then $X$ is maximal ($\omega$)hyperconnected and minimal ($\omega$)door.

Proof. Let $\{J'_n\}$ be an ($\omega$)topology on $X$ stronger than $\{J_n\}$ such that $(X, \{J'_n\})$ is ($\omega$)hyperconnected. If possible, suppose $G$ be a non-empty set with $G \in J'_m$ for some $m$ and $G \notin \bigcup J_n$. Since $X$ is ($\omega$)door, $X - G \in J_l$ for some $l$. Hence $X - G \in \bigcup J'_n$. This contradicts the fact that $(X, \{J'_n\})$ is ($\omega$)hyperconnected. Thus $G \in \bigcup J_n$. Therefore $\bigcup J'_n = \bigcup J_n$. 

\[
(\omega)\text{hyperconnectedness}
\]
Again let $(X, \{J'_n\})$ be an $(\omega)$door space such that $\bigcup_n J'_n \subset \bigcup_n J_n$. Suppose, if possible, $G$ is a non-empty set with $G \in \bigcup_n J_n$ and $G \notin \bigcup_n J'_n$. But then $X - G \in \bigcup_n J'_n$. So $X - G \in \bigcup_n J_n$ which contradicts the $(\omega)$hyperconnectedness of $(X, \{J_n\})$. Therefore $\bigcup_n J'_n = \bigcup_n J_n$.

**Definition 8.3.5.** $X$ is said to be $(\omega)$submaximal if every $(\omega)$dense subset of $X$ is $(\omega)$open.

**Theorem 8.3.9.** $X$ is maximal $(\omega)$hyperconnected if and only if it is $(\omega)$submaximal and $(\omega)$hyperconnected.

**Proof.** Suppose $X$ is maximal $(\omega)$hyperconnected. Let $E \subset X$ be $(\omega)$dense. By Corollary 8.3.6, $\left( \bigcup_n J_n \right) - \{\emptyset\}$ is an ultrafilter. Therefore $E$ must be $(\omega)$open. For, if $E$ is not $(\omega)$open, then $E^c$ must be $(\omega)$open, since $\left( \bigcup_n J_n \right) - \{\emptyset\}$ is an ultrafilter. Therefore $E$ is $(\omega)$closed and hence $(\omega)cl E = E$. Again since $E$ is $(\omega)$dense, $(\omega)cl E = X$. Therefore $E = X$. Thus $X$ is $(\omega)$submaximal.

Conversely, suppose $X$ is $(\omega)$submaximal and $(\omega)$hyperconnected. Let $(X, \{J'_n\})$ be $(\omega)$hyperconnected with $\bigcup_n J'_n \supset \bigcup_n J_n$. If $G \in \bigcup_n J'_n$ be a non-empty set, then since $(X, \{J'_n\})$ is $(\omega)$hyperconnected, $(\omega)cl G$ (the $(\omega)$closure of $G$ in $(X, \{J'_n\})$) coincides with $X$. This implies that $(\omega)cl G$(the $(\omega)$closure of $G$ in $(X, \{J_n\})$) = $X$ (since $(\omega)cl G$ (w.r.t $(X, \{J_n\})$) $\supset (\omega)cl G$ (w.r.t $(X, \{J'_n\})$)), and so it follows that $G$ is $(\omega)$dense in $(X, \{J_n\})$. Hence $G \in \bigcup_n J_n$. Thus $\bigcup_n J'_n = \bigcup_n J_n$. 
