Chapter VII

Near \((\omega)\)compactness

In this chapter, we define near \((\omega)\)compactness and almost \((\omega)\)paracompactness and obtain some characterizations of these notions.

7.1 Introduction

Singal and Mathur ([36], [37]) introduced and studied the notion of nearly-compact spaces. Singal and Arya [35] also defined almost-paracompact spaces. In this chapter we extend these concepts to an \((\omega)\)topological space.

In the second section of this chapter, we introduce and study the notion of nearly \((\omega)\)compact spaces. Some of the results obtained are as follows:

Theorem 7.2.1 gives a set of characterization of nearly \((\omega)\)compact spaces similar to those obtained by Singal and Mathur [36] for nearly compact spaces.

A semi-(\omega)regular, nearly \((\omega)\)compact space is \((\omega)\)compact.

The content of the chapter is taken from our communicated paper [45].
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(Theorem 7.2.2).

An almost \((\omega)\)regular, almost \((\omega)\)compact space is nearly \((\omega)\)compact (Theorem 7.2.3).

An \((\omega)\)Hausdorff, nearly \((\omega)\)compact space is almost \((\omega)\)regular (Theorem 7.2.5).

In the final section, we introduce and study the notion of almost \((\omega)\)paracompact spaces. We prove the following results on this notion.

If for each \(n \in \mathbb{N}\), \((X, \mathcal{J}_n)\) is a regular topological space and every \((\omega)\)open cover of \(X\) has for some \(l \in \mathbb{N}\) a \((\mathcal{J}_l)\)open refinement which covers \(X\), then \(X\) is almost \((\omega)\)paracompact if and only if it is \((\omega)\)paracompact (Theorem 7.3.1).

Every strongly regularly \((\omega)\)closed subset of an almost \((\omega)\)paracompact space is almost \((\omega)\)paracompact (Theorem 7.3.2).

### 7.2 Nearly \((\omega)\)compact spaces

We introduce the following definition.

**Definition 7.2.1.** \(X\) is said to be *nearly \((\omega)\)compact* if every \((\omega)\)open cover \(\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}\) of \(X\) has a finite subcollection \(\mathcal{U}_0 = \{U_\alpha_1, U_\alpha_2, ..., U_\alpha_m\}\) with \(\bigcup_{k=1}^{m} (\mathcal{J}_{n_\alpha_k}) \cap \bigcap_{k=1}^{m} (\mathcal{J}_{n_\alpha_k}) \cap \bigcup_{k=1}^{m} \text{int}(\mathcal{J}_{n_\alpha_k}) \cap \text{cl} U_\alpha_k = X\), where \(n_{\alpha_k}\) is any natural number such that \(U_\alpha_k \in \mathcal{J}_{n_{\alpha_k}}\).
Remark 7.2.1. From the definitions of \((\omega)\)compactness (Definition 2.3.2), almost \((\omega)\)compactness (Definition 6.3.3), and near \((\omega)\)compactness it is clear that any nearly \((\omega)\)compact space is almost \((\omega)\)compact while every \((\omega)\)compact space is nearly \((\omega)\)compact. However, the converse relations are not true as shown by the following examples.

Example 7.2.1. The \((\omega)\)topological space considered in Example 6.3.1 is almost \((\omega)\)compact as shown below.

Let \(U\) be an \((\omega)\)open cover of \(X\), and \(U\) be any member in \(U\). If \(m\) is the largest integer present in \(U\), obviously \((J_m)clU\) contains all but a finite number of natural numbers. In fact \((J_m)clU\) does not contain at most the numbers 1, 2, ..., \(m - 1\). Taking at most \(m - 1\) more members from \(U\) we get the requisite finite subcollection.

We now see that it is not nearly \((\omega)\)compact as for any \(n\), the \((J_n)\)interior of the \((J_n)\)closure of any \((\omega)\)open set except \(\mathbb{N}\) is always finite and so it is impossible to cover \(\mathbb{N}\) using the \((J_n)\)interiors of the \((J_n)\)closures of a finite number of \((\omega)\)open sets.

Example 7.2.2. Consider the \((\omega)\)topological space \((\mathbb{N}, \{J_n\})\) in which the topologies \(\{J_n\}\) are defined as follows:

\[
J_n = \{\emptyset, \mathbb{N}\} \cup \{E \subset \{1, 2, ..., n\} \mid 1 \in E\}.
\]

The space is nearly \((\omega)\)compact as for any \(n\), the \((J_n)\)closure of any \((\omega)\)open set is \(\mathbb{N}\).
However, it is not \((\omega)\)compact as the \((\omega)\)open cover 
\[\{1, 2, \ldots, n\} \mid n \in \mathbb{N}\] does not have any finite subcover.

The following theorem gives a set of characterizations of nearly \((\omega)\)compactness.

**Theorem 7.2.1.** In an \((\omega)\)topological space \(X\), the following statements are equivalent:

(a) \(X\) is nearly \((\omega)\)compact.

(b) Every basic \((\omega)\)open cover \(U = \{U_\alpha \mid \alpha \in A\}\) of \(X\) possesses a finite subcollection \(U_0 = \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}\) with 
\[\bigcup_{k=1}^{m} (\mathcal{J}_{n_{\alpha_k}}) \text{int}((\mathcal{J}_{n_{\alpha_k}}) \text{cl}U_{\alpha_k}) = X,\] where \(n_{\alpha_k}\) is any natural number such that \(U_{\alpha_k} \in \mathcal{J}_{n_{\alpha_k}}\).

(c) Every regularly \((\omega)\)open cover of \(X\) has a finite subcover.

(d) Every regularly \((\omega)\)closed collection with finite intersection property has non-empty intersection.

(e) Every \((\omega)\)closed collection \(\mathcal{F} = \{F_\alpha \mid \alpha \in A\}\) with the property that for any finite subfamily \(\{F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_m}\}\) of \(\mathcal{F}\), 
\[\bigcap_{k=1}^{m} (\mathcal{J}_{n_{\alpha_k}}) \text{cl}((\mathcal{J}_{n_{\alpha_k}}) \text{int}F_{\alpha_k}) \neq \emptyset,\] has non-empty intersection.

**Proof.** (a) \(\Rightarrow\) (b): Obvious.

(b) \(\Rightarrow\) (c): Let \(\mathcal{G} = \{G_\alpha \mid \alpha \in A\}\) be a regularly \((\omega)\)open cover of \(X\) with \(G_\alpha \in \mathcal{J}_{n_\alpha}\) and \(G_\alpha = (\mathcal{J}_{n_\alpha}) \text{int}((\mathcal{J}_{n_\alpha}) \text{cl}G_\alpha)\). For each \(\alpha \in A\), let \(G_\alpha\) be replaced by basic \((\mathcal{J}_{n_\alpha})\)open sets which make
the \((\mathcal{J}_{n_a})\) open set \(G_\alpha\). We get a basic \((\omega)\) open cover \(\mathcal{U}\) of \(X\). By \((b)\), we obtain a finite subcollection \(\mathcal{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}\) with
\[
\bigcup_{k=1}^{m} (\mathcal{J}_{n_{\alpha_k}}) \text{int}((\mathcal{J}_{n_{\alpha_k}}) \text{cl}U_{\alpha_k}) = X,
\]
where \(U_{\alpha_k} \in \mathcal{J}_{n_{\alpha_k}}\) if \(U_{\alpha_k} \subset G_{\alpha_k}\). Since \(U_{\alpha_k} \subset G_{\alpha_k}\), we get
\[
\bigcup_{k=1}^{m} (\mathcal{J}_{n_{\alpha_k}}) \text{int}((\mathcal{J}_{n_{\alpha_k}}) \text{cl}G_{\alpha_k}) = X
\]
\[
\Rightarrow \bigcup_{k=1}^{m} G_{\alpha_k} = X.
\]

\((c) \Rightarrow (d)\): Let \(\mathcal{F} = \{F_\alpha \mid \alpha \in A\}\) be a regularly \((\omega)\) closed collection of subsets of \(X\) with

\((i)\) \(X - F_\alpha \in \mathcal{J}_{n_\alpha}\) and \((\mathcal{J}_{n_\alpha}) \text{int}((\mathcal{J}_{n_\alpha}) \text{cl}(X - F_\alpha)) = X - F_\alpha\),

\((ii)\) for any \(m \in \mathbb{N}\), \(\bigcap_{k=1}^{m} F_{\alpha_k} \neq \emptyset\).

If possible, suppose \(\cap \{F_\alpha \mid \alpha \in A\} = \emptyset\), then \(\{X - F_\alpha \mid \alpha \in A\}\) is a regularly \((\omega)\) open cover of \(X\). Therefore by \((c)\), \(\{X - F_\alpha \mid \alpha \in A\}\) has a finite subcover \(\{X - F_{\alpha_1}, X - F_{\alpha_2}, \ldots, X - F_{\alpha_m}\}\). So \(\bigcap_{k=1}^{m} F_{\alpha_k} = \emptyset\). This contradicts our assumption. Thus \(\cap \{F_\alpha \mid \alpha \in A\} \neq \emptyset\).

\((d) \Rightarrow (e)\): Let \(\mathcal{F} = \{F_\alpha \mid \alpha \in A\}\) be an \((\omega)\) closed collection of subsets of \(X\) having the property mentioned in \((e)\). Then \(\{(\mathcal{J}_{n_\alpha}) \text{cl}((\mathcal{J}_{n_\alpha}) \text{int}F_\alpha) \mid \alpha \in A\}\) is a collection of regularly \((\omega)\) closed sets having finite intersection property. So by \((d)\)
\[
\cap_{\alpha \in A} (\mathcal{J}_{n_\alpha}) \text{cl}((\mathcal{J}_{n_\alpha}) \text{int}F_\alpha) \neq \emptyset
\]
which implies \(\cap_{\alpha \in A} F_\alpha \neq \emptyset\).
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\((e) \Rightarrow (a)\): Let \(\mathcal{U} = \{U_\alpha \mid \alpha \in A\}\) be an \((\omega)\)open cover of \(X\). If possible, suppose \(X\) is not nearly \((\omega)\)compact, then for every finite subcollection \(\mathcal{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}\) of \(\mathcal{U}\),

\[
\bigcup_{k=1}^{m} (\mathcal{J}_{n_{\alpha_k}}) \text{int}((\mathcal{J}_{n_{\alpha_k}}) \text{cl}U_{\alpha_k}) \neq X.
\]

Thus

\[
\bigcap_{k=1}^{m} (X - (\mathcal{J}_{n_{\alpha_k}}) \text{int}((\mathcal{J}_{n_{\alpha_k}}) \text{cl}U_{\alpha_k})) \neq \emptyset.
\]

But

\[
X - (\mathcal{J}_{n_{\alpha_k}}) \text{int}((\mathcal{J}_{n_{\alpha_k}}) \text{cl}U_{\alpha_k}) \subset (\mathcal{J}_{n_{\alpha_k}}) \text{cl}((\mathcal{J}_{n_{\alpha_k}}) \text{int}(X - U_{\alpha_k})).
\]

Therefore

\[
\bigcap_{k=1}^{m} (\mathcal{J}_{n_{\alpha_k}}) \text{cl}((\mathcal{J}_{n_{\alpha_k}}) \text{int}(X - U_{\alpha_k})) \neq \emptyset.
\]

Thus \(\{X - U_\alpha \mid \alpha \in A\}\) is an \((\omega)\)closed collection of subsets of \(X\) satisfying the property mentioned in \((e)\). Hence \(\cap_{\alpha}(X - U_\alpha) \neq \emptyset\) which implies \(\cup_{\alpha}U_\alpha \neq X\) which contradicts the fact that \(\{U_\alpha \mid \alpha \in A\}\) is an \((\omega)\)open cover of \(X\). Hence \(X\) is nearly \((\omega)\)compact. \(\square\)

**Theorem 7.2.2.** A semi-(\(\omega)\)regular, nearly \((\omega)\)compact space is \((\omega)\)compact.

**Proof.** Let \(X\) be a semi-(\(\omega)\)regular, nearly \((\omega)\)compact space and \(\mathcal{U} = \{U_\alpha \mid \alpha \in A\}\) be an \((\omega)\)open cover of \(X\). For each \(x \in X\), there exists an \(\alpha_x \in A\) such that \(x \in U_{\alpha_x}\). Since \(X\) is semi-(\(\omega)\)regular, there exists for some \(n \in \mathbb{N}\) a regularly (\(\mathcal{J}_n\))open set \(G_x\) such that \(x \in G_x \subset U_{\alpha_x}\). \(\{G_x \mid x \in X\}\) is a regularly
(ω)open cover of $X$ and has therefore a finite subcover $\{G_{x_i} \mid i = 1, 2, ..., m\}$. Then $\{U_{\alpha_{x_i}} \mid i = 1, 2, ..., m\}$ is a finite subcover of $U$. Hence $X$ is (ω)compact.

**Theorem 7.2.3.** An almost (ω)regular, almost (ω)compact space is nearly (ω)compact.

**Proof.** Let $X$ be an almost (ω)regular, almost (ω)compact space and $U = \{U_\alpha \mid \alpha \in A\}$ be a regularly (ω)open cover of $X$. For each $x \in X$, there exists an $\alpha_x \in A$ such that $x \in U_{\alpha_x}$. Since $X$ is almost (ω)regular, by Theorem 6.2.1 there exists for some $n_x \in \mathbb{N}$, a (J,$n_x$)open set $V_x$ such that

$$x \in V_x \subset (J_{n_x})clV_x \subset U_{\alpha_x}.$$ 

Then $\mathcal{V} = \{V_x \mid x \in X\}$ forms an (ω)open cover of $X$. Since $X$ is an almost (ω)compact space, there exists a finite subfamily $\{V_{x_i} \mid i = 1, 2, ..., m\}$ of $\mathcal{V}$ such that $\bigcup_{k=1}^{m} (J_{n_{x_i}})clV_{x_i} = X$, where $V_{x_i} \in J_{n_{x_i}}, n_{x_i} \in \mathbb{N}$. Clearly $\{U_{\alpha_{x_i}} \mid i = 1, 2, ..., m\}$ is a finite subcover of $U$. Therefore by Theorem 7.2.1, $X$ is nearly (ω)compact.

To prove the next theorem, we require the following lemma:

**Lemma 7.2.4.** Each (ω)open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of a regularly (ω)closed subset $Y$ of a nearly (ω)compact space $X$ has a finite subcollection $\mathcal{U}' = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ such that $\bigcup_{i=1}^{m} (J_{n_{\alpha_i}})int((J_{n_{\alpha_i}})clU_{\alpha_i}) \supset Y$, where $n_{\alpha_i} \in \mathbb{N}$ is any natural number such that $U_{\alpha_i} \in J_{n_{\alpha_i}}$. 


Proof. Let \( U = \{ U_\alpha \mid \alpha \in A \} \) be an \((\omega)\)open cover of a regularly \((\omega)\)closed subset \( Y \) of a nearly \((\omega)\)compact space \( X \). Since \( Y \) is a regularly \((\omega)\)closed set, \( X - Y \) is a regularly \((\omega)\)open subset of \( X \). Hence the collection \( U \cup \{ X - Y \} \) forms an \((\omega)\)open cover of \( X \). As \( X \) is nearly \((\omega)\)compact, this \((\omega)\)open cover has a finite subfamily \( U' = \{ U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m} \} \) such that 

\[
\bigcup_{i=1}^{m} (J_{n_{\alpha_i}}) \text{int}((J_{n_{\alpha_i}}) \text{cl}U_{\alpha_i}) = X,
\]

where \( U_{\alpha_i} \in J_{n_{\alpha_i}}, n_{\alpha_i} \in \mathbb{N} \). If \( U' \) does not contain the set \( X - Y \) then \( U' \) is the required subcover of \( U \). However if \( X - Y \in U' \) then since \( X - Y \) is regularly \((\omega)\)open, there exists an \( n_{\alpha_i} \in \mathbb{N}, i = 1, 2, \ldots, m \) such that 

\[
(J_{n_{\alpha_i}}) \text{int}((J_{n_{\alpha_i}}) \text{cl}(X - Y)) = X - Y.
\]

So obviously the remaining members of \( U' \) form the required subcover of \( U \).

\[
\square
\]

**Theorem 7.2.5.** An \((\omega)\)Hausdorff, nearly \((\omega)\)compact space is almost \((\omega)\)regular.

Proof. Let the space \( X \) be \((\omega)\)Hausdorff and nearly \((\omega)\)compact. Suppose \( y \) is a point of \( X \) and \( A \) is a regularly \((\omega)\)open subset of \( X \) containing \( y \). For \( x \in X - A \), there exists an \( n_x \in \mathbb{N} \) such that for some \( U_x, V_x \in J_{n_x} \), we have \( x \in U_x, y \in V_x \) and \( U_x \cap V_x = \emptyset \). Then \( U = \{ U_x \mid x \in X - A \} \) is an \((\omega)\)open cover of the regularly \((\omega)\)closed set \( X - A \). Therefore by the Lemma 7.2.4, \( U \) has a finite subcover \( U' = \{ U_{x_i} \mid i = 1, 2, \ldots, m \} \) such that 

\[
X - A \subset \bigcup_{i=1}^{m} (J_{n_{x_i}}) \text{int}((J_{n_{x_i}}) \text{cl}U_{x_i}),
\]
where $n_x$ is any natural number such that $U_x \in \mathcal{J}_{n_x}$. Let
\[ G = \bigcup_{i=1}^{m} (\mathcal{J}_{n_x} \cap \text{int}(\mathcal{J}_{n_x}^{cl} U_x)) \text{ and } \]
\[ H = \bigcap_{i=1}^{m} V_x. \]
Obviously $G$ and $H$ are disjoint $(\mathcal{J}_{n_0})$ open sets, where $n_0 = \max\{n_x, n_{x_2}, \ldots, n_{x_m}\}$.

Also $y \in H$ and $X - A \subset G \subset X - (\mathcal{J}_{n_0}^{cl} H)$. So $(\mathcal{J}_{n_0}^{cl} H) \subset A$. Thus
\[ y \in H \subset (\mathcal{J}_{n_0}^{cl} H) \subset A \]
and $X$ is almost $(\omega)$ regular. 

\section*{7.3 Almost $(\omega)$ paracompact spaces}

We introduce the notion almost $(\omega)$ paracompact spaces analogous to the notion of almost paracompact spaces [35]. For this section we make the following modification in the notion of refinement of a cover.

\textbf{Definition 7.3.1.} Let $\mathcal{U}$ be cover of the $(\omega)$ topological space $X$, a collection $\mathcal{V}$ of sets is said to be a refinement of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some set of $\mathcal{U}$. Here it is not necessary for a refinement to be a cover as well.

\textbf{Definition 7.3.2.} $X$ is said to be almost $(\omega)$ paracompact if every $(\omega)$ open cover $\mathcal{U}$ of $X$ has for some $n$, a $(\mathcal{J}_n)$ locally finite $(\mathcal{J}_n)$ open refinement $\mathcal{V}$ such that the family of $(\mathcal{J}_n)$ closures of the members of $\mathcal{V}$ covers $X$. 
Obviously an \((\omega)\)paracompact space is almost \((\omega)\)paracompact but the converse is not true as seen below.

**Example 7.3.1.** The \((\omega)\)topological space considered in Example 6.2.1 is almost \((\omega)\)compact (Remark 6.3.3). Therefore this \((\omega)\)topological space is almost \((\omega)\)paracompact. But it is not \((\omega)\)paracompact as we see below.

The \((\omega)\)open cover \(U = \{\{1, 2, ..., n\} | n \in \mathbb{N}\}\) does not have for any \(n\), a \((J_n)\)open refinement which covers the space.

**Theorem 7.3.1.** Suppose, for each \(n \in \mathbb{N}\), \((X, J_n)\) is a regular topological space and every \((\omega)\)open cover of \(X\) has for some \(l \in \mathbb{N}\) a \((J_l)\)open refinement covering \(X\). Then \(X\) is almost \((\omega)\)paracompact if and only if it is \((\omega)\)paracompact.

**Proof.** Let \(U\) be any \((\omega)\)open cover of \(X\). \(U\) has for some \(l\) a \((J_l)\)open refinement \(V\) covering \(X\). Since \((X, J_l)\) is regular, for each \(x \in X\) and any \(V \in V\) with \(x \in V\), there exists an \((J_l)\)open set \(V_x\) such that \(x \in V_x \subset (J_l)clV_x \subset V\). Then \(V' = \{V_x | x \in X\}\) is a \((J_l)\)open and hence \((\omega)\)open cover of \(X\). By almost \((\omega)\)paracompactness of \(X\), \(V'\) has for some \(n\), a \((J_n)\)locally finite \((J_n)\)open refinement \(S\) such that the family of \((J_n)\)closures of the members of \(S\) covers \(X\). Let us consider the family

\[
P = \{(J_l)clV_x \cap (J_n)clS | V_x \in V', S \in S\}.
\]

\(P\) is a \((J_k)\)locally finite \((J_k)\)closed refinement of \(V\) which in
turn is a refinement of $\mathcal{U}$, where $k = \max\{n, l\}$. Thus by Theorem 3.3.1, $X$ is $(\omega)$paracompact.

The converse is obvious. \qed

**Theorem 7.3.2.** If a space $X$ is almost $(\omega)$paracompact then every strongly regularly $(\omega)$closed subset of $X$ is almost $(\omega)$paracompact.

*Proof.* Suppose $X$ is almost $(\omega)$paracompact. Let $F$ be any strongly regularly $(\omega)$closed subset of $X$ and $\{F \cap G_\alpha \mid \alpha \in A\}$ be an $(\omega)$open cover of $F$, where each $G_\alpha$ is $(\omega)$open in $X$. Then $\{G_\alpha \mid \alpha \in A\} \cup \{X - F\}$ is an $(\omega)$open cover of $X$. Since $X$ is almost $(\omega)$paracompact, there exists for some $n$, a $(\mathcal{J}_n)$locally finite $(\mathcal{J}_n)$open refinement $\{R_\beta \mid \beta \in B\}$ of $\{G_\alpha \mid \alpha \in A\} \cup \{X - F\}$ such that

$$\bigcup \{(\mathcal{J}_n)clR_\beta \mid \beta \in B\} = X.$$ 

Since $F$ is strongly regularly $(\omega)$closed, $F = (\mathcal{J}_n)clG$ for some $G \in \mathcal{J}_n$. It is easy to see that

$$(\mathcal{J}_n)cl(X - (\mathcal{J}_n)clG) \cap G = \emptyset$$

and so $\{R_\beta \mid \beta \in B\}$ may be considered to be a refinement of $\{G_\alpha \mid \alpha \in A\}$, satisfying the property $G \subset \bigcup_{\beta \in B}(\mathcal{J}_n)clR_\beta$. Again since $G$ is $(\mathcal{J}_n)$open,

$$\big(J_n\big)cl(G \cap (\bigcup_{\beta \in B} R_\beta)) = (\mathcal{J}_n)cl((\mathcal{J}_n)cl(G \cap (\bigcup_{\beta \in B} R_\beta))). \hspace{1cm} (7.3.1)$$

Further, as $\{R_\beta \mid \beta \in B\}$ is $(\mathcal{J}_n)$locally finite, therefore

$$(\mathcal{J}_n)cl(\bigcup_{\beta \in B} R_\beta) = \bigcup_{\beta \in B}(\mathcal{J}_n)clR_\beta. \hspace{1cm} (7.3.2)$$
Then

\[ F = (\mathcal{J}_n)clG = (\mathcal{J}_n)cl(G \cap (\bigcup_{\beta \in B} (\mathcal{J}_n)clR_\beta)) \]

\[ = (\mathcal{J}_n)cl(G \cap (\mathcal{J}_n)cl(\bigcup_{\beta \in B} R_\beta)) \text{ (by (7.3.2))} \]

\[ = (\mathcal{J}_n)cl(G \cap (\bigcup_{\beta \in B} R_\beta)) \text{ (by (7.3.1))} \]

\[ = (\mathcal{J}_n)cl(\bigcup_{\beta \in B} (G \cap R_\beta)) \]

\[ \subset (\mathcal{J}_n)cl(\bigcup_{\beta \in B} (F \cap R_\beta)). \]

Hence \( \{ F \cap R_\beta \mid \beta \in B \} \) is a \((\mathcal{J}_n)\)locally finite \((\mathcal{J}_n)\)open refinement of \( \{ F \cap G_\alpha \mid \alpha \in A \} \) the \((\mathcal{J}_n)\)closures of whose members cover \( F \). Hence \( F \) is almost \((\omega)\)paracompact. \( \square \)