CHAPTER-V

ONE-DIMENSIONAL SOLUTE DISPERSION ALONG TRANSIENT FLOW WITH CAUCHY TYPE BOUNDARY CONDITIONS: DISPERSION COEFFICIENT IS DIRECTLY PROPORTIONAL TO SQUARE OF SEEPAGE VELOCITY

5.1 Introduction

The attenuation of pollutant’s mass concentration in the groundwater system is explained by hydraulic mixing process, in which it is transported down the stream, away from the source of pollution attenuated with position and time, and is known as hydrodynamic dispersion. The space time distribution behavior of the contaminant concentration is described by a partial differential equation of parabolic type, derived on the principle of conservation of mass and Fick’s laws of diffusion, known as advection-dispersion (AD) equation. In one dimension its general form is

\[
\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left\{ D(x,t) \frac{\partial C}{\partial x} - u(x,t)C \right\},
\]

where \( C \) is the pollutant’s concentration at a position \( x \) at time \( t \), \( D(x,t) \) is referred to as dispersivity parameter and \( u(x,t) \) as the velocity of the convective medium. Mathematical modelers use the AD equation to describe the concentration level at different position and time, away from its source, through its analytical and numerical solutions. The literature presenting several analytical methods is reviewed in a recent work (Guerrero et al., 2009). Solutions of one-dimensional AD equation are obtained under various initial and boundary conditions. A summary of these can be seen in the Table 5.1.

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<tr>
<th>Sl. No.</th>
<th>Author</th>
<th>I.C.</th>
<th>U.B.C.</th>
<th>L.B.C.</th>
<th>Types of Aquifer</th>
<th>Aquifer Properties</th>
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<tbody>
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<td>Van Genuchten and Alves (1982)</td>
<td>( f(x) )</td>
<td>( c(x,t) = c_0 )</td>
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<td>Isotropic, Anisotropic Homogeneous Non-homogeneous</td>
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<td>5</td>
<td>Singh et al. (2008, 2009, 2010)</td>
<td>(c_i)</td>
<td>(c(x,t) = c_0 [1 + \exp(-qt)])</td>
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<td>Singh et al. (2012)</td>
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<td>Jaiswal and Kumar (2011)</td>
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<td>(-D(x,t) \frac{\partial C}{\partial x} + u(x,t) C(x,t) = \begin{cases} u_0 C_0; &amp; 0 &lt; t \leq t_0 \ 0, &amp; t &gt; t_0 \end{cases})</td>
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<td>Homogeneous and Heterogeneous</td>
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Table 5.1 Analytical Solutions for different initial and boundary conditions
Analytical solutions are usually derived using simplifications but are free from numerical dispersion and other truncation errors that often occur in numerical simulations (Zheng and Bennett, 1995). With the use of analytical solutions for estimating the movement of contaminant plumes, one can save a lot of effort to guide and collect data and monitor water quality despite complexities of hydrogeologic systems (Wexler, 1992). Analytical solutions of advection-diffusion equation with a spatially variable coefficient in up to three dimensions were discussed by Zoppou and Knight (1999). Analytical solutions of transient unsaturated transport of water and contaminants through horizontal porous media were presented by Sander and Braddock (2005). Analytical solution for transport of decaying solutes in rivers with transient storage was derived by Smedt (2006). Analytical solutions for sequentially coupled one-dimensional reactive transport problems were explored by Srinivasan and Clement (2008). Many such models concern homogeneous media, but in reality the ability of the mass to permeate though the medium of air, soil or groundwater varies with position, which is referred to as heterogeneity. Early efforts to describe heterogeneity were achieved by making the use of stratification and defining porosity – distance relationship (Coats and Smith 1964; Shamir and Harleman 1967; Lin 1977; Valocchi 1989).

The relationship between the dispersion parameter $D$ and the velocity of flow $u$ occurring in the advection-dispersion equation attracted many scientists to analyze. Scheidegger (1957) summarized his analysis of two possible relationships: (1) $D = \alpha u^2$, where $\alpha$ is a constant of the porous medium, and (2) $D = \beta u$, where $\beta$ is another constant of the porous medium. Taylor (1953) landed on a relation that $D$ proportional to $u^2$, whereas Bear and Todd (1960) suggested $D$ as proportional to $u$. Freeze and Cherry (1979) observed that $D$ is proportional to a power $n$ of velocity $u$ which ranges between 1 and 2. In the Indian context, Ghosh and Sharma (2006) experimentally observed that $D$ is proportional to a power $n$ of velocity $u$ which ranges between 1 and 1.2.

The present study invoke the relation that $D$ is proportional to $u^2$, as suggested by Scheidegger (1957) along transient groundwater flow in a homogeneous semi-infinite aquifer. The aquifer is assumed to be polluted initially and is an exponentially decreasing
function of space variable together with a constant initial concentration. Analytical solution is obtained for one-dimensional ADE having an input concentration in the form of Cauchy type boundary condition injected at the origin and the contaminant concentration at an infinite distance away from the source is supposed to be zero at all times. Laplace transformation technique is used to obtain analytical solutions. Graphs are plotted using MATLAB to have a deep insight of the problem. Also a sensitivity analysis has been done using the input variables, dispersion coefficient $D$, seepage velocity of flow $u$, and decay rate coefficients $q$.

5.2 Mathematical Formulation

Let $C(x,t)$ be the concentration of contaminants in the aquifer $[ML^{-3}]$, $u$ be the groundwater velocity of the porous medium transporting the solute particles $[LT^{-1}]$, and $D$ be the solute dispersion known as dispersion coefficient if it is independent of the independent variable $[L^2T^{-1}]$ at any time $t[T]$.

Then the linear advection-dispersion equation in one dimensional form can be written as:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left\{ D(x,t) \frac{\partial C}{\partial x} - u(x,t) C \right\}$$  (5.1)

The dispersion coefficient and flow velocity can be expressed as:

$$D(x,t) = D_0 \left[ f(mt) \right]^2 ; u(x,t) = u_0 f(mt)$$  (5.2)

where $D_0$ is the constant initial dispersion coefficient and $u_0$ is the initial groundwater velocity. Here, $m$ is the flow resistance coefficient whose dimension is inverse of the dimension of time $t$, i.e., of dimension $(T^{-1})$. Hence, $f(mt)$ is an expression for a non-dimensional variable.

The system is assumed to be polluted initially, i.e., some initial contaminant exists in aquifer at $t = 0$ and the initial concentration is an exponentially decreasing function of
the space variable. Also time-dependent source contaminant concentration in the form of Cauchy type boundary condition is injected at the origin of the aquifer and the contaminant concentration gradient at an infinite distance is supposed to be zero for all times. The system is assumed to be of semi-infinite extent.

Thus the initial and boundary conditions for the problem can be expressed as:

\[ C(x,t) = C_i \exp(-\gamma x); \quad t = 0, x \geq 0 \]  
\[ -D(x,t) \frac{\partial C}{\partial x} + u(x,t)C = \frac{uC_0}{2} \left[1 + \exp(-qt)\right]; \quad x = 0, t > 0 \]  
\[ \frac{\partial C}{\partial x} = 0; \quad x \to \infty, t \geq 0 \]

where \( C_i \) is the initial solute concentration \([\text{ML}^{-3}]\) included with an exponentially decreasing function of space describing the distribution of concentration at all points of the flow domain, i.e., at \( t = 0 \), and \( \gamma \) is a constant coefficient parameter whose dimension is the inverse of the space variable, i.e., \([\text{L}^{-1}]\). Here, \( C_0 \) is the solute concentration \([\text{ML}^{-3}]\), and \( q \) is the decay rate coefficient \([\text{T}^{-1}]\).

### 5.2.1 Analytical Solution

Using Eq. (5.2), Eq. (5.1) and boundary conditions in Eq. (5.3) to (5.5) are converted as:

\[ \frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left\{ D_0 \left[f(mt)\right]^2 \frac{\partial C}{\partial x} - u_0 f(mt) C \right\} \]  
\[ C(x,t) = C_i \exp(-\gamma x); \quad t = 0, x \geq 0 \]  
\[ -D_0 \left[f(mt)\right]^2 \frac{\partial C}{\partial x} + u_0 f(mt) C = \frac{u_0 f(mt) C_0}{2} \left[1 + \exp(-qt)\right]; \quad x = 0, t > 0 \]
\( \frac{\partial C}{\partial x} = 0; \quad x \to \infty, t \geq 0 \) \hspace{1cm} (5.9)

Introducing a new independent variable, \( X \) with the transformation (Jaiswal et al., 2009):

\[
\frac{dX}{dx} = \frac{1}{f(mt)} \quad \text{or,} \quad X = \int \frac{dx}{f(mt)} = \frac{x}{f(mt)} \hspace{1cm} (5.10)
\]

Eqs. (5.6) to (5.9) can be written as

\[
\frac{\partial C}{\partial t} - \frac{xmf'(mt)}{[f(mt)]^2} \frac{\partial C}{\partial X} = D_0 \frac{\partial^2 C}{\partial X^2} - u_0 \frac{\partial C}{\partial X} \hspace{1cm} (5.11)
\]

\[
C(X,t) = C_i \exp\{-X\gamma f(mt)\}; \quad t = 0, X \geq 0 \hspace{1cm} (5.12)
\]

\[
-D_0 \frac{\partial C}{\partial X} + u_0 C = \frac{u_0 C_0}{2} \left[ 1 + \exp(-qt) \right]; \quad X = 0, t > 0 \hspace{1cm} (5.13)
\]

\[
\frac{\partial C}{\partial X} = 0; \quad X \to \infty, t \geq 0 \hspace{1cm} (5.14)
\]

The function \( f(mt) \) is chosen as the sinusoidal form of temporally dependent dispersion, \( f(mt) = 1 - \sin(mt) \), and the exponential form of temporally dependent dispersion, \( f(mt) = \exp(mt) \), in such a way, that \( f(mt) = 1 \) for \( m = 0 \) or \( t = 0 \). Now, Eqs. (5.11) to (5.14) can transform as follows:

\[
\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial X^2} - u_0 \frac{\partial C}{\partial X} \hspace{1cm} (5.15)
\]

\[
C(X,t) = C_i \exp(-\gamma X); \quad t = 0, X \geq 0 \hspace{1cm} (5.16)
\]

\[
-D_0 \frac{\partial C}{\partial X} + u_0 C = \frac{u_0 C_0}{2} \left[ 1 + \exp(-qt) \right]; \quad X = 0, t > 0 \hspace{1cm} (5.17)
\]
\[ \frac{\partial C}{\partial X} = 0; \quad X \to \infty, t \geq 0 \quad (5.18) \]

Now we introduce a transformation:

\[ C(X, t) = K(X, t) \exp \left[ \frac{u_0 X}{2D_0} - \frac{u_0^2 t}{4D_0} \right] \quad (5.19) \]

With the use of this transformation, the initial and boundary value problem is transformed to the following equations:

\[ \frac{\partial K}{\partial t} = D_0 \frac{\partial^2 K}{\partial X^2} \quad (5.20) \]

\[ K(X, t) = C_0 \exp \left\{ - \left( \frac{u_0}{2D_0} + \gamma \right) X \right\}; \quad X \geq 0, t = 0 \quad (5.21) \]

\[ D_0 \frac{\partial K}{\partial X} - \frac{u_0}{2} K = - \frac{u_0 C_0}{2} \left[ 2 \exp \left( \frac{u_0^2 t}{4D_0} \right) - q t \times \exp \left( \frac{u_0^2 t}{4D_0} \right) \right]; \quad X = 0, t > 0 \quad (5.22) \]

\[ \frac{\partial K}{\partial X} = - \frac{u_0}{2D_0} K; \quad X \to \infty, t \geq 0 \quad (5.23) \]

Applying the Laplace transform (Sneddon, 1974) to Eqs. (5.20) to (5.23), we obtain

\[ \frac{d^2 \overline{K}}{dX^2} - \frac{p}{D_0} \overline{K} = - \frac{C_0}{D_0} \exp(-\eta X) \quad (5.24) \]

\[ D_0 \frac{d \overline{K}}{dX} - \frac{u_0}{2} \overline{K} = - \frac{u_0 C_0}{2} \left[ \frac{2}{(p - \alpha^2)} - \frac{q}{(p - \alpha^2)^2} \right]; \quad X = 0 \quad (5.25) \]

\[ \frac{d \overline{K}}{dX} = - \frac{u_0}{2D_0} \overline{K}; \quad X \to \infty, t \geq 0 \quad (5.26) \]
where \( \eta = \left( \frac{u_0}{2D_0} + \gamma \right) \), \( \alpha^2 = \frac{u_0^2}{4D_0} \) and \( \overline{K}(X,p) = \int_0^K(X,t) \exp(-pt) dt \).

Solving Eq. (5.24), we obtain

\[
\text{C.F.} = C_1 \exp \left( -X \sqrt{\frac{p}{D_0}} \right) + C_2 \exp \left( X \sqrt{\frac{p}{D_0}} \right)
\]

and P.I. = \( \frac{C_i}{p - D_0 \eta^2} \exp(-\eta X) \)

The general solution of Eq. (5.24) can be written as

\[
\overline{K}(X,p) = C_1 \exp \left( -X \sqrt{\frac{p}{D_0}} \right) + C_2 \exp \left( X \sqrt{\frac{p}{D_0}} \right) + \frac{C_i}{p - D_0 \eta^2} \exp(-\eta X) \tag{5.27}
\]

Differentiating Eq. (5.27) with respect to \( X \), we get

\[
\frac{d\overline{K}}{dX} = -C_1 \sqrt{\frac{p}{D_0}} \exp \left( -X \sqrt{\frac{p}{D_0}} \right) + C_2 \sqrt{\frac{p}{D_0}} \exp \left( X \sqrt{\frac{p}{D_0}} \right) - \frac{\eta C_i}{p - D_0 \eta^2} \exp(-\eta X)
\]

Now,

\[
D_0 \frac{d\overline{K}}{dX} - \frac{u_0}{2} \overline{K} = -C_1 \sqrt{pD_0} \exp \left( -X \sqrt{\frac{p}{D_0}} \right) + C_2 \sqrt{pD_0} \exp \left( X \sqrt{\frac{p}{D_0}} \right)
\]

\[
- \frac{\eta D_0 C_i}{p - D_0 \eta^2} \exp(-\eta X) - \frac{u_0}{2} \left[ C_1 \exp \left( -X \sqrt{\frac{p}{D_0}} \right) + C_2 \exp \left( X \sqrt{\frac{p}{D_0}} \right) \right]
\]

Using the condition given in Eq. (5.25) and putting \( X = 0 \) in above, we have
\[-\frac{u_0 C_0}{2} \left[ \frac{2}{(p - \alpha^2)} - \frac{q}{(p - \alpha^2)^2} \right] = -C_1 \sqrt{pD_0} + C_2 \sqrt{pD_0} - \frac{\eta D_0 C_i}{p - D_0 \eta^2} \]

\[-\frac{u_0}{2} \left[ C_1 + C_2 + \frac{C_i}{p - D_0 \eta^2} \right] \]

\[= C_1 \left\{ -\sqrt{pD_0} - \frac{u_0}{2} \right\} + C_2 \left\{ \sqrt{pD_0} - \frac{u_0}{2} \right\} \]

\[-\frac{C_i}{p - D_0 \eta^2} \left\{ \eta D_0 + \frac{u_0}{2} \right\} \]

(5.28)

Again,

\[\frac{d\bar{K}}{dX} = -C_1 \sqrt{\frac{p}{D_0}} \exp \left( -X \sqrt{\frac{p}{D_0}} \right) + C_2 \sqrt{\frac{p}{D_0}} \exp \left( X \sqrt{\frac{p}{D_0}} \right) - \frac{\eta C_i}{p - D_0 \eta^2} \exp (-\eta X)\]

Similarly using the condition given in Eq. (5.26) and putting $X \to \infty$ in above, we obtain $C_2 = 0$.

Hence putting $C_2 = 0$ in Eq. (5.28), we have

\[C_1 = \frac{-u_0 C_0}{2 \left( \sqrt{pD_0} + \frac{u_0}{2} \right)} \left[ \frac{2}{(p - \alpha^2)} - \frac{q}{(p - \alpha^2)^2} \right] - \frac{C_i \left( \eta D_0 + \frac{u_0}{2} \right)}{\left( \sqrt{pD_0} + \frac{u_0}{2} \right) (p - D_0 \eta^2)}\]

Therefore, putting the values of $C_1$ and $C_2$ in Eq. (5.27), we have the general solution as
\[
\bar{K}(X, p) = \exp\left(-X \sqrt{\frac{p}{D_0}}\right) \left\{ \begin{array}{c}
\frac{u_0 C_0}{2\left(\sqrt{p D_0} + \frac{u_0}{2}\right)} \left[ \frac{2}{(p - \alpha^2)} - \frac{q}{(p - \alpha^2)^2} \right] \\
C_i \left( \eta D_0 + \frac{u_0}{2} \right) - \frac{C_i}{\left(\sqrt{p D_0} + \frac{u_0}{2}\right)\left(p - D_0 \eta^2\right)} \\
+ \frac{C_i}{\left(p - D_0 \eta^2\right)} \exp\left(-\eta X\right)
\end{array} \right.
\]

i.e.,

\[
\bar{K}(X, p) = \frac{u_0 C_0}{2\sqrt{D_0}} \left[ \frac{2 \exp\left(-X \sqrt{\frac{p}{D_0}}\right)}{\left(\sqrt{p + \frac{u_0}{2D_0}}\right)\left(p - \frac{u_0^2}{4D_0}\right)} - \frac{q \times \exp\left(-X \sqrt{\frac{p}{D_0}}\right)}{\left(\sqrt{p + \frac{u_0}{2D_0}}\right)\left(p - \frac{u_0^2}{4D_0}\right)^2} \right] - \frac{C_i \left( \eta D_0 + \frac{u_0}{2} \right)}{\sqrt{D_0} \left(\sqrt{p + \frac{u_0}{2\sqrt{D_0}}}\right)\left(p - D_0 \eta^2\right)} \exp\left(-X \sqrt{\frac{p}{D_0}}\right) + \frac{C_i}{\left(p - D_0 \eta^2\right)} \exp\left(-\eta X\right)
\]

\text{(5.29)}

Taking inverse Laplace Transform (Bateman, 1954) of Eq. (5.29), we obtain
Using the transform given in Eq. (5.19) backward, we can obtain the solution as follows:

\[ C(X,t) = F(X,t) - G(X,t) - H(X,t) + I(X,t) \]  

(5.30)

where
\[ F(X,t) = \frac{C_0}{2} \left[ 2u_0 \sqrt{\frac{t}{\pi D_0}} \exp \left( \frac{u_0 X}{2D_0} - \frac{X^2}{4D_0 t} - \frac{u_0^2 t}{4D_0} \right) + \text{erfc} \left( \frac{X-u_0 t}{2\sqrt{D_0 t}} \right) \right] \]  
\[ - \left( 1 + \frac{u_0 X}{D_0} + \frac{u_0^2 t}{D_0} \right) \exp \left( \frac{u_0 X}{D_0} \right) \text{erfc} \left( \frac{X+u_0 t}{2\sqrt{D_0 t}} \right) \]  
\[ = \frac{qC_0}{2} \left[ \frac{1}{2u_0} \sqrt{\frac{t}{\pi D_0}} (2D_0 + u_0 X + u_0^2 t) \exp \left( \frac{u_0 X}{2D_0} - \frac{X^2}{4D_0 t} - \frac{u_0^2 t}{4D_0} \right) \right. \]  
\[ + \left. \frac{1}{2u_0} (u_0^2 t - u_0 X - D_0) \text{erfc} \left( \frac{X-u_0 t}{2\sqrt{D_0 t}} \right) - \frac{1}{2u_0^2} \left( u_0^2 t - D_0 + \frac{u_0^2}{2} \left( \frac{X+u_0 t}{\sqrt{D_0}} \right)^2 \right) \right] \]  
\[ \times \exp \left( \frac{u_0 X}{D_0} \right) \text{erfc} \left( \frac{X+u_0 t}{2\sqrt{D_0 t}} \right) \]  
\[ (5.31a) \]

\[ G(X,t) = \frac{qC_0}{2} \left[ \frac{1}{2u_0} \sqrt{\frac{t}{\pi D_0}} (2D_0 + u_0 X + u_0^2 t) \exp \left( \frac{u_0 X}{2D_0} - \frac{X^2}{4D_0 t} - \frac{u_0^2 t}{4D_0} \right) \right. \]  
\[ + \left. \frac{1}{2u_0} (u_0^2 t - u_0 X - D_0) \text{erfc} \left( \frac{X-u_0 t}{2\sqrt{D_0 t}} \right) - \frac{1}{2u_0^2} \left( u_0^2 t - D_0 + \frac{u_0^2}{2} \left( \frac{X+u_0 t}{\sqrt{D_0}} \right)^2 \right) \right] \]  
\[ \times \exp \left( \frac{u_0 X}{D_0} \right) \text{erfc} \left( \frac{X+u_0 t}{2\sqrt{D_0 t}} \right) \]  
\[ (5.31b) \]

\[ H(X,t) = \frac{1}{2} I(X,t) \left[ \text{erfc} \left( \frac{X - (2\gamma D_0 + u_0) t}{2\sqrt{D_0 t}} \right) - \left( \gamma D_0 + u_0 \right) \exp \left( \frac{2\gamma + u_0}{D_0} X \right) \right. \]  
\[ \times \text{erfc} \left( \frac{X + (2\gamma D_0 + u_0) t}{2\sqrt{D_0 t}} \right) \]  
\[ + \frac{u_0 C_0}{2\gamma D_0} \exp \left( \frac{u_0 X}{D_0} \right) \text{erfc} \left( \frac{X+u_0 t}{2\sqrt{D_0 t}} \right) \]  
\[ (5.31c) \]

\[ I(X,t) = C_i \exp \left( \left( \gamma^2 D_0 + \gamma u_0 \right) t - \gamma X \right) \]  
\[ (5.31d) \]
5.3 **Illustrative Example and Discussion**

An analytical solution for temporally dependent dispersion with transient velocity given by Eq. (5.30) is computed for an input data of $C_i = 0.01$, $C_o = 1.0$, $D_0 = 0.25(\text{km}^2/\text{year})$, $u_0 = 0.025(\text{km}/\text{year})$, $m = 0.1(\text{/year})$, $\gamma = 0.01(\text{/km})$, and $q = 0.001(\text{/year})$ in order to depict the variation of contaminant concentration along transient groundwater flow in the space domain $0 \leq x \leq 1(\text{km})$ and the time domain $0.4 \leq t \leq 2.2(\text{years})$, for the Cauchy type input condition. The solution is computed with temporally dependent dispersion along with the time varying velocity for sinusoidal form, i.e., $f(mt) = 1 - \sin(mt)$, and the exponential form, i.e., $f(mt) = \exp(mt)$. In tropical regions in Indian sub-continent, groundwater velocity and water level may exhibit seasonally sinusoidal behavior in aquifers (Kumar and Kumar, 1998; Thangarajan, 2006), as the groundwater velocity and water level are minimum during the peak of the summer season (the period of greatest pumping), which falls in the month of June, just before rainy season and also it attains the maximum values during the peak of winter season around December, after the rainy season (the period of lowest pumping). In these regions, groundwater infiltration is from rainfall and rivers. In case of aquifers nearby industries, the contaminant infiltration will be regular and hence it may exhibit an increasing tendency. To depict this condition mathematically, an exponentially increasing function of time has been taken for dispersion as well as flow velocity. It is observed that the non-reactive solute concentration increases with time near the source but the solute concentration decreases with time away from the source. However, the non-reactive solute concentration decreases with distance and goes on decreasing to reach the minimum or harmless concentration in the domain of the aquifer. This representation is shown graphically in Fig. 5.1. Curves 1-4 in Fig. 5.1 represent the contaminant concentration in 0.4, 1.0, 1.6 and 2.2 years, respectively. The concentration values at each position and in each of the curve are higher in the case of exponential form of velocity expression than that of sinusoidal form of velocity expression.
Sensitivity Analysis:

The analytical solution given by Eq. (5.30) is computed using a MATLAB program and the concentration values thus obtained are shown in Table 5.2(a-d) with input values of sensitive parameters like dispersion, seepage velocity and decay rate coefficients in the range of $D_0 = 0.2 - 0.25 (km^2/year)$, $u_0 = 0.02 - 0.025 (km/year)$, and $q = 0.0003 - 0.001 (/year)$. The concentration values at different positions are obtained in row (i) and (ii) for the exponential form of expression and sinusoidal form of expression respectively. It can be observed that the concentration values in both the forms, i.e., sinusoidal and exponential are equal at the origin but as we move on the concentration values in sinusoidal form of the velocity are less than that of the exponential form of velocity at each position with respect to time and space. Also it can be observed that the concentration values increase initially when we increase the values of sensitive parameters $D_0, u_0$ and $q$ however the concentration values decrease with respect to distance.

When we decrease the values of $D_0, u_0$ and $q$ beyond the range mentioned above, the concentration values show anomaly. For example, if we go on decreasing the values of sensitive parameters less than the values mention above then initially the concentration values decrease with the distance but they start increasing at the other end of the system. Hence, it will obviously distort the nature of the solution in this way and must be bound with some suitable values. With this need we fixed a range of values for sensitive parameters mentioned above and found that our solution for the present problem is bounded and acceptable for this particular range of values of sensitive parameters.
Table 5.2(a) Contaminant Concentration with $D_0 = 0.25$, $u_0 = 0.025$ and $q = 0.001$ for

(i) Exponential form (ii) Sinusoidal form

Table 5.2(b) Contaminant Concentration with $D_0 = 0.22$, $u_0 = 0.022$ and $q = 0.0007$ for

(i) Exponential form (ii) Sinusoidal form
Table 5.2(c) Contaminant Concentration with $D_0 = 0.21$, $u_0 = 0.021$ and $q = 0.0005$ for

(i) Exponential form (ii) Sinusoidal form

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Table 5.2(d) Contaminant Concentration with $D_0 = 0.2$, $u_0 = 0.02$ and $q = 0.0003$ for

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5.4 Conclusion

Laplace Transform Technique is used for the analytical solution of one-dimensional solute transport along transient groundwater flow with time dependent dispersion in a homogeneous semi-infinite aquifer. An input concentration in the form of Cauchy type boundary condition has been taken at the origin. The solution obtained with the Cauchy type boundary condition is more realistic as the linear combination of concentration and its gradients are prescribed on the boundary. The results obtained for two expressions of temporally dependent dispersion, such as sinusoidally and exponentially increasing forms, are more realistic, because the time-dependent input concentration is considered at the source.
Fig. 5.1 Time-dependent contaminant concentration along transient groundwater flow in a semi-infinite aquifer subject to temporally dependent dispersion with Cauchy type input condition.