CHAPTER-IV
ONE-DIMENSIONAL SOLUTE DISPERSION ALONG TRANSIENT FLOW WITH
DIRICHLET TYPE BOUNDARY CONDITIONS: DISPERSION COEFFICIENT IS
DIRECTLY PROPORTIONAL TO SQUARE OF SEEPAGE VELOCITY

4.1 Introduction

Groundwater, in general, is considered mostly pure and cleaner than surface water and in many developing countries, such as India, it is not even treated before use. However, this pristine source of water is under increasing threat of contamination. In a country like India, groundwater at many places is nowadays being contaminated by a host of human activities, such as unscientific sewage disposal, refuse dumps, pesticide and chemical fertilizer contamination, industrial effluent discharges, and toxic waste disposal (Sharma and Reddy, 2004; Rausch et al., 2005; Thangarajan, 2006). The advective-dispersive equation is widely used in describing pollutant distribution in aquifers, oil reservoirs, rivers, lakes and air. This equation is also used in describing similar phenomena in bio-physics and bio-medical sciences. In the context of pollutant dispersion in groundwater, this equation was solved analytically and numerically using appropriate initial and boundary conditions. Earlier studies like those of Banks and Ali (1964), Ogata (1970), Lai and Jurinak (1971) considered stationary flow through a homogeneous and isotropic porous domain. Many such studies were reviewed by Van Genuchten and Alves, (1982) and Lindstrom and Boersma (1989).

Analytical solutions are usually derived using simplifications but are free from numerical dispersion and other truncation errors that often occur in numerical simulations (Zheng and Bennett, 1995). With the use of analytical solutions for estimating the movement of contaminant plumes, one can save a lot of effort to guide and collect data and monitor water quality despite complexities of hydrogeologic systems (Wexler, 1992). There is a plethora of studies on one-dimensional solute transport studies. Chrysikopoulos and Sim (1996) discussed virus transport in homogeneous porous media with a time dependent distribution coefficient. Analytical solutions of advection-diffusion equation with a
spatially variable coefficient in up to three dimensions were discussed by Zoppou and Knight (1999). Analytical solutions of transient unsaturated transport of water and contaminants through horizontal porous media were presented by Sander and Braddock (2005). Analytical solution for transport of decaying solutes in rivers with transient storage was derived by Smedt (2006). Longitudinal dispersion with time dependent source concentration in a semi-infinite aquifer was investigated by Singh et al. (2008). Analytical solutions for sequentially coupled one-dimensional reactive transport problems were explored by Srinivasan and Clement (2008).

For analyses two possible relationships between parameters, such as dispersion $D$ and seepage velocity $u$, have been proposed: (i) $D$ is proportional to $u$ and (ii) $D$ is proportional to $u^2$ (Scheidegger, 1957; Freeze and Cherry, 1979; Zoppou and Knight, 1999; Ghosh and Sharma, 2006; Jaiswal et al., 2011). The objective of the present studies that to use $D$ is proportional to $u^2$ relationship and explain the possible analytical solution with two different approaches: (i) stationary dispersion with transient velocity and (ii) temporally dependent dispersion with transient velocity. The time-dependent dispersivity behaviour of non-reactive solutes in a system of parallel structures was described by Kumar et al. (2006). In a recent study, an analytical solution for conservative solute transport in one-dimensional homogeneous porous formations with time-dependent velocity was derived by Singh et al. (2009). Analytical solution to one-dimensional advection-diffusion equation with variable coefficient in semi-infinite media was explored by Kumar et al. (2010) with the use of the above assumptions. A solute transport along temporally and spatially dependent flow through horizontal semi-infinite media in which dispersion is proportional to the square of velocity concept has been used by Jaiswal et al. (2011).

4.2 Mathematical Formulation

We consider a one-dimensional advective-dispersive equation in a homogeneous isotropic semi-infinite aquifer. Let there be some initial contaminant concentration in the aquifer system which is represented by an exponentially decreasing function in space at $t = 0$. Also, a time-dependent source contaminant concentration is injected at the head of
the aquifer, i.e., at $x=0$ and let the contaminant concentration gradient at an infinite distance away from the source be zero at all times. This reveals that the contaminant concentration at an infinite extent is supposed to be uniform or zero at all times. Let $C(x,t)$ be the contaminant concentration in the aquifer [ML$^{-3}$] at any time $t$ [T]. Let $u$ be the groundwater velocity in the porous medium transporting the solute particles [LT$^{-1}$], and $D$ be the solute dispersion [L$^2$T$^{-1}$]. The dispersion directly proportional to square of the seepage velocity is used. The molecular diffusion may depend on the solute but the effect of molecular diffusion in this study is not taken into account, because the value of molecular diffusion does not vary significantly for different soils and contaminant behaviors and it ranges from $1\times10^{-9}$ to $2\times10^{-9}$ m$^2$/s (Mitchell, 1976).

### 4.2.1 Case-I: Uniform dispersion with transient velocity

The advective-dispersive problem in one-dimensional form can be written as

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left[ D(x,t) \frac{\partial C}{\partial x} - u(x,t) C \right]$$  \hspace{1cm} (4.1)

Let us assume:

$$D(x,t) = D_0; \quad u(x,t) = u_0 f(mt)$$  \hspace{1cm} (4.2)

where $D_0$ is the constant initial dispersion coefficient, and $u_0$ is the initial groundwater velocity. Here, $f(mt)$ is an arbitrary function.

The initial and boundary conditions can now be written as:

$$C(x,t) = \exp\left(-\gamma x\right); \quad t = 0, x \geq 0$$  \hspace{1cm} (4.3)

$$C(x,t) = \frac{C_0}{2} \left\{ 1 + \exp\left(-q t\right) \right\}; \quad x = 0; t > 0$$  \hspace{1cm} (4.4)

$$\frac{\partial C}{\partial x} = 0; \quad x \to \infty; t \geq 0$$  \hspace{1cm} (4.5)


where $C_i$ is the initial solute concentration [$ML^{-3}$] included with an exponentially decreasing function of space describing the distribution of concentration at all points in the flow domain, i.e., at $t = 0$, and $\gamma$ is a constant coefficient parameter whose dimension is the inverse of the space variable, i.e., $[L^{-1}]$. Here, $C_0$ is the non-reactive solute concentration [$ML^{-3}$], and $q$ is the decay rate coefficient [$T^{-1}$]. In most of the solute transport problems dealt with by the various researchers, the initial condition is assumed to be zero only to avoid the complexity of the solution of the problem. But this is not the case always in physical situations which can easily be understood with an example. Any geological formation, for example, a semi-infinite or finite aquifer which is assumed initially solute free meaning no solute concentration may not be so. It may happen that some initial background concentration in the aquifer exists which was ignored by the previous researchers and it is now taken into account by incorporating an exponentially decreasing function of space variable as an initial condition. The waste material or industrial effluents from the Jharia coal field, Dhanbad, is discharged into Damoder River which is being polluted with time. The location is south eastern part of Jharia coal field south of Damoder River bounded by latitude (N) 23° 39' 24" to 23° 40' 0" and longitude (E) 86° 24' 01" to 86° 25' 02" . The time-dependent input source concentration is considered as a boundary condition. However, many of the researchers assumed a stationary source of input concentration for mathematical convenient.

### 4.2.2 Analytical Solution

Eq. (4.1) and Eq. (4.2) can be combined as:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left[ D_0 \frac{\partial C}{\partial x} - u_0 f (m t) C \right] \tag{4.6}$$

Introducing a new independent variable, $X$, with the transformation (Jaiswal et al. 2009):

$$\frac{dX}{dx} = -\frac{1}{f (m t)} \quad \text{or} \quad X = -\int_{0}^{x} \frac{dx}{f (m t)} = -\frac{x}{f (m t)} \tag{4.7}$$
Eq. (4.6) can now be written as follows:

\[
\left[ f(m \, t) \right]^2 \left[ \frac{\partial C}{\partial t} \frac{\partial T}{\partial t} + \frac{\partial C}{\partial X} \frac{\partial X}{\partial t} \right] = D_0 \frac{\partial^2 C}{\partial X^2} + u_0 \left[ f(m \, t) \right]^2 \frac{\partial C}{\partial X}
\]

This can further simplify under the following transformation (Crank 1975)

\[
T = \int_0^t \left[ f(m \, t) \right]^2
\]

as

\[
\left[ f(m \, t) \right]^2 \left[ \frac{\partial C}{\partial T} \left[ f(m \, t) \right]^2 + \frac{1}{\left[ f(m \, t) \right]^2} \frac{x m f'(m \, t)}{\left[ f(m \, t) \right]^2} \frac{\partial C}{\partial X} \right] = D_0 \frac{\partial^2 C}{\partial X^2} + u_0 \left[ f(m \, t) \right]^2 \frac{\partial C}{\partial X}
\]

i.e., using Eq. (4.7), we get

\[
\frac{\partial C}{\partial T} - X m f'(m \, t) f(m \, t) \frac{\partial C}{\partial X} = D_0 \frac{\partial^2 C}{\partial X^2} + u_0 \left[ f(m \, t) \right]^2 \frac{\partial C}{\partial X}
\]

\[
\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial X^2} + u_0 \left[ f(m \, t) \right]^2 \frac{\partial C}{\partial X} + m f'(m \, t) f(m \, t) X \frac{\partial C}{\partial X}
\]

i.e.,

\[
\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial X^2} + u_0 \left[ f(m \, t) \right]^2 \frac{\partial C}{\partial X} + m f'(m \, t) f(m \, t) X \frac{\partial C}{\partial X}
\]  

Here, \( f(m \, t) \) is chosen such that \( f(m \, t) = 1 \) for \( m = 0 \) which represents the temporally independent dispersion, i.e., stationary non-reactive solute dispersion from a point source along a stationary flow domain or \( t = 0 \) which corresponds to the initial stage. Here, \( m \) is the flow resistance coefficient whose dimension is the inverse of time, i.e., [T\(^{-1}\)]. Accordingly, \( f'(m \, t) = 0 \) as \( f(m \, t) = 1 \) is assumed and hence the 3\(^{rd}\) term of r. h. s of Eq. (4.9) is eliminated for mathematical convenience.
Finally, Eq. (4.9) can be written as follows:

\[
\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial X^2} + u_0 \left[ f(mt) \right]^2 \frac{\partial C}{\partial X} \tag{4.10}
\]

Since we assumed that dispersion is directly proportional to the square of seepage velocity, i.e.,

\[D = bu^2\tag{4.11}\]

where \(b\) is a constant coefficient [T] (Scheidegger, 1957; Freeze and Cherry, 1979; Zoppou and Knight, 1999; Ghosh and Sharma, 2006; Jaiswal et al., 2011).

Using Eq. (4.2), Eq. (4.11) becomes

\[
D_0 = bu_0^2 \left[ f(mt) \right]^2 \tag{4.12a}
\]

\[
\frac{D_0}{bu_0} = u_0 \left[ f(mt) \right]^2 \tag{4.12b}
\]

\[
U_0 = u_0 \left[ f(mt) \right]^2 \tag{4.12c}
\]

where \(U_0 = \frac{D_0}{bu_0}\). \tag{4.13}

For stationary values, \(f(mt) = 1\) and so Eq. (4.12c) becomes \(U_0 = u_0\) and hence by using Eq. (4.13) we can get

\[u_0 = \frac{D_0}{bu_0} \Rightarrow u_0^2 = \frac{D_0}{b} \Rightarrow u_0 = \sqrt{\frac{D_0}{b}}.\tag{4.14}\]

Using Eq. (4.12c), Eq. (4.10) can be written as

\[
\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial X^2} + U_0 \frac{\partial C}{\partial X} \tag{4.15}
\]
Using Eqs. (4.7) and (4.8), the conditions expressed by Eqs. (4.3) to (4.5) can be written as

\[ C(X, T) = C_i \exp(\gamma X); \quad T = 0; -\infty < X \leq 0 \]  
\[ C(X, T) = \frac{C_0}{2} \{1 + \exp(-qT)\}; \quad X = 0; T > 0 \]  
\[ \frac{\partial C}{\partial X} = 0; \quad X \to -\infty; T \geq 0 \]

Now introduce another new independent variable:

\[ Z = -X \]

and a new dependent variable using the following transformation defined as:

\[ C(Z, T) = K(Z, T) \exp \left( \frac{U_0}{2D_0} Z - \frac{U_0^2}{4D_0} T \right) \]

Using Eq. (4.19), we obtain

\[ \frac{\partial C}{\partial X} = \frac{\partial C}{\partial Z} \frac{\partial Z}{\partial X} = -\frac{\partial C}{\partial Z} \]

\[ \frac{\partial^2 C}{\partial X^2} = \frac{\partial^2 C}{\partial Z^2} \frac{\partial Z}{\partial X} = \frac{\partial^3 C}{\partial Z^3} \]

Using above results, Eq. (4.15), (4.16), (4.17), and (4.18) are transformed to

\[ \frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial Z^2} - U_0 \frac{\partial C}{\partial Z} \]  
\[ C(Z, T) = C_i \exp(-\gamma Z); \quad T = 0, Z \geq 0 \]  
\[ C(Z, T) = \frac{C_0}{2} \left[1 + \exp(-qT)\right]; \quad Z = 0; T > 0 \]
\[
\frac{\partial C}{\partial Z} = 0; \quad Z \to \infty; T \geq 0 \quad \text{(4.21d)}
\]

Now, using Eq. (4.20) we obtain

\[
\frac{\partial C}{\partial Z} = \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \frac{\partial K}{\partial Z} + \frac{U_0}{2D_0} \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] K
\]

\[
= \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ \frac{\partial K}{\partial Z} + \frac{U_0}{2D_0} K \right]
\]

\[
\frac{\partial^2 C}{\partial Z^2} = \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ \frac{\partial^2 K}{\partial Z^2} + \frac{U_0}{2D_0} \frac{\partial K}{\partial Z} + \frac{U_0^2}{4D_0} \right] + \frac{U_0}{2D_0} \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ \frac{\partial K}{\partial Z} + \frac{U_0}{2D_0} K \right]
\]

\[
= \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ \frac{\partial^2 K}{\partial Z^2} + \frac{U_0}{D_0} \frac{\partial K}{\partial Z} + \frac{U_0^2}{4D_0} K \right]
\]

\[
\frac{\partial C}{\partial T} = \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \frac{\partial K}{\partial T} + \left( -\frac{U_0^2}{4D_0} \right) \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] K
\]

\[
= \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ \frac{\partial K}{\partial T} - \frac{U_0^2}{4D_0} K \right]
\]

Using these transformations in Eqs. (4.21a) to (4.21d), we have:

Eq. (4.21a) becomes

\[
\exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ \frac{\partial K}{\partial T} - \frac{U_0^2}{4D_0} K \right] = \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ D_0 \frac{\partial^2 K}{\partial Z^2} + \frac{U_0}{4D_0} \frac{\partial K}{\partial Z} + \frac{U_0^2}{4D_0} K \right]
\]

\[
- \exp \left[ \frac{U_0 - Z - U_0^2}{2D_0} T \right] \left[ \frac{U_0}{2D_0} \frac{\partial K}{\partial Z} + \frac{U_0^2}{4D_0} K \right]
\]

or,

\[
\frac{\partial K}{\partial T} - \frac{U_0^2}{4D_0} K = D_0 \frac{\partial^2 K}{\partial Z^2} + \frac{U_0}{4D_0} \frac{\partial K}{\partial Z} + \frac{U_0^2}{4D_0} K - \frac{U_0}{2D_0} \frac{\partial K}{\partial Z} - \frac{U_0^2}{2D_0} K
\]
\[
\frac{\partial K}{\partial T} = D_0 \frac{\partial^2 K}{\partial Z^2}
\] (4.22)

Eq. (4.21b) becomes

\[
K(Z,T) \exp \left[ \frac{U_0}{2D_0} Z \right] = C_i \exp(-\gamma Z); \quad Z \geq 0; T = 0
\]

\[
K(Z,T) = C_i \exp \left\{ - \left( \frac{U_0}{2D_0} + \gamma \right) Z \right\}; \quad Z \geq 0; T = 0
\] (4.23)

Eq. (4.21c) becomes

\[
K(Z,T) \exp \left[ - \frac{U_0^2}{4D_0} T \right] = \frac{C_0}{2} \left[ 2 - qT \right]
\]

\[
K(Z,T) = \frac{C_0}{2} \left\{ 2 \exp \left( \frac{U_0^2}{4D_0} T \right) - qT \exp \left( \frac{U_0^2}{4D_0} T \right) \right\}; \quad Z = 0; T > 0
\] (4.24)

Eq. (4.21d) becomes

\[
\frac{\partial K}{\partial Z} = - \frac{U_0}{2D_0} K; \quad Z \rightarrow \infty; T \geq 0
\] (4.25)

Applying the Laplace transform (Sneddon, 1974) to Eqs. (4.22) to (4.25), we obtain

\[
\frac{d^2 \overline{K}}{dZ^2} - \frac{p}{D_0} \overline{K} = - \frac{C_i}{D_0} \exp \left( - \xi Z \right)
\] (4.26)

\[
\overline{K}(Z, p) = \frac{C_0}{2} \left\{ \frac{2}{(p - \beta^2)} - \frac{q}{(p - \beta^2)^2} \right\}; \quad Z = 0, T > 0
\] (4.27)

\[
\frac{d \overline{K}}{dZ} = - \frac{U_0}{2D_0} \overline{K}; \quad Z \rightarrow \infty; T \geq 0
\] (4.28)
where \( \xi = \left( \frac{U_0}{2D_0} + \gamma \right) \), \( \beta^2 = \frac{U_0^2}{4D_0} \) and \( \overline{K}(Z,p) = \int_0^\infty K(Z,T) \exp(-pT) dT \).

The general solution of Eq. (4.26) can be written as

\[
\overline{K}(Z,p) = C_1 \exp\left(-Z \sqrt{\frac{p}{D_0}} \right) + C_2 \exp\left(Z \sqrt{\frac{p}{D_0}} \right) + \frac{C_i}{p - D_0 \xi^2} \exp(-\xi Z) \tag{4.29}
\]

Putting \( Z = 0 \) in Eq. (4.29) and using the condition in Eq. (4.27), we obtain:

\[
C_1 + C_2 = \frac{C_0}{2} \left( \frac{2}{(p - \beta^2)} - \frac{q}{(p - \beta^2)^2} \right) - \frac{C_i}{p - D_0 \xi^2} \tag{4.30}
\]

From Eq. (4.29), we have

\[
\frac{d\overline{K}}{dZ} = -\sqrt{\frac{p}{D_0}} C_1 \exp\left(-Z \sqrt{\frac{p}{D_0}} \right) + C_2 \sqrt{\frac{p}{D_0}} \exp\left(Z \sqrt{\frac{p}{D_0}} \right) - \frac{\xi C_i}{p - D_0 \xi^2} \exp(-\xi Z) \tag{4.31a}
\]

Putting \( Z \to \infty \) and using Eq. (4.28) in Eq. (4.31a), we obtain \( C_2 = 0 \), and then putting this value of \( C_2 \) in Eq. (4.30), we obtain the value of \( C_1 \) as

\[
C_1 = \frac{C_0}{2} \left( \frac{2}{(p - \beta^2)} - \frac{q}{(p - \beta^2)^2} \right) - \frac{C_i}{p - D_0 \xi^2} \tag{4.31b}
\]

Inserting these values of \( C_1 \) and \( C_2 \) in Eq. (4.29), we have the general solution as

\[
\overline{K}(Z,p) = \left[ \frac{C_0}{2} \left( \frac{2}{(p - \beta^2)} - \frac{q}{(p - \beta^2)^2} \right) - \frac{C_i}{p - D_0 \xi^2} \right] \exp\left(-Z \sqrt{\frac{p}{D_0}} \right) + \frac{C_i}{p - D_0 \xi^2} \exp(-\xi Z) \tag{4.32}
\]

Taking inverse Laplace Transform (Bateman, 1954) of Eq. (4.32), we obtain
\[ K(Z,T) = \frac{C_0}{2} \left\{ \exp\left( \frac{U_0^2 T}{4D_0} - \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z - U_0 T}{2\sqrt{D_0 T}} \right) + \exp\left( \frac{U_0^2 T}{4D_0} + \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z + U_0 T}{2\sqrt{D_0 T}} \right) \right\} \\
- \frac{gC_0}{2} \left\{ \frac{1}{2U_0} (U_0 T - Z) \exp\left( \frac{U_0^2 T}{4D_0} - \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z - U_0 T}{2\sqrt{D_0 T}} \right) \right.
+ \left. \frac{1}{2U_0} (U_0 T + Z) \exp\left( \frac{U_0^2 T}{4D_0} + \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z + U_0 T}{2\sqrt{D_0 T}} \right) \right\} \\
- \frac{C_i}{2} \left\{ \exp\left( D_0 \xi^2 T - \bar{\xi} Z \right) \text{erfc}\left( \frac{Z}{2\sqrt{D_0 T}} - \bar{\xi} \sqrt{D_0 T} \right) \right.
+ \exp\left( D_0 \xi^2 T + \bar{\xi} Z \right) \text{erfc}\left( \frac{Z}{2\sqrt{D_0 T}} + \bar{\xi} \sqrt{D_0 T} \right) \right\} \\
+ C_i \exp(-\bar{\xi} Z) \exp\left( \xi^2 D_0 T \right) \right. \\
\] \\
or, \\
\[ K(Z,T) = \frac{C_0}{2} \left\{ \exp\left( \frac{U_0^2 T}{4D_0} - \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z - U_0 T}{2\sqrt{D_0 T}} \right) + \exp\left( \frac{U_0^2 T}{4D_0} + \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z + U_0 T}{2\sqrt{D_0 T}} \right) \right\} \\
- \frac{gC_0}{2} \left\{ \frac{1}{2U_0} (U_0 T - Z) \exp\left( \frac{U_0^2 T}{4D_0} - \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z - U_0 T}{2\sqrt{D_0 T}} \right) \right. \\
+ \left. \frac{1}{2U_0} (U_0 T + Z) \exp\left( \frac{U_0^2 T}{4D_0} + \frac{U_0}{2D_0} Z \right) \text{erfc}\left( \frac{Z + U_0 T}{2\sqrt{D_0 T}} \right) \right\} \\
- \frac{C_i}{2} \left\{ \exp\left( D_0 \xi^2 T - \bar{\xi} Z \right) \text{erfc}\left( \frac{Z}{2\sqrt{D_0 T}} - \bar{\xi} \sqrt{D_0 T} \right) \right. \\
+ \exp\left( D_0 \xi^2 T + \bar{\xi} Z \right) \text{erfc}\left( \frac{Z}{2\sqrt{D_0 T}} + \bar{\xi} \sqrt{D_0 T} \right) \right\} \\
+ C_i \exp(-\bar{\xi} Z) \exp\left( \xi^2 D_0 T \right) \right. \\
\] \\

(4.33) \\
Using the transform given in Eq. (4.20) backward, we get the solution as follows:
\[ C(Z,T) = \frac{C_0}{2} \left[ \text{erfc} \left( \frac{Z-U_0T}{2\sqrt{D_0T}} \right) + \exp \left( \frac{U_0Z}{D_0} \right) \text{erfc} \left( \frac{Z+U_0T}{2\sqrt{D_0T}} \right) \right] \]

\[ -\frac{qC_0}{4U_0} \left( (U_0T-Z) \text{erfc} \left( \frac{Z-U_0T}{2\sqrt{D_0T}} \right) + (U_0T+Z) \exp \left( \frac{U_0Z}{D_0} \right) \text{erfc} \left( \frac{Z+U_0T}{2\sqrt{D_0T}} \right) \right) \]

\[ -\frac{C}{2} \exp \left( \gamma^2 D_0 - \gamma U_0 T - \gamma Z \right) \left\{ \text{erfc} \left( \frac{Z-(U_0 + 2D_0\gamma)T}{2\sqrt{D_0T}} \right) + \exp \left( \frac{U_0}{D_0} + 2\gamma \right) Z \text{erfc} \left( \frac{Z+(U_0 + 2D_0\gamma)T}{2\sqrt{D_0T}} \right) \right\} \]

\[ + C \exp \left( \gamma^2 D_0 - \gamma U_0 T - \gamma Z \right) \]

(4.34)

Again, using the transforms given in Eqs. (4.19) and (4.7) backward, we obtain the solution as follows:

\[ C(x,T) = F(x,T) - G(x,T) - H(x,T) + I(x,T) \]

(4.35)

where

\[ F(x,T) = \frac{C_0}{2} \left[ \text{erfc} \left( \frac{x}{2\sqrt{D_0T}} - \frac{U_0T}{f(mt)} \right) + \exp \left( \frac{U_0x}{D_0f(mt)} \right) \text{erfc} \left( \frac{x}{2\sqrt{D_0T}} + \frac{U_0T}{f(mt)} \right) \right] \]

(4.36a)

\[ G(x,T) = \frac{qC_0}{4U_0} \left\{ \left( U_0T - \frac{x}{f(mt)} \right) \text{erfc} \left( \frac{x}{2\sqrt{D_0T}} - \frac{U_0T}{f(mt)} \right) + \left( U_0T + \frac{x}{f(mt)} \right) \exp \left( \frac{U_0x}{D_0f(mt)} \right) \text{erfc} \left( \frac{x}{2\sqrt{D_0T}} + \frac{U_0T}{f(mt)} \right) \right\} \]

(4.36b)
\[
H(x, T) = \frac{1}{2} I(x, T) \left[ \text{erfc} \left( \frac{x}{f(mt)} \left( \frac{U_0 + 2D_0\gamma}{2\sqrt{D_0T}} \right) \right) 
+ \exp \left( \frac{U_0}{D_0} + 2\gamma \right) \left( \frac{x}{f(mt)} \right) \times \text{erfc} \left( \frac{x}{f(mt)} + \left( \frac{U_0 + 2D_0\gamma}{2\sqrt{D_0T}} \right) \right) \right]
\]

(4.36c)

\[
I(x, T) = C_0 \exp \left( \left( \gamma^2 D_0 - \gamma U_0 \right) T - \gamma \frac{x}{f(mt)} \right)
\]

(4.36d)

**Particular Case:**

If we put \( q = 0 \) in Eq. (4.35), then the solution of the problem can be written as

\[
C(x, T) = F(x, T) - H(x, T) + I(x, T)
\]

(4.37)

where

\[
F(x, T) = \frac{C_0}{2} \left[ \text{erfc} \left( \frac{x}{f(mt)} - \frac{U_0T}{2\sqrt{D_0T}} \right) + \exp \left( \frac{U_0x}{D_0f(mt)} \right) \text{erfc} \left( \frac{x}{f(mt)} + \frac{U_0T}{2\sqrt{D_0T}} \right) \right]
\]

(4.38a)
\[ H(x,T) = \frac{1}{2} I(x,T) \left[ \text{erfc} \left( \frac{x}{f(mt)} - \left( \frac{U_0 + 2D_0\gamma}{2\sqrt{D_0T}} \right) T \right) \right] \]

\[ + \exp \left( \frac{U_0}{D_0} + 2\gamma \right) \left( \frac{x}{f(mt)} \right) \times \text{erfc} \left( \frac{x}{f(mt)} + \left( \frac{U_0 + 2D_0\gamma}{2\sqrt{D_0T}} \right) T \right) \]  

\[ (4.38b) \]

\[ I(x,T) = C_i \exp \left\{ \left( \gamma^2 D_0 - \gamma U_0 \right) T - \gamma \frac{x}{f(mt)} \right\} \]  

\[ (4.38c) \]

### 4.2.3 Case-II :- Time varying dispersion with transient velocity

The advective-dispersive equation in one-dimensional form can be written as

\[ \frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left[ D(x,t) \frac{\partial C}{\partial x} - u(x,t) C \right] \]  

\[ (4.39) \]

Let \( D(x,t) = D_0 \left[ f(m,t) \right]^2; \ u(x,t) = u_0 f(m,t) \)  

\[ (4.40) \]

where \( D_0 \) is the constant initial dispersion coefficient, and \( u_0 \) is the initial groundwater velocity. Here, \( f(m,t) \) is an arbitrary function.

The function \( f(m,t) \) is chosen such that the following condition is satisfied as in the previous cases i.e.,

\[ f(m,t) = 1 \] for \( m = 0 \) or \( t = 0 \).  

\[ (4.41) \]
The initial and boundary conditions can now be expressed as:

\[ C(x, t) = C_i \exp(-\gamma x); \quad t = 0, x \geq 0 \]  
\[ C(x, t) = \frac{C_0}{2} \{1 + \exp(-qt)\}; \quad x = 0; t > 0 \]  
\[ \frac{\partial C}{\partial x} = 0; \quad x \to \infty; t \geq 0 \]

where \( C_i \) is the initial non-reactive solute concentration \([\text{ML}^{-3}]\) included with an exponentially decreasing function of space describing the distribution of concentration at all points of the flow domain, i.e., at \( t = 0 \), and \( \gamma \) is a constant coefficient parameter whose dimension is the inverse of the space variable, i.e., \([\text{L}^{-1}]\). Here, \( C_0 \) is the solute concentration \([\text{ML}^{-3}]\), and \( q \) is the decay rate coefficient \([\text{T}^{-1}]\).

### 4.2.4 Analytical Solution

Using the same procedure as in case-I, we can obtain the solution as follows:

\[ C(x, t) = F(x, t) - G(x, t) - H(x, t) + I(x, t) \]  

where,

\[ F(x, t) = \frac{C_0}{2} \left[ \text{erfc} \left( \frac{x - u_0 t}{2\sqrt{D_0 t}} \right) \right] + \exp \left( \frac{u_0 x}{D_0 f(mt)} \right) \text{erfc} \left( \frac{x + u_0 t}{2\sqrt{D_0 t}} \right) \]  

\[ (4.46a) \]
\[
G(x,t) = \frac{qC_0}{4u_0} \left[ u_0 t - \frac{x}{f(mt)} \text{erfc} \left( \frac{x}{2\sqrt{D_0 t}} \right) \right] \\
+ \left[ u_0 t + \frac{x}{f(mt)} \right] \exp \left( \frac{u_0 x}{D_0 f(mt)} \right) \text{erfc} \left( \frac{x}{2\sqrt{D_0 t}} + u_0 t \right) \quad (4.46b)
\]

\[
H(x,t) = \frac{1}{2} I(x,t) \left[ \frac{x}{f(mt)} - (u_0 + 2D_0 \gamma) t \right] \text{erfc} \left( \frac{x}{2\sqrt{D_0 t}} \right) \\
+ \exp \left( \frac{u_0}{D_0} + 2\gamma \right) \left( \frac{x}{f(mt)} \right) \times \text{erfc} \left( \frac{x}{2\sqrt{D_0 t}} + (u_0 + 2D_0 \gamma) t \right) \quad (4.46c)
\]

\[
I(x,t) = C \exp \left( \gamma^2 D_0 - \gamma u_0 \right) t - \gamma \frac{x}{f(mt)} \quad (4.46d)
\]

**Particular Case:**

If we put \( q = 0 \) in Eq. (4.45), then the solution can be written as

\[
C(x,t) = F(x,t) - H(x,t) + I(x,t) \quad (4.47)
\]

where,

\[
F(x,t) = \frac{C_0}{2} \left[ \text{erfc} \left( \frac{x}{2\sqrt{D_0 t}} \right) + \exp \left( \frac{u_0 x}{D_0 f(mt)} \right) \text{erfc} \left( \frac{x}{2\sqrt{D_0 t}} + u_0 t \right) \right] \quad (4.48a)
\]
\[ H(x,t) = \frac{1}{2} I(x,t) \left[ \text{erfc} \left( \frac{x}{f(mt)} - (u_0 + 2D_0\gamma)t \right) \right] \]

\[ + \exp \left( \frac{u_0}{D_0} + 2\gamma \right) \left( \frac{x}{f(mt)} \right) \times \text{erfc} \left( \frac{x}{f(mt)} + (u_0 + 2D_0\gamma)t \right) \]

\[ (4.48b) \]

\[ I(x,t) = C_i \exp \left( (\gamma^2 D_0 - \gamma u_0)t - \gamma \frac{x}{f(mt)} \right) \]

\[ (4.48c) \]

4.3 Illustrative Example and Discussion

To represent the temporally dependent dispersion, we consider \( f(mt) \) in general and sinusoidal and exponential forms of \( f(mt) = 1 - \sin(mt) \), and \( f(mt) = \exp(mt) \), respectively, in particular. Some other similar types of expressions satisfying the condition for \( f(mt) \) i.e., \( f(mt) = 1 \) for \( m = 0 \) which represents the temporally independent dispersion, i.e., stationary non-reactive solute dispersion from a point source along a stationary flow domain or \( t = 0 \) which corresponds to the initial stage. The time-dependent dispersion along a stationary flow was considered to derive an analytical solution using the method of superposition principle (Aral and Liao, 1996).

An analytical solution for stationary dispersion with transient velocity given by Eq. (4.35) was computed for an input data of \( C_i = 0.01 \), \( C_0 = 1.0 \), \( D_0 = 0.14(km^2/year) \), \( u_0 = 0.25(km/year) \), \( m = 0.1(1/year) \), \( \gamma = 0.001(1/km) \), \( b = 2.24(year) \), and \( q = 0.054(1/year) \) in order to depict the variation of contaminant concentration along transient groundwater flow in the space domain \( 0 \leq x \leq 2(km) \) and the time domain \( 0.4 \leq t \leq 1.6(years) \). The solution was computed with stationary dispersion along with the sinusoidal form of time varying velocity, \( u = u_0 f(mt) = u_0 [1 - \sin(mt)] \) and the exponential form of time varying velocity, \( u = u_0 f(mt) = u_0 \exp(mt) \). It was observed that the non-
reactive solute concentration decreased with time near the source but the solute concentration increased with time away from the source. However, the non-reactive solute concentration decreased with distance and went on decreasing to reach the minimum or harmless concentration in the domain of the aquifer extended up to $x = 2.0(km)$. This representation is shown graphically in Fig. 4.1(a). Curves 1-5 in Fig. 4.1(a) represent the contaminant concentration in 0.4, 0.7, 1.0, 1.3 and 1.6 years, respectively. The values of concentration at each position were lower for the sinusoidal form of velocity than for the exponential form of velocity. This means that we may observe less solute concentration with sinusoidal form of velocity which often represents the tropical regions in Indian subcontinent. In aquifers in tropical regions, groundwater velocity and water level may exhibit seasonally sinusoidal behavior (as noted by Kumar and Kumar, 1998; Thangarajan, 2006). In tropical regions in India, groundwater velocity and water level are minimum during the peak of the summer season (the period of greatest pumping), which falls in the month of June, just before rainy season. Maximum values are observed during the peak of winter season around December, after the rainy season (the period of lowest pumping). In these regions, groundwater infiltration is from rainfall and rivers. Groundwater flow depends on aquifer properties, such as porosity, permeability, hydraulic conductivity, etc. For a homogeneous aquifer, its properties are spatially invariant. In particular, if we can remove the decay rate coefficient, i.e., $q = 0$, then the problem becomes a stationary source of concentration given by Eq. (4.37) and the representation is shown graphically in Fig. 4.1(b) in which we can observe the same decreasing tendency of contaminant concentration.

The analytical solution for temporally dependent dispersion with transient velocity given by Eq. (4.45) was computed for the same set of input values $C_i = 0.01$, $C_o = 1.0$, $D_o = 0.14(km^2/year)$, $u_o = 0.25(km/year)$, $m = 0.1(/year)$, $\gamma = 0.001(/km)$, and $q = 0.054(/year)$. The representation is shown graphically in Fig. 4.2(a) which depicts a similar distribution pattern of contaminant concentration. However, the difference between the concentration values for both types of velocity expressions is little wider in comparison to Fig. 4.1(a). In particular, if we can remove the decay rate coefficient, i.e., $q = 0$, then the problem becomes a stationary source of concentration given by Eq. (4.47) and the
representation is shown graphically in Fig. 4.2(b) in which the distribution pattern of contaminant concentration is also depicted. Figs. 4.1(a,b) - 4.2(a,b) are depicted up to scale \( x = 2.0(km) \) in which the minimum or harmless concentration can be observed beyond the distance \( x = 1.0(km) \).

4.4 Conclusion

Analytical solutions for one-dimensional non-reactive solute dispersion along transient groundwater flow with stationary and time varying dispersion in a semi-infinite aquifer are obtained using the Laplace Transform Technique. The solutions describe the nature of the contaminant concentration in space and time. The results obtained for two expressions of time varying velocity expressions, such as sinusoidally and exponentially increasing forms, are more significant for the time dependent input concentration rather than a stationary source of input concentration, considered at the source. A smaller value of the decay rate coefficient \( q \) approaches the case of time dependent form towards the stationary source of input concentration by considering \( q = 0 \). In this work, advective-dispersive problems are taken into consideration with two different cases: 1) stationary dispersion with transient velocity and 2) time varying dispersion with transient velocity. This study shows the impact of dispersion theory established i.e., dispersion is directly proportional to square of the seepage velocity.
Fig. 4.1(a) Time-dependent contaminant concentration along transient groundwater flow for sinusoidal form of expression, $f(mt)$ (Solid Line) and exponential form of expression, $f(mt)$ (Dotted Line) subject to uniform dispersion in a semi-infinite aquifer.
Fig. 4.1(b) Constant source of contaminant concentration along transient groundwater flow for sinusoidal form of expression, \( f(mt) \) (Solid Line) and exponential form of expression, \( f(mt) \) (Dotted Line) subject to uniform dispersion in a semi-infinite aquifer.
Fig. 4.2(a) Time-dependent contaminant concentration along transient groundwater flow for sinusoidal form of expression, $f(mt)$ (Solid Line) and exponential form of expression, $f(mt)$ (Dotted Line) subject to time varying dispersion in a semi-infinite aquifer.
Fig. 4.2(b) Constant source of contaminant concentration along transient groundwater flow for sinusoidal form of expression, $f(mt)$ (Solid Line) and exponential form of expression, $f(mt)$ (Dotted Line) subject to time varying dispersion in a semi-infinite aquifer.