CHAPTER 3

Statistical Analysis of Crosstalk in Distributed Raman Amplifier

3.1. INTRODUCTION

Wavelength division multiplexing (WDM) scheme is used to multiplex different information carrying wavelength channels on a single fiber (Agrawal, 2001). WDM scheme has many advantages in terms of effective cost and efficient performance. However, it is limited by various factors such as attenuation, dispersion and nonlinear effects. Stimulated Raman scattering is one such nonlinear effect which greatly limits the system performance (Toulouse, 2005; Wu and Way, 2004; Wang et. al., 1998; Ding et. al., 2002; Chraplyvy, 1983; Chraplyvy, 1984; Zhang et. al., 1994; Wang et. al., 1998; Cotter and Hill, 1984; Norimatsu and Yamamoto, 2001). Stimulated Raman scattering (SRS) is an inelastic effect that causes transitions in energy states due to interactions of photons with molecules. In WDM system SRS causes power transfer from a lower wavelength to a higher wavelength resulting in crosstalk. Cross Phase Modulation (XPM) and Self Phase Modulation (SPM) are two other nonlinear effects also known as optical Kerr effect. The change in phase of channel is proportional to its own intensity in SPM and to the intensity of other channels in XPM. The power transfer due to SRS and phase deviation due to XPM and SPM has been denoted as crosstalk.

Crosstalk in WDM system with lumped amplifier has been studied both theoretically and experimentally in previous works (Forghieri et. al., 1995; Christodoulides and Jander, 1996; Mazurczyk et. al., 2000). In one of such studies, soliton propagation in optical fiber communication systems was considered taking into
account the effects of delayed Raman response and the random character of pulse sequences (Peleg, 2007). It was concluded that in such systems the Raman-induced energy exchange due to collisions leads to lognormal statistics for the pulse amplitudes. The PDF of SRS crosstalk has been approximated by lognormal distribution (Forghieri et. al., 1995; Peleg, 2004; Chung and Peleg, 2005; Ho 2000). In Ref. (Ho, 2000), it is shown that in order to use PDF for system performance evaluation, methods to evaluate standard deviation is essential. In the current work, standard deviation of crosstalk due to SRS, XPM and SPM in DRA is calculated which will be supportive for characterizing the PDF of crosstalk.

Chromatic dispersion is referred to broadening of input signal as it travels down the fiber length. It is the second derivative of optical phase with respect to optical frequency. The interaction between nonlinearity and dispersion is an important issue in the design of Lightwave system. Phase modulation of signals in WDM system due to SPM and XPM gets converted to intensity modulation through dispersion and thus results in waveform distortions. Depending on fiber chromatic dispersion and its management, XPM induced nonlinear phase shift may become very detrimental for WDM signals (Ho, 2005; Ho and Wang, 2006). In WDM transmission systems, XPM induces a broadening of the signal spectrum and so wider optical filter bandwidth is required at the receiver. This degrades the system performance, because more spontaneous emission noise enters the receiver (Bellotti et. al.,1997). Moreover, the damaging effects of dispersion become more dominant on wider spectrum of signal and hence system performance degrades further. It has been found that XPM-induced signal broadening is similar to the one induced by laser phase noise. In the decision circuit of the receiver (Agrawal, 2002), the phase fluctuations cause error in decision making. The phase fluctuation due to laser phase noise is minimized by using semiconductor laser whose line width is a small fraction of bit rate. But phase fluctuation due to XPM induced signal broadening still remains and causes error in the signal detection.

Electronic predistortion (EPD) of chromatic dispersion using digital signal processing is a cost-effective alternative to conventional optical dispersion compensation (ODC) using inline dispersion compensating fiber (Weber et. al., 2009). The inline dispersion compensation in ODC systems is replaced by an EPD transmitter
that pre-compensates the individual WDM channels for the chromatic dispersion of the entire transmission distance. EPD has been experimentally demonstrated for 10 Gbps single channel and WDM system (Birk et. al., 2006; McGhan et. al., 2005). Research experiments (Klekamp et. al., 2006; Essiambre and Winzer, 2005) and simulations (Chandrashekhar et. al., 2006) have demonstrated that EPD systems are strongly degraded by SPM and XPM compared to ODC systems.

Optical amplifiers (OA) such as Erbium-doped fiber amplifiers (EDFA) and distributed Raman amplifier (DRA) are used to compensate attenuation of signals which is a major limiting factor in optical communication. In presence of OA, the nonlinear effects are greatly enhanced. In DRA high power pump co and counter propagate with the signal providing continuous gain all along the transmission line. Raman pumps also interact among themselves but the spacing between them is in the present work has been considered to be around 10 nm. At such large separation only SRS will be the dominant nonlinear effect and will cause power transfer among the pumps. XPM dominate at smaller interchannel spacing and will not be a detrimental effect. Another nonlinear effect FWM between pump-pump and pump-signal causes generation of new wavelengths under the condition of phase matching. These new wavelengths can be in the range of both signal wavelengths and pump wavelengths. By appropriate selection of fiber dispersion minimizes FWM. Moreover, the effect has been reported to be more dominant in broad band amplifiers made from Non Zero Dispersion Fiber (NZDF) with zero dispersion wavelength located between pump and signal bands (Bromage, 2004; Boutellier et. al., 2004; Leng et. al., 2005). In our analysis we have assumed ideal pumping configuration not affected by nonlinear effects.

It has been found in research that when multiple pumps at different wavelength are launched into the fiber, the gain magnitude of shorter and longer pumps decreases and increases respectively. In our analysis we have simplified differences in gain by assuming equal gain magnitude for all pumps. Similar approach is available in some previous works (Kavally et. al., 2009; Kavally et. al., 2010; Fludger and Mears, 2001; Parolari et. al., 2003) where, undepleted pump approximation was assumed as the intensity of pump is much higher than that of signal. This assumption is generally true for many practical cases and allows one to solve the analytical equations of evolution of
pump power and signal power for a variety of pumping configurations. In the present work, undepleted pump approximation has been assumed for the derivation of analytical expression of crosstalk.

In our analysis, noise power evolution and interpulse collision has not been considered. In Ref. – (Bromage, 2004), the author gives a detailed discussion on different types of noise occurring in DRA such as amplified spontaneous noise, signal-spontaneous beat noise, multi-path interference (MPI) noise, transfer of relative intensity of noise (RIN) from pump to signal. Some work has been done in this field such as Ref. – (Kavally et. al., 2009) that gives an analytical study of RIN, Ref. – (Kavally et. al., 2010) where MPI in DRA has been discussed. The present work based on statistical method (Ho, 2000; Yamamoto and Norimatsu, 2003; Ho, 2006) is evolved to study the signal crosstalk in DRA. The amplified noise is not considered in present analysis as it is presumed that transfer of noise from one channel to other will be low and will affect the overall noise in the channel instead of affecting the signal degradation due to crosstalk. Hence the equations developed in the present work do not incorporate the effect of noise (Premartne, 2004) but is valid for evaluation of crosstalk in DRA. Again in Ref. – (Premaratne, 2004), the author gives an analytical characterization of evolution of signal power and noise figure in forward pumped DRA. As can be seen in literature, consideration of noise requires detailed analysis and has hence been deferred by the authors for future. It has been found in research that for complete collisions, the collision-induced frequency shift of a pulse is negligible, whereas its position shift is significant. For strong dispersion management it has been found that incomplete collisions can be neglected, whereas for dispersion management system the contribution of the incomplete collisions can be significant (Kaup et. al., 1998; Kaup et. al., 1999). As the collision induced frequency shift is negligible, so there is no significant shift in frequency of signal. Moreover dispersion management is not considered in the analysis hence the impact of collision rate on crosstalk performance is neglected without loss of significant change in result.

In literature, results are available that show that mechanism for error generation can be due to interplay between SRS induced cross frequency shift (XFS) and SRS crosstalk rather than SRS crosstalk alone (Peleg, 2007; Chung and Peleg, 2008).
Moreover perturbations like cross phase modulation (XPM), Raman self frequency shift (SFS) and Raman cross frequency shift (XFS) also contribute to signal degradation and sets a bound on the minimum frequency spacing for stable transmission due to Raman induced interplay between amplitude and frequency dynamics (Nguyen and Peleg, 2010). In Luis and Cartaxo (2005), the influence of residual dispersion and SPM on the limitations imposed by XPM on the performance of dispersion-compensated systems is assessed. In Jiang and Fan (2003), the author discusses the interplay between SRS and XPM and the role of dispersion in the enhancement or suppression of SRS and XPM. It has been reported that interaction between XPM and SRS depends on the relative wavelength location between pump and probe as well as dispersion management scheme, and may add either constructively or destructively. The above effects in DRA are not considered in the present work.

In the present chapter, statistical analysis of crosstalk due to SRS, XPM and SPM has been applied for the first time to WDM system employing DRA. In distributed Raman amplifiers, high power pump co and counter propagates with the WDM signals providing a continuous gain to compensate for the attenuation of signals. Using statistical methods closed-form formulae have been derived to study crosstalk in single segment of fiber link employing DRA. Since the gain is continuously increasing along the transmission line, crosstalk standard deviation has been calculated for variable gain along the entire length of single mode fiber. The analytical equation has been applied to WDM pumped Raman amplifiers in forward and backward configurations to obtain crosstalk for different DRA parameters such as input power, wavelength separation and bit rate of the system. These results are then used to determine system bound for a typical WDM system using DRA as amplifier.

3.2. THEORY

In this section, crosstalk variance is derived in a distributed Raman amplifier for bi-directional pumping scheme. A single span of DRA of length L with no repeater is considered. Thus optical power of the \(i^{th}\) channel after transmission through distance \(z\) is given as (Ho, 2000; Yamamoto and Norimatsu, 2003)

\[
P_i(z, t) = P_i(0, \tau) \exp\{-\alpha z - x_i(z, t) - j\phi_i(z, t)\} \tag{3.1}\]
\( P_1(0, \tau) \) is the input at \( z = 0 \) and \( \tau = t - \frac{z}{v_i} \) with \( v_i \) being the group velocity of \( i^{th} \) channel and \( \alpha \) is the fiber attenuation co-efficient. In (3.1), it has been assumed that fiber dispersion causes just pulse walk-off and no pulse distortion or inter pulse collision (Christodoulides et. al.1996, Ho 2000).

\[
x_i(z, t) = \sum_{j=1}^{N} x_{ij}(z, t)
\]  

(3.2)

\[
\phi_i(z, t) = \sum_{j=1}^{N} \phi_{ij}(z, t)
\]  

(3.3)

\( x_i(z, t) \) is the power change in the \( i^{th} \) channel due to remaining (N-1) channels induced by SRS. \( \phi_i(z, t) \) is the phase change in the \( i^{th} \) channel due to remaining (N-1) channels induced by XPM. In (3.2) and (3.3),

\[
x_{ij}(z, t) = \sum_{k=-\infty}^{+\infty} b_k q_{ij} \left( t - \frac{z}{v_i} - kT \right)
\]  

(3.4)

\[
\phi_{ij}(z, t) = \sum_{k=-\infty}^{+\infty} b_k q_{ij} \left( t - \frac{z}{v_i} - kT \right)
\]  

(3.5)

where \( b_k = \{0,1\} \) is random transmission data and thus \( x_{ij} \) and \( \phi_{ij} \) are considered to be a random variable and \( T \) is bit period. The term \( q_{ij}(t) \) can be represented as

\[
q_{ij}(t) = K \int_0^L p(t - d_i z) e^{-aq^*} g_F(z') g_B(z') dz'
\]  

(3.6)

where \( K \) in SRS is,

\[
K = \frac{g_R}{\Psi_{\text{eff}} f_i f_j} \left( f_i - f_j \right) \quad (|f_i - f_j| \leq 15 \text{ THz})
\]

(3.7)

\[
= 0 \quad (|f_i - f_j| > 15 \text{ THz})
\]

(3.8)

and \( K \) in XPM is,

\[
K = 2\gamma
\]  

(3.9)
In (3.6), \( p(t) \) is pulse shape and \( d_{ij} (= 1/v_j - 1/v_i) \) is propagation time difference between the two different channels \((i, j)\) during a unit length transmission. In (3.7), the Raman gain function is approximated by triangular function and \( g_R \) is the average slope of the triangular profile (Yamamoto and Norimatsu, 2003). The frequency of \( i^{th} (j^{th}) \) channel is \( f_i \) \((f_j)\), polarization constant \( \psi \) is equal to 2 and \( A_{\text{eff}} \) is effective area of cross-section. The parameter \( K \) takes a positive (negative) value if \( f_i \) is greater (less) than \( f_j \) which implies that optical power \( P_i \) is depleted (amplified) by the \( j^{th} \) channel. If \( f_i = f_j \) then \( K \) becomes zero and there is no power exchange between the channels. In (3.9), \( \gamma \) is the nonlinear co-efficient.

When forward pumped DRA is used in optical transmission systems, the optical signal at the Stokes frequency exists from the beginning. So, pump depletion is considered in the analysis. Forward amplification gain (Kao and Wu, 1989) is

\[
g_F(z')dz = \exp \left( \frac{\gamma_p \left( \sum \frac{\lambda_s}{A_P} P_{so} + P_{Po} \right)}{A_{\text{eff}} \alpha} \left( 1 - e^{-\alpha z} \right) \right)
\]

where peak Raman gain \( \gamma_p = 6.5 \times 10^{-14} \text{ m/W} \), \( P_{Po} \) and \( P_{so} \) are the initial pump and signal power respectively, \( \lambda_s (\lambda_p) \) is signal (Pump) wavelength.

When backward pumped DRA is used in optical transmission systems, the system is designed not to induce pump depletion because the signal level will fluctuate if pump depletion occurs. So, pump depletion is neglected. In this case, backward amplification gain (Kao and Wu, 1989) is

\[
g_B(z')dz = \exp \left( \frac{\gamma_p \left( \sum \frac{\lambda_s}{A_P} P_{so} + P_{PL} \right)}{A_{\text{eff}} \alpha} \left( e^{\alpha(z'-L)} - e^{-\alpha L} \right) \right)
\]

Where \( P_{PL} \) is initial backward pump power, \( L \) is the length of DRA and other parameters are same as in (3.10).

In the expression of forward and backward gain it has been assumed that the attenuation constant of pump and signal wavelength is same and equal to 0.2 dB/km. Variation in attenuation constant with wavelength has been neglected.
3.2.1. VARIANCE OF CROSSTALK FOR BI-DIRECTIONAL PUMPED DRA

The variance of crosstalk is calculated for the bi-directional pumped DRA. The variance of crosstalk is given as (Ho, 2004; Yamamoto and Norimatsu, 2003)

\[ \alpha_{ij}^2(i,j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Q_{ij}(\omega)|^2 d\omega \]  \hspace{1cm} (3.12)

\( Q_{ij}(\omega) \) is the Fourier transform of \( q_{ij}(t) \) and is given as:

\[ Q_{ij}(\omega) = K \int_0^L P(\omega)e^{-jd_{ij}z'\omega} e^{-ax'} e^{-K'(1-e^{-az'})} e^{(K''(e^{a(z'-L)}-e^{-aL})} dz' \] \hspace{1cm} (3.13)

\[ = KP(\omega) \int_0^L e^{-(\alpha+d_{ij}\omega)z'} e^{K'(1-e^{-az'})} e^{(K''(e^{a(z'-L)}-e^{-aL})} dz' \] \hspace{1cm} (3.14)

where,

\[ K' = \frac{\gamma_p (\sum \frac{\lambda_s}{\lambda_p} P_{so} + P_{po})}{\alpha A_{eff}} \]

and

\[ K'' = \frac{\gamma_p (\sum \frac{\lambda_s}{\lambda_p} P_{so} + P_{pl})}{\alpha A_{eff}} \]

substituting \( z' = \ln(t) \), \( dz' = \frac{1}{t} dt \)

The Eq. (3.14) becomes

\[ Q_{ij}(\omega) = KP(\omega)e^{K' e^{-\alpha L}} \int_1^{e^{L}} e^{-(\alpha + d_{ij}\omega)} \frac{t^{-\alpha}}{t} e^{-K't^{-\alpha} + K''t^{aL}} dt \] \hspace{1cm} (3.15)
In (3.15) the exponential $e^{-K't^{-a} + K''t^a e^{-atL}}$ is binomially expanded to be the sum of two terms $(1 - K't^{-a} + K''t^a e^{-atL})$ and the higher order terms are neglected which contribute insignificantly in the series expansion.

$$Q_{ij}(\omega) = CP(\omega) \int e^{t} \frac{1 - K't^{-a} + K''t^a e^{-atL}}{t^{(a+1+jd_{ij})\omega}} dt$$

Where $C = Ke^{K''t^a e^{-atL}}$. Integrating and substituting the limits we get,

$$Q_{ij}(\omega) = CP(\omega) \left[ \frac{1 - e^{-(\alpha+jd_{ij}\omega)L}}{\alpha + jd_{ij}\omega} \right] - CK'P(\omega) \left[ \frac{1 - e^{-(2\alpha+jd_{ij}\omega)L}}{2\alpha + jd_{ij}\omega} \right]$$

$$+ CP(\omega)K''e^{-atL} \left[ \frac{1 - e^{-(jd_{ij}\omega)L}}{jd_{ij}\omega} \right]$$

(3.17)

Where $K'$ and $K''$ are found in Eq. (3.14)

A rectangular NRZ pulse is given as $p(t) = \begin{cases} \frac{2P_0}{T} & \text{for} \ |t| < \frac{T}{2} \\ 0 & \text{for} \ |t| > \frac{T}{2} \end{cases}$

(3.18)

Fourier transform of $p(t)$ is given as $|P(\omega)| = 2P_0T\frac{\sin^{\omega T}}{\omega T}$

(3.19)

An RZ pulse of duty cycle $\tau_b$ is given by same equation as above with $T$ replaced by $\frac{T}{\tau_b}$

$\tau_b = 2$ for 50% duty cycle

$= 3$ for 33% duty cycle

The expression of $Q_{ij}(\omega)$ is manipulated algebraically before substituting in the equation of variance.

$$|Q_{ij}(\omega)|^2 = |P(\omega)|^2(|M + N + O|^2$$

$$= |P(\omega)|^2((M + N + O) \times (M + N + O)^*)$$

$$= |P(\omega)|^2(|M|^2 + |N|^2 + |O|^2 + MN^* + M^*N + MO^* + M^*O + ON^* + O^*N)$$

$$= |P(\omega)|^2(|M|^2 + |N|^2 + |O|^2 + 2 \times \text{Re}(MN^*) + 2 \times \text{Re}(MO^*) + 2 \times \text{Re}(NO^*))$$

(3.20)
Substituting the values of M, N and O from equation of $Q_{ij}(\omega)$ and simplifying we get the following terms

\[
M = C \left[ \frac{1 - e^{-(\alpha + j\delta_{ij}\omega)L}}{\alpha + j\delta_{ij}\omega} \right] \tag{3.24}
\]

\[
N = -CK' \left[ \frac{1 - e^{-(2\alpha + j\delta_{ij}\omega)L}}{2\alpha + j\delta_{ij}\omega} \right] \tag{3.25}
\]

\[
O = CK'' e^{-\alpha L} \left[ \frac{1 - e^{-(j\delta_{ij}\omega)L}}{j\delta_{ij}\omega} \right] \tag{3.26}
\]

where $K'$, $K''$ are given in (3.14) and C is given in (3.16).

\[
|M|^2 = M \times M^* \tag{3.27}
\]

\[
M^* = C \left[ \frac{1 - e^{-(\alpha - j\delta_{ij}\omega)L}}{\alpha - j\delta_{ij}\omega} \right] \tag{3.28}
\]

\[
M \times M^* = C \left[ \frac{1 - e^{-(\alpha + j\delta_{ij}\omega)L}}{\alpha + j\delta_{ij}\omega} \right] \times C \left[ \frac{1 - e^{-(\alpha - j\delta_{ij}\omega)L}}{\alpha - j\delta_{ij}\omega} \right] \tag{3.29}
\]

\[
= C^2 \left[ \frac{1 + e^{-2\alpha L} - e^{-\alpha L} \left( e^{j\delta_{ij}\omega L} + e^{-j\delta_{ij}\omega L} \right)}{\alpha^2 + (\delta_{ij}\omega)^2} \right] \tag{3.30}
\]

\[
= C^2 \left[ \frac{1 + e^{-2\alpha L} - e^{-\alpha L} \left( 2 \cos(\delta_{ij}\omega L) \right)}{\alpha^2 + (\delta_{ij}\omega)^2} \right] \tag{3.31}
\]

\[
= C^2 \left[ \frac{1 + e^{-2\alpha L} - e^{-\alpha L} \left( 2 - 2 \sin^2 \left( \frac{\delta_{ij}\omega L}{2} \right) \right)}{\alpha^2 + (\delta_{ij}\omega)^2} \right] \tag{3.32}
\]

\[
= C^2 \left[ \frac{1 + e^{-2\alpha L} - e^{-\alpha L} \left( 2 - 4 \sin^2 \left( \frac{\delta_{ij}\omega L}{2} \right) \right)}{\alpha^2 + (\delta_{ij}\omega)^2} \right] \tag{3.33}
\]

\[
= C^2 \left[ \frac{1 + e^{-2\alpha L} - 2e^{-\alpha L} + 4e^{-\alpha L} \sin^2 \left( \frac{\delta_{ij}\omega L}{2} \right)}{\alpha^2 + (\delta_{ij}\omega)^2} \right] \tag{3.34}
\]
\[ C^2 \left[ \frac{(1 - e^{-\alpha L})^2 + 4e^{-\alpha L} \sin^2 \left( \frac{d_{ij} \omega L}{2} \right)}{\alpha^2 + (d_{ij} \omega)^2} \right] \]  

(3.35)

where \( C = Ke^{K'}e^{-\alpha L} \)

Thus,

\[ |M|^2 = \frac{C^2}{\alpha^2 + (d_{ij} \omega)^2} \left[ (1 - e^{-\alpha L})^2 + 4e^{-\alpha L} \sin^2 \left( \frac{d_{ij} \omega L}{2} \right) \right] \]  

(3.36)

Similarly,

\[ |N|^2 = \frac{C^2K'^2}{4\alpha^2 + (d_{ij} \omega)^2} \left[ (1 - e^{-2\alpha L})^2 + 4e^{-2\alpha L} \sin^2 \left( \frac{d_{ij} \omega L}{2} \right) \right] \]  

(3.37)

\[ |O|^2 = \frac{C^2K'^2e^{-2\alpha L}}{(d_{ij} \omega)^2} \left[ 4 \sin^2 \left( \frac{d_{ij} \omega L}{2} \right) \right] \]  

(3.38)

\[ 2\text{Re}(MN^*) = 2\text{Re} \left[ C \left( \frac{1 - e^{-(\alpha+jd_{ij} \omega)L}}{\alpha + jd_{ij} \omega} \right) x(-CK') \left( \frac{1 - e^{-(2\alpha-jd_{ij} \omega)L}}{2\alpha - jd_{ij} \omega} \right) \right] \]  

(3.39)

\[ = (-2C^2K') \times \frac{1 - e^{-(\alpha+jd_{ij} \omega)L} - e^{-(2\alpha-jd_{ij} \omega)L} + e^{-3\alpha L}}{2\alpha^2 + (d_{ij} \omega)^2 + j\alpha d_{ij} \omega} \]  

(3.40)

\( \{ \) Here \( N^* \) represents the complex conjugate of \( N \) and is given by \( CK' \left[ \frac{1 - e^{-(2\alpha-jd_{ij} \omega)L}}{2\alpha - jd_{ij} \omega} \right] \) \( \}\)  

Algebraic manipulation of \( 2\text{Re}(MN^*) \):

\[ \text{Re} \left[ \frac{a + jb}{c + jd} \right] = \text{Re} \left[ \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} \right] = \text{Re} \left[ \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \right] = \frac{(ac + bd)}{c^2 + d^2} \]  

(3.41)

manipulating the expression of \( 2\text{Re}(MN^*) \) to get it in the form of \( \frac{a+jb}{c+jd} \). Numerator of \( 2\text{Re}(MN^*) \) :-
\[1 - e^{-(\alpha+jd_{ij}\omega)L} - e^{-(2\alpha-jd_{ij}\omega)L} + e^{-3\alpha L}\]  
\[= + e^{-3\alpha L} - e^{-\alpha L}(\cos(d_{ij}\omega L) - j \sin(d_{ij}\omega L)) - e^{-2\alpha L}(\cos(d_{ij}\omega L) + j \sin(d_{ij}\omega L))\]  
\[= 1 + e^{-3\alpha L} - \cos(d_{ij}\omega L)(e^{-\alpha L} + e^{-2\alpha L}) + j \sin(d_{ij}\omega L)(e^{-\alpha L} - e^{-2\alpha L})\]  
Therefore,
\[a = 1 + e^{-3\alpha L} - \cos(d_{ij}\omega L)(e^{-\alpha L} + e^{-2\alpha L})\]  
\[b = \sin(d_{ij}\omega L)(e^{-\alpha L} - e^{-2\alpha L})\]  
\[c = 2\alpha^2 + (d_{ij}\omega L)^2\]  
\[d = \alpha d_{ij}\omega\]  
Substituting a, b, c and d in the expression of \(2\text{Re}(MN^*)\), we get
\[2\text{Re}(MN^*) = \frac{2\alpha^2 + d_{ij}^2\omega^2}{(\alpha^2 + d_{ij}^2\omega^2)(4\alpha^2 + d_{ij}^2\omega^2)} \left[1 + e^{-3\alpha L} - \cos(d_{ij}\omega L)(e^{-\alpha L} + e^{-2\alpha L})\right]\]  
\[+ \frac{A \alpha d_{ij} L \sin d_{ij}\omega L}{(\alpha^2 + d_{ij}^2\omega^2)(4\alpha^2 + d_{ij}^2\omega^2)} \left[e^{-\alpha L} - e^{-2\alpha L}\right]\]  
where constant \(A = (-2C^2 K')\)
\[2\text{Re}(MO^*) = 2\text{Re} \times \left[C \frac{1 - e^{-(\alpha+jd_{ij}\omega)L}}{\alpha + jd_{ij}\omega} \times CK'' e^{-\alpha L} \left[1 - e^{(jd_{ij}\omega)L}/(-jd_{ij}\omega)\right]\right]\]  
\[= (2C^2 K'' e^{-\alpha L}) \times \left[\frac{1 - e^{-(\alpha+jd_{ij}\omega)L} - e^{(jd_{ij}\omega)L} + e^{-\alpha L}}{(d_{ij}\omega)^2 + j\alpha d_{ij}\omega}\right]\]  
\[= (2C^2 K'' e^{-\alpha L}) \times \left[1 - e^{(jd_{ij}\omega)L}/(-jd_{ij}\omega)\right]\]  
\[\{\text{Here } O^* \text{ represents the complex conjugate of } O \text{ and is given by } CK'' e^{-\alpha L} \left[1 - e^{(jd_{ij}\omega)L}/(-jd_{ij}\omega)\right]\}\]

Algebraic manipulation of \(2\text{Re}(MO^*)\):-
\[ \text{Re}\left[ \frac{a + jb}{c + jd} \right] = \text{Re}\left[ \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} \right] = \text{Re}\left[ \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \right] = \frac{(ac + bd)}{c^2 + d^2} \]  

(3.52)

Manipulating the expression of \(2\text{Re}(\text{MO}^*)\) to get it in the form of \(\frac{a + jb}{c + jd}\)

Numerator of \(2\text{Re}(\text{MN}^*)\) :-

\[
1 - e^{-(\alpha + jd\omega)l} - e^{(jd\omega)l} + e^{-\alpha l}
\]

(3.53)

\[=1 + e^{-\alpha l} - e^{-\alpha l}\left(\cos(d_{ij}\omega L) - j \sin(d_{ij}\omega L)\right) - \left(\cos(d_{ij}\omega L) + j \sin(d_{ij}\omega L)\right)\]  

(3.54)

\[= 1 + e^{-\alpha l} - \cos(d_{ij}\omega L)(e^{-\alpha l} + 1) + j \sin(d_{ij}\omega L) (e^{-\alpha l} - 1) \]

(3.55)

\[= (1 + e^{-\alpha l})(1 - \cos(d_{ij}\omega L)) + j \sin(d_{ij}\omega L) (e^{-\alpha l} - 1) \]

(3.56)

Therefore,

\[a = (1 + e^{-\alpha l})(1 - \cos(d_{ij}\omega L)) \]

(3.57)

\[b = \sin(d_{ij}\omega L) (e^{-\alpha l} - 1) \]

(3.58)

\[c = (d_{ij}\omega)^2 \]

(3.59)

\[d = \alpha d_{ij}\omega \]

(3.60)

Substituting \(a\), \(b\), \(c\) and \(d\) in the expression of \(2\text{Re}(\text{MO}^*)\), we get

\[
2\text{Re}(\text{MO}^*) = \left[\frac{A'}{(\alpha^2 + d_{ij}^2\omega^2)(d_{ij}^2\omega^2)}\right] \left\{ (1 + e^{-\alpha l})d_{ij}^2\omega^2(1 - \cos d_{ij}\omega L) + \alpha d_{ij}\omega \sin d_{ij}\omega L (1 - e^{-\alpha l}) \right\} \]

(3.61)

where constant \(A' = 2C^2K'e^{-\alpha l}\) and \(K'\) is given in (3.14).
using similar technique we get the expressions of \(2\text{Re}(\text{NO}')\) by replacing \(\alpha\) by \(2\alpha\) in the expression of \(2\text{Re}(\text{MN}')\).

\[
2\text{Re}(\text{NO}') = \left[ \frac{2c_2k'k''e^{-2\alpha L}}{(4\alpha^2 + d_{ij}^2\omega^2)(d_{ij}^2\omega^2)} \right] \left( (1 + e^{-2\alpha L})d_{ij}^2\omega^2(1 - \cos d_{ij}\omega L) + 2\alpha d_{ij}\omega \sin d_{ij}\omega L (1 - e^{-2\alpha L}) \right)
\]

(3.62)

Substituting the terms of \(|Q_{ij}(\omega)|^2\) in the equation of variance and performing the infinite integration:

Integration of \(|M|^2\):

\[
\frac{|P(\omega)|^2}{2\pi} \int_{-\infty}^{\infty} |M|^2 d\omega = \frac{4\text{P}_0^2T^2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 + (d_{ij}\omega)^2} |M|^2 d\omega
\]

(3.63)

\[
= \frac{4\text{P}_0^2T^2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 + (d_{ij}\omega)^2} \left( \frac{C^2}{\alpha^2 + (d_{ij}\omega)^2} \right) \left( 1 - e^{-\alpha L} \right)^2 + 4e^{-\alpha L}\sin^2 \left( \frac{d_{ij}\omega L}{2} \right) d\omega
\]

(3.64)

\[
I_1 = \frac{(1 - e^{-\alpha L})^2 \times C^2 \sin^2 \frac{\omega T}{2}}{\alpha^2 + (d_{ij}\omega)^2 \omega^2 T^2 / 4}
\]

(3.65)

\[
\int_{-\infty}^{\infty} I_1 d\omega = \frac{C^2\text{P}_0^2}{\alpha^3L_w} (1 - e^{-\alpha L})^2 P_1(\alpha)
\]

(3.66)

Where

\[
P_1(\alpha) = (e^{-\alpha L_w} + \alpha L_w - 1)
\]

(3.67)

For integration we have used the formula (Gradshetyn and Ryzhik, 2000)

\[
\int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 + \left( \frac{\alpha}{d_{ij}} \right)^2} d\omega = \frac{2\pi}{4} e^{-\alpha L_w} + \alpha L_w - 1
\]

(3.68)

\[
I_2 = \frac{4e^{-\alpha L}\sin^2 \left( \frac{d_{ij}\omega L}{2} \right) \times C^2 \sin^2 \frac{\omega T}{2}}{\alpha^2 + (d_{ij}\omega)^2 \omega^2 T^2 / 4}
\]

(3.69)
\[
\int_{-\infty}^{\infty} I_2(\omega) \, d\omega = \frac{C^2 P_0^2}{\alpha^3 L_w} e^{-\alpha L} P_2(\alpha) \tag{3.70}
\]

\[
P_2(\alpha) = 2(e^{-\alpha L} + e^{-\alpha L} - 1) - (e^{-\alpha (L+L_w)} + e^{-\alpha (L-L_w)} + \alpha (|L+L_w| - |L-L_w|)) \tag{3.71}
\]

For integration following trigonometric formulae and integrals are used (Gradshetyn and Ryzhik, 2000)

\[
\sin^2 \frac{\omega T}{2} \times \sin^2 \left(\frac{d_{ij} \omega L}{2}\right) = \frac{1}{4} \left[ 2 \sin^2 \frac{\omega T}{2} + 2 \sin^2 \left(\frac{d_{ij} \omega L}{2}\right) - \sin^2 \left(\frac{2T + d_{ij} L}{2}\right) - \sin^2 \left(\frac{2T - d_{ij} L}{2}\right) \right] \tag{3.72}
\]

\[
\int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 \left(\omega^2 + \left(\frac{\alpha}{d_{ij}}\right)^2\right)} \, d\omega = \frac{2\pi}{4 \left(\frac{\alpha}{d_{ij}}\right)} (e^{-\alpha L} + \alpha L - 1) \tag{3.73}
\]

Thus, integration of |M|^2

\[
\frac{|P(\omega)|^2}{2\pi} \int_{-\infty}^{\infty} |M|^2 d\omega = \frac{C^2 P_0^2}{\alpha^3 L_w} \left\{(1 - e^{-\alpha L})^2 P_1(\alpha) + e^{-\alpha L} P_2(\alpha)\right\} \tag{3.74}
\]

Similarly, integration of |N|^2:

\[
\frac{|P(\omega)|^2}{2\pi} \int_{-\infty}^{\infty} |N|^2 d\omega \tag{3.75}
\]

\[
= \frac{4P_0^2 P_2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} |N|^2 d\omega \tag{3.76}
\]

\[
= \frac{4P_0^2 P_2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{4\omega^2 T^2} \left(\frac{C^2 K^2}{4\omega^2 + (d_{ij} \omega L)} \left(1 - e^{-2\alpha L})^2 + 4e^{-2\alpha L} \sin^2 \left(\frac{d_{ij} \omega L}{2}\right)\right) \right) d\omega \tag{3.77}
\]
\[
\frac{C^2K^2P_0^2}{8\alpha^3L_w} \left\{ (1 - e^{-2\alpha L})^2 P_1(2\alpha) + e^{-2\alpha L} P_2(2\alpha) \right\} \quad (3.78)
\]

where \( K \) is given in (3.14)

Integration of \(|O|^2\):

\[
\frac{|P(\omega)|^2}{2\pi} \int_{-\infty}^{\infty} |O|^2 d\omega = \frac{4P_0^2T^2}{8\pi^2T^2} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} |O|^2 d\omega \quad (3.79)
\]

\[
= \frac{4P_0^2T^2}{8\pi^2T^2} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} \left( \frac{C^2K^2 e^{-2\alpha L}}{2} \left( \frac{\sin^2 \left( \frac{d_{ij}\omega L}{2} \right)}{2} \right) \right) d\omega \quad (3.80)
\]

\[
= \frac{8C^2K^2P_0^2 e^{-2\alpha L}}{3d_{ij}^2T^2} \min \left( \frac{T^2}{4}, \frac{d_{ij}^2L^2}{4} \right) \left\{ 3 \max \left( \frac{T}{2}, \frac{d_{ij}L}{2} \right) - \min \left( \frac{T}{2}, \frac{d_{ij}L}{2} \right) \right\} \quad (3.81)
\]

For integration we have used the following formula (Gradshetyn and Ryzhik, 2000)

\[
\int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^4} dx = \frac{\pi}{6} \min(a^2, b^2) [3 \max(a, b) - \min(a, b)] \quad (3.82)
\]

Integration of \(2\text{Re}(MN^*)\):

\[
2\text{Re}(MN^*) = \frac{A \left( 2\alpha^2 + d_{ij}^2 \omega^2 \right)}{(\alpha^2 + d_{ij}^2 \omega^2)(4\alpha^2 + d_{ij}^2 \omega^2)} \left[ 1 + e^{-3\alpha L} - \cos \left( d_{ij}\omega L \right) (e^{-\alpha L} + e^{-2\alpha L}) \right]
\]

\[
+ \frac{A \alpha d_{ij}\omega \sin d_{ij}\omega L}{(\alpha^2 + d_{ij}^2 \omega^2)(4\alpha^2 + d_{ij}^2 \omega^2)} [e^{-\alpha L} - e^{-2\alpha L}] \quad (3.83)
\]

Algebraic manipulation of first part i.e.

\[
\frac{A \left( 2\alpha^2 + d_{ij}^2 \omega^2 \right)}{(\alpha^2 + d_{ij}^2 \omega^2)(4\alpha^2 + d_{ij}^2 \omega^2)} \left[ 1 + e^{-3\alpha L} - \cos \left( d_{ij}\omega L \right) (e^{-\alpha L} + e^{-2\alpha L}) \right] \quad (3.84)
\]

where \( A \) is given in (3.49).
separating the term \( \frac{(2\alpha^2 + d_{ij}^2\omega^2)}{(\alpha^2 + d_{ij}^2\omega^2)(4\alpha^2 + d_{ij}^2\omega^2)} \) into two parts using the method of separation of variables.

\[
\frac{(2\alpha^2 + d_{ij}^2\omega^2)}{(\alpha^2 + d_{ij}^2\omega^2)(4\alpha^2 + d_{ij}^2\omega^2)} = \frac{B}{(\alpha^2 + d_{ij}^2\omega^2)} + \frac{D}{(4\alpha^2 + d_{ij}^2\omega^2)}
\]

(3.86)

\[
(2\alpha^2 + d_{ij}^2\omega^2) = B \left(4\alpha^2 + d_{ij}^2\omega^2\right) + D \left(\alpha^2 + d_{ij}^2\omega^2\right)
\]

(3.87)

\[
B = \frac{1}{3} ; D = \frac{2}{3}
\]

(3.88)

In the expression \( [1 + e^{-3\alpha L} - \cos(d_{ij}\omega L)(e^{-\alpha L} + e^{-2\alpha L})] \), replacing \( \cos(d_{ij}\omega L) \) by \( \left(1 - 2\sin^2\left(\frac{d_{ij}\omega L}{2}\right)\right) \) we get:

\[
[1 + e^{-3\alpha L} - \cos(d_{ij}\omega L)(e^{-\alpha L} + e^{-2\alpha L})]
\]

(3.90)

\[
= 1 + e^{-3\alpha L} - (e^{-\alpha L} + e^{-2\alpha L}) \left(1 - 2\sin^2\left(\frac{d_{ij}\omega L}{2}\right)\right)
\]

(3.91)

\[
= (1 + e^{-3\alpha L} - e^{-\alpha L} - e^{-2\alpha L}) + 2(e^{-\alpha L} + e^{-2\alpha L}) \sin^2\left(\frac{d_{ij}\omega L}{2}\right)
\]

(3.92)

Replacing \( (1 + e^{-3\alpha L} - e^{-\alpha L} - e^{-2\alpha L}) \) by \( C_1 \), \( (e^{-\alpha L} + e^{-2\alpha L}) \) by \( C_2 \) and \( (e^{-\alpha L} - e^{-2\alpha L}) \) by \( C_3 \) in the expression of \( 2\text{Re}(MN^*) \) we get

\[
2\text{Re}(MN^*) =
\]

\[
A \left[ \left(\frac{1}{3(\alpha^2 + d_{ij}^2\omega^2)} + \frac{2}{3(4\alpha^2 + d_{ij}^2\omega^2)}\right) \left(C_1 + 2C_2 \sin^2\left(\frac{d_{ij}\omega L}{2}\right)\right) \right] + A \left(\frac{C_3 \alpha d_{ij}\omega}{(\alpha^2 + d_{ij}^2\omega^2)(4\alpha^2 + d_{ij}^2\omega^2)} \right)
\]

(3.93)

where \( A \) is given in (3.49).
separating \( \frac{1}{(a^2 + d_i^2 \omega^2)(4a^2 + d_i^2 \omega^2)} \) into two terms we get:

\[
\frac{1}{(a^2 + d_i^2 \omega^2)(4a^2 + d_i^2 \omega^2)} = \frac{1}{3a^2} \left[ \frac{1}{(a^2 + d_i^2 \omega^2)} - \frac{1}{(4a^2 + d_i^2 \omega^2)} \right]
\]

(3.94)

Therefore,

\[ 2 \text{Re}(MN^*) = \]

\[
A \left[ \frac{C_1}{3(a^2 + d_i^2 \omega^2)} + \frac{2C_1}{3(4a^2 + d_i^2 \omega^2)} + \frac{2C_2 \sin^2 \left( \frac{d_i \omega L}{2} \right)}{3(a^2 + d_i^2 \omega^2)} + \frac{4C_2 \sin^2 \left( \frac{d_i \omega L}{2} \right)}{3(4a^2 + d_i^2 \omega^2)} \right.
\]

\[ + \frac{(C_3) \alpha d_i \omega \sin d_i \omega L}{3a^2(4a^2 + d_i^2 \omega^2)} - \frac{(C_3) \alpha d_i \omega \sin d_i \omega L}{3a^2(4a^2 + d_i^2 \omega^2)} \]

\]

(3.95)

where,

\[
A = (-2C^2K')
\]

\[
C_1 = (1 + e^{-3aL} - e^{-aL} - e^{-2aL})
\]

\[
C_2 = (e^{-aL} + e^{-2aL})
\]

\[
C_3 = (e^{-aL} - e^{-2aL})
\]

Integrating each term:

\[
\frac{1}{8 \pi T} \int \frac{AC_1}{3(a^2 + d_i^2 \omega^2)} \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^2} = \frac{(AC_1)^2 P_0^2}{3a^2L_w} P_1(\alpha)
\]

(3.96)

\[
\frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{2}{3} \left( \frac{AC_1}{4a^2 + d_i^2 \omega^2} \right) \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^2} = \frac{2(AC_1)^2 P_0^2}{3a^2L_w} P_1(2\alpha)
\]

(3.97)

where \( AC_1 = (-2C^2K') (1 + e^{-3aL} - e^{-aL} - e^{-2aL}) \)

\[
\frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{2}{3} \left( \frac{AC_2}{\omega^2} \right) \times \sin^2 \left( \frac{d_i \omega L}{2} \right) \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^2} = \frac{(AC_2)^2 P_0^2}{6a^2L_w} P_2(\alpha)
\]

(3.98)

\[
\frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{4}{3} \left( \frac{AC_2}{4a^2 + d_i^2 \omega^2} \right) \times \sin^2 \left( \frac{d_i \omega L}{2} \right) \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^2} = \frac{(AC_2)^2 P_0^2}{3a^2L_w} P_2(2\alpha)
\]

(3.99)

where \( AC_2 = (-2C^2K') (e^{-aL} + e^{-2aL}) \)
\[ \frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{(C_3A) \alpha d_{ij} \omega \sin d_{ij} \omega L}{3 \alpha^2 (\alpha^2 + d_{ij}^2 \omega^2)} \times \frac{16 p_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega} \] (3.100)

\[ = \frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{(C_3A) \alpha d_{ij} \omega \sin d_{ij} \omega L}{3 \alpha (4 \alpha^2 + d_{ij}^2 \omega^2)} \times \frac{16 p_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega} \] (3.101)

\[ = \frac{(AC_3)^2 p_0^2}{6 \alpha^3 L_w} p_3(\alpha) \] (3.102)

\[ = \frac{(AC_3)^2 p_0^2}{24 \alpha^3 L_w} p_3(2\alpha) \] (3.103)

where, \( AC_3 = (-2C^2k) (e^{-\alpha L} - e^{-2\alpha L}) \). For integration, we have used the following formula (Gradshetyn and Ryzhik, 2000)

\[ \int_{0}^{\infty} \frac{\sin^2 ax \sin bx}{x(x^2 + c^2)} dx \]

\[ = \frac{\pi}{8|c|^2} \left[ \text{sign}(b) \{2 - 2e^{-|b||c|}\} + \text{sign}(2a - b) \{1 - e^{-|2a-b||c|}\} ight] \]

\[ - \text{sign}(2a + b) \{1 - e^{-|2a+b||c|}\} \] (3.106)

and the trigonometric equation

\[(\sin \alpha \sin \beta)^2 = \frac{1}{4} (2 \sin^2 \alpha + 2 \sin^2 \beta - \sin^2 (\alpha + \beta) - \sin^2 (\alpha - \beta)) \] (3.107)

Integration of \(2 \text{Re}(M_0^+):\)

\[ 2 \text{Re}(M_0^+) = \left[ \frac{A'}{\left(\alpha^2 + d_{ij}^2 \omega^2\right)\left(d_{ij}^2 \omega^2\right)\left(1 + e^{-\alpha L}\right)} \left(1 - \cos d_{ij} \omega L\right) + \alpha d_{ij} \omega \sin d_{ij} \omega L \left(1 - e^{-\alpha L}\right)\right] \] (3.108)
\[ A' \left( \frac{1}{\alpha^2 + d_{ij}^2 \omega^2} \right) \left( \frac{1 + e^{-\alpha L}}{d_{ij}^2 \omega^2} \right) \left( 1 - \cos d_{ij} \omega L \right) \]

\[ + \frac{A'}{\left( \frac{1}{\alpha^2 + d_{ij}^2 \omega^2} \right) \left( d_{ij}^2 \omega^2 \right)} \left( \alpha d_{ij} \omega \sin d_{ij} \omega L \left( 1 - e^{-\alpha L} \right) \right) \]

\[ = \frac{A'}{\left( \frac{1}{\alpha^2 + d_{ij}^2 \omega^2} \right) \left( d_{ij}^2 \omega^2 \right)} \left( 1 + e^{-\alpha L} \right) \left( 1 - \cos d_{ij} \omega L \right) \]

\[ + \frac{A'}{\left( \frac{1}{\alpha^2 + d_{ij}^2 \omega^2} \right) \left( d_{ij}^2 \omega^2 \right)} \alpha \sin d_{ij} \omega L \left( 1 - e^{-\alpha L} \right) \]

Integration of each term:

\[ \frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{A'}{\left( \frac{1}{\alpha^2 + d_{ij}^2 \omega^2} \right) \left( 1 + e^{-\alpha L} \right) \left( 1 - \cos d_{ij} \omega L \right)} \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^2} \]

\[ = \frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{A'}{\left( \frac{1}{\alpha^2 + d_{ij}^2 \omega^2} \right) \left( 1 + e^{-\alpha L} \right) \sin^2 \left( \frac{d_{ij} \omega L}{2} \right)} \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^2} \]

\[ = \frac{1}{2} \left( 1 + e^{-\alpha L} \right) P_2(\alpha) \]

\[ = \frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{A'}{\left( \frac{1}{\alpha^2 + d_{ij}^2 \omega^2} \right) d_{ij} \omega} \alpha \sin d_{ij} \omega L \left( 1 - e^{-\alpha L} \right) \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^2} \]

\[ = \frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{A'}{d_{ij}^3} \left( \frac{1}{\alpha^2 + \omega^2} \right) \alpha \sin d_{ij} \omega L \left( 1 - e^{-\alpha L} \right) \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^3} \]

Separating the fraction:

\[ \frac{1}{\left( \frac{\alpha}{d_{ij}} \right)^2 + \omega^2} \omega^3 = \frac{1}{\omega^3} - \left( \frac{\alpha}{d_{ij}} \right)^2 + \omega^2) \omega \]

Integrating each term of the separated function:

\[ \frac{1}{8 \pi T} \int_{-\infty}^{\infty} \frac{A'}{d_{ij}^3} \left( \alpha \sin d_{ij} \omega L \left( 1 - e^{-\alpha L} \right) \right) \times \frac{16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{\omega^3} \]
\[ K_1 = \frac{1}{8 \pi T} \times \frac{A' \times 16 P_0^2}{(d_{ij}^3)} \{ \alpha (1 - e^{-\alpha L}) \} \]

For integration we have used the formula (Gradshetyn and Ryzhik, 2000)

\[
\int_0^\infty \frac{\sin^2 ax \sin bx}{x^3} dx = \pi (\text{sign}(b + 2a) \left( \frac{b^2}{8} + \frac{ab}{2} + \frac{a^2}{2} \right) - \text{sign}(b) \frac{b^2}{4} + \text{sign}(-b + 2a) \left( \frac{b^2}{8} + \frac{ba}{2} - \frac{a^2}{2} \right) \] \tag{3.118}

\[
= \frac{1}{8 \pi T} \int_{-\infty}^\infty \frac{A' \times 16 P_0^2 \times \sin^2 \frac{\omega T}{2}}{(\frac{\alpha}{d_{ij}})^2 + \omega^2} \{ \alpha \sin d_{ij} \omega L \left( 1 - e^{-\alpha L} \right) \}
\]

\[
= K_1 \times \left[ \text{sign}(d_{ij} L)(2 - 2e^{-\alpha L}) + \text{sign}(T - d_{ij} L)(1 - e^{-\alpha|L-L_w|}) - \text{sign}(T + d_{ij} L)(1 - e^{-\alpha|L+L_w|}) \right] \] \tag{3.119}

Integration of 2Re(NO⁺):-

Using similar technique we can get the expressions of integration 2Re(NO⁺) by replacing \( \alpha \) by \( 2\alpha \) in the expression of 2Re(MN⁺). The detailed integration performed can be summarized as:-

\[
\frac{4P_0^2 T^2}{8\pi T} \int_{-\infty}^\infty \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} |M|^2 = \frac{C^2 P_0^2}{\alpha^3 L_w} \{ (1 - e^{-\alpha L})^2 P_1(\alpha) + e^{-\alpha L} P_2(\alpha) \} \tag{3.120}
\]
\[
\frac{4P_0^2T^2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} |N|^2 = \frac{C^2 K' P_0^2}{8\alpha^3 L_w} \left\{ (1 - e^{-2\alpha L})^2 p_1(2\alpha) + e^{-2\alpha L} p_2(2\alpha) \right\}
\]

(3.121)

\[
\frac{4P_0^2T^2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} |O|^2
\]

\[
= \frac{8C^2 K' P_0^2 e^{-2\alpha L}}{3d_{ij} T} \min \left( \frac{T^2}{4}, \frac{d_{ij}^2 L^2}{4} \right) \left\{ 3 \max \left( \frac{T}{2}, \frac{d_{ij} L}{2} \right) - \min \left( \frac{T}{2}, \frac{d_{ij} L}{2} \right) \right\}
\]

(3.122)

\[
\frac{4P_0^2T^2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} 2\text{Re}(M N^*)
\]

\[
= \frac{-C^2 K' P_0^2}{\alpha^3 L_w} \left[ \left( 1 + e^{-3\alpha L} - e^{-2\alpha L} - e^{-\alpha L} \right) \left( \frac{P_1(\alpha)}{3} + \frac{2 P_2(\alpha)}{3} \right) + \frac{1}{2} (e^{-\alpha L})^2 \right]
\]

\[
+ \frac{C^2 K' P_0^2}{6\alpha^3 L_w} \left[ \frac{P_3(2\alpha)}{4} - \frac{P_3(\alpha)}{4} \right]
\]

(3.123)

\[
\frac{4P_0^2T^2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} 2\text{Re}(M O^*)
\]

\[
= \frac{C^2 K'' e^{-\alpha L} P_0^2}{\alpha^3 L_w} \left[ (1 + e^{-\alpha L}) p_2(\alpha) - (1 - e^{-\alpha L}) p_3(\alpha) \right]
\]

\[
+ \frac{C^2 K'' e^{-\alpha L} P_0^2 T}{\alpha d_{ij}} \left[ (1 - e^{-\alpha L}) P_4(T, d_{ij} L) \right]
\]

(3.124)

where \(K'\) and \(K''\) are given in (3.14).

\[
\frac{4P_0^2T^2}{8\pi T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\omega T}{2}}{\omega^2 T^2} 2\text{Re}(N O^*)
\]

\[
= \frac{C^2 K'' e^{-2\alpha L} P_0^2}{8\alpha^3 L_w} \left[ (1 + e^{-2\alpha L}) p_2(2\alpha) - (1 - e^{-2\alpha L}) p_3(2\alpha) \right]
\]

\[
+ \frac{C^2 K'' e^{-2\alpha L} P_0^2 T}{2\alpha d_{ij}} \left[ (1 - e^{-2\alpha L}) P_4(T, d_{ij} L) \right]
\]

(3.125)

where,

\[
P_1(\alpha) = (e^{-\alpha L_w} + \alpha L_w - 1)
\]

\[
P_2(\alpha) = 2(e^{-\alpha L_w} + e^{-\alpha L} - 1) - (e^{-\alpha L_w} + e^{-\alpha L} - 1)\alpha(l + L_w) + \alpha(|L + L_w| - |L - L_w|)
\]

\[
P_3(\alpha) = \left[ \text{sign}(d_{ij} L)(2 - 2e^{-\alpha L}) + \text{sign}(T - d_{ij} L)(1 - e^{-\alpha L}) \right]
\]

\[
- \text{sign}(T + d_{ij} L)(1 - e^{-\alpha L}) \right]
\]

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In N-channel WDM system, each channel is modulated independently by random data. Thus, the variance can be approximated as

\[ \sigma^2_X(i) = \sum_{j=1}^{N} \sigma^2_X(i,j)(i \neq j) \]  

(3.126)

XPM and SPM are nonlinear Kerr effects that originate from intensity dependence of the refractive index. While SPM refers to self induced phase shifts experienced by an optical signal during its propagation in optical fiber, XPM refers to nonlinear phase shift of optical field induced by co-propagating field at different wavelength. From the equation of nonlinear phase shift (Agrawal, 2003), it can be said that for equally intense optical field, the contribution of XPM to the nonlinear phase shift is twice compared to that of SPM. The variance of SPM induced nonlinear phase shift can be calculated directly from the expression of XPM by making interchannel separation equal to zero as is given in Ref. – (Ho, 2004) Eq. (5) and (6)). This is due to the fact that in SPM, change in phase of the signal is proportional to its own intensity. Though SPM causes XPM induced PM (Phase Modulation) to PM conversion and IM (Intensity modulation) conversion, the analytical characterization of which has been done in Ref. – (Luis and Cartaxo, 2005) but this conversion can be neglected for approximate XPM characterization in WDM system. In the final calculation, the total nonlinear phase shift of a signal in an N-channel WDM system is calculated by the sum of XPM induced phase shift from remaining N-1 channels and SPM induced phase shift from the same channel. The summation of variance can be traced back to earlier papers like Ref. – (Wu and Way, 2004) (Eq. (2)) where the total variance of nonlinear effect is given by the sum of variance of XPM and SPM. The variance of SPM induced crosstalk is calculated by taking the interchannel separation (\(\Delta \lambda\)) equal to zero in the expression of variance of XPM (Ho, 2004). Hence, for \(\Delta \lambda = 0\),

\[ \sigma^2_{SPM}(i) = \frac{1}{4} \sigma^2_{XPM}(\Delta \lambda = 0) \]  

(3.127)
The factor of \( \frac{1}{4} \) comes from the fact that phase shift induced by XPM is twice as large as SPM for the same intensity of the signal (Ho, 2004). Thus total variance of XPM and SPM induced crosstalk is given as (Wu and Way, 2004)

\[
\sigma_X^2(i) = \sigma_{SPM}^2(i) + \sigma_{XPM}^2(i)
\]

(3.128)

The crosstalk standard deviation is evaluated on decibel scale by using the formulae

\[
\sigma_X (\text{dB}) = -10 \log_{10} e^{-\sigma_X} = \sigma_X \cdot 10 \log_{10} e
\]

(3.129)

A typical DRA configuration is shown in Fig. 3.1. It shows number of signals at different wavelengths multiplexed in a single DRA. Each wavelength represents an independent communication channel being amplified by the DRA and ultimately demultiplexed at the output. The interacting channels of WDM system are assumed to transmit 0 dBm power in wavelength range of 1515 – 1575 nm with inter-channel separation of 1 nm. The multi-pumps consisting of 6 pumps are in wavelength range of 1420 -1470 nm, separated by 10 nm and each having power of 20 dBm yielding wideband and flat gain spectrum in signal wavelength range of 1515 – 1575 nm. Each pump provides peak Raman gain to signal at a shifted wavelength of 100 nm from pump wavelength and has a gain spectrum of 10 nm.

Fig. 3.1 Schematic diagram of multi-pumped DRA
Fig. 3.2 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for SMF (with pump) and SMF (without pump).

For analysis purpose standard SMF fiber is used with following parameters: $\gamma$ (Nonlinear Co-efficient) = 1.18 W$^{-1}$km$^{-1}$, $g'_{R}/A_{\text{eff}}$ (Slope of Raman gain curve/Effective Area) = 0.57 x $10^{-16}$ m$^{-1}$W$^{-1}$Hz$^{-1}$, $A_{\text{eff}}$ (Effective Area) = 80 µm², Zero Dispersion Wavelength = 1265.5 nm, Dispersion Slope = 0.058 x $10^{3}$ ps/nm/nm/km, $\alpha$ (attenuation co-efficient) = 0.2 dB/km, L (length) = 80 km. The configuration of bi-directional pumped DRA shown in Fig. 1 reduces to a single span of single mode fiber by taking forward and backward gain ($g_F$ and $g_B$) as zero. In Fig. 3.2, green curve shows the results obtained by integrating Eq. (3.12) for zero gain and coincides with that obtained by using equation developed by Yamamoto and Norimatsu (2003) for single span of single mode for SRS crosstalk. This shows the validity of our approach. Pumps interact among themselves in a similar way as signals propagating in a fiber i.e. pumps at lower wavelengths lose power to higher wavelengths. The interaction among pumps is neglected in our analysis. Our work focuses on crosstalk between signals as they are continuously pumped while propagating in the fiber.
3.3. ANALYSIS OF CROSSTALK

The expression for variance of crosstalk that has been derived for bi-directional pumped DRA can be used to investigate crosstalk in forward pumped DRA by making backward pump $P_{PL}$ strength equal to zero. Similarly to study crosstalk in backward pumped DRA, the strength of forward pumps $P_{P0}$ is made equal to zero. Fig. 3.3 shows variation of SRS crosstalk standard deviation with signal wavelength for all the three pumping configuration. It can be observed from the figure that in all the three pumping configurations minima occur around 1545 nm i.e. the central channel of the WDM system. The reason for central channel suffering minimum crosstalk is that it gains power from lower wavelength channel and loses power to higher wavelength channel. Hence net power transfer due to SRS is smallest in central channel. It can also be observed from the figure that backward pumped DRA experiences minimum crosstalk among the three pumping configuration. The reason for backward pumped DRA suffering minimum crosstalk stems from the fact that in forward and bi-directional pumped DRA, high power pump propagate simultaneously from the start of signal transmission. Hence due to high signal power crosstalk suffered is higher for these two pumping configuration. On the contrary, in backward pumped DRA, signal strength is low at the start of transmission and starts getting pumped after travelling through a certain distance in the fiber. Moreover pulse walk off is more dominant in forward and bidirectional pumped DRA in which signal and pump co-propagate in the same direction compared to backward pumped DRA in which signal and pump counter-propagate. So, backward pumped DRA suffers minimum crosstalk.

Fig. 3.4 shows variation of combined XPM and SPM induced crosstalk standard deviation with signal wavelength for all the three pumping configurations. In all the three pumping configuration maxima occurs around 1545 nm i.e. the central channel of the WDM system. The reason for central channel suffering maximum crosstalk is that in XPM nearby channels i.e. channels with smaller interchannel separation cause greater phase deviation. Hence net phase deviation due to XPM is large in central channel as both lower and higher wavelength channels in the WDM system influence it. It can also be observed from the figure that similar to SRS, backward pumped DRA experiences minimum crosstalk among the three pumping configuration. The reason can be attributed to the same factor as SRS.
Fig. 3.3 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for forward, backward and bi-directional pumped DRA.

Fig. 3.4 Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for forward, backward and bi-directional pumped DRA.
It was also observed that crosstalk due to phase deviation is much higher than due to power transfer. Thus XPM and SPM are much detrimental compared to SRS in WDM system employing DRA.

The transmission capacity of a wavelength division multiplexing optical system can be increased by increasing the data rate per wavelength or increasing number of wavelength channels in fixed optical bandwidth. The first method has been illustrated by the increase in data rate per wavelength from 2.5 Gbps to 10 Gbps and then to 40 Gbps (Wu and Way, 2004). In the second method, the number of wavelengths in a fixed optical band is significantly increased by decreasing the spacing between adjacent wavelengths (Wu and Way, 2004). The effects of variation in the above mentioned factors were investigated in DRA.

Fig. 3.5, 3.6 and 3.7 shows the variation of SRS crosstalk standard deviation for forward pumped DRA, backward pumped DRA and bi-directional pumped DRA for different values of wavelength separation. For the three values of wavelength separation i.e. 1.0 nm, 1.5 nm and 2.0 nm, keeping the wavelength range (1515 nm -1575 nm) and other parameters same as previous analysis, it can be seen that crosstalk decreases with the increase in wavelength separation. Similar behavior was observed for XPM and SPM induced crosstalk as seen from Fig. 3.8, 3.9 and 3.10. This is because number of channels in the system decreases from 60 channels for 1.0 nm spacing to 40 channels for 1.5 nm spacing and 30 channels for 2.0 nm spacing. The same pumping configuration is used in all the three analysis i.e. 6 pumps in the wavelength range of 1420 nm to 1470 nm was used. But as the number of channels in the WDM system decreases, each pump is now pumping 10, 7 and 5 channels for 1 nm, 1.5 nm and 2.0 nm wavelength separation respectively. The gain of the signals increased but total input power decreases. Hence in the trade-off between signal strength and signal gain, the former dominates the crosstalk performance.
Fig. 3.5 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for forward pumped DRA for $\Delta \lambda = 1.0\text{ nm}, 1.5\text{ nm}, 2.0\text{ nm}$.

Fig. 3.6 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for backward pumped DRA for $\Delta \lambda = 1.0\text{ nm}, 1.5\text{ nm}, 2.0\text{ nm}$.
Fig. 3.7 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for bidirectional pumped DRA for $\Delta\lambda = 1.0 \text{ nm}, 1.5 \text{ nm}, 2.0 \text{ nm}$.

Fig. 3.8 Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for forward pumped DRA for $\Delta\lambda = 1.0 \text{ nm}, 1.5 \text{ nm}, 2.0 \text{ nm}$.
Fig. 3.9 Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for backward pumped DRA for $\Delta \lambda = 1.0$ nm, 1.5 nm, 2.0 nm.

Fig. 3.10 Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for bidirectional pumped DRA for $\Delta \lambda = 1.0$ nm, 1.5 nm, 2.0 nm.
On the contrary, when the wavelength separation was increased keeping the number of channel in WDM system fixed to 60 without keeping the wavelength range fixed, i.e. 1515 nm - 1575 nm for 1 nm separation, 1515 nm - 1605 nm for 1.5 nm separation and 1515 nm - 1635 nm for 2.0 nm, it was found that SRS crosstalk increases. This is because SRS crosstalk is directly proportional to variation in interchannel separation i.e. the constant $K$ in the expression of variance. With the increase in interchannel separation, the value of proportionality constant increases and hence crosstalk increases. On the contrary XPM and SPM induced crosstalk decreases with the increase in interchannel separation. The reason for the behavior stems from the fact that XPM induced crosstalk is inversely proportional to interchannel separation i.e. smaller is the separation greater is the phase deviation. Hence crosstalk decreases with the increase in interchannel separation.

It is observed from Fig. 3.11, 3.12 and 3.13 that with the increase in bit rate of the system, SRS crosstalk decreases for forward, backward and bidirectional pumped DRA. The SRS crosstalk suffered by each channel almost halves for increase in bit rate of system from 2.5 Gbps to 10 Gbps and from 10 Gbps to 40 Gbps for forward pumped DRA and backward pumped DRA. Similarly, XPM crosstalk decreases with the increase in bit rate of the system as seen from Fig. 3.14, 3.15 and 3.16. This is because the walk-off of length $L_W$ given as $\frac{T}{|D\Delta\lambda|}$, where $T$ is the time period and is equal to inverse of bit rate of the system, $D$ is the dispersion co-efficient of the fiber and $\Delta\lambda$ is the interchannel separation. As seen from the expression, walk-off length varies inversely with fiber dispersion co-efficient, interchannel separation and bit rate of the system. Hence it is expected that in WDM system employing (DRA), crosstalk will decrease with the increase in bit rate of the system for constant fiber dispersion co-efficient and interchannel separation.
Fig. 3.11 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for forward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps

Fig. 3.12 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for backward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps
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Fig. 3.13 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for bidirectional pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps

Fig. 3.14 Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for forward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps
Fig. 3.15 Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for backward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.

Fig. 3.16 Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for bidirectional pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.
Fig. 3.17, 3.18 and 3.19 show variation of SRS crosstalk standard deviation with input power of the signals for forward, backward and bi-directional pumped DRA respectively. Similarly, Fig. 3.20, 3.21 and 3.22 show variation of XPM crosstalk standard deviation with input power of the signals for forward, backward and bi-directional pumped DRA respectively. The values of input power considered are 0 dBm, 3 dBm and 5 dBm. With the increase in signal strength, crosstalk increases for all the three configurations of DRA as the expression of variance is directly proportional to power of the signal.

Fig. 3.17 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for forward pumped DRA for signal power = 0 dBm, 3 dBm, 5 dBm
Fig. 3.18 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for backward pumped DRA for signal power = 0 dBm, 3 dBm, 5 dBm

Fig. 3.19 Variation of SRS Crosstalk (dB) with signal wavelength (nm) for backward pumped DRA for signal power = 0 dBm, 3 dBm, 5 dBm
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Using these results, system bound for a typical DRA configuration given in Fig. 3.1 is evaluated theoretically and is given in Fig. 3.23, 3.24 and 3.25 for SRS crosstalk and 3.26, 3.27 and 3.28 for XPM crosstalk. It can be seen from figures that for all three pumping configuration, the limits of average input power increases with the increase in bit rate of the system. For example, if SRS crosstalk of 0.1 dB is the tolerable limit for WDM system, then from Fig. 3.23, it can be observed that the limit of input power for 2.5 Gbps system is 5 dBm, 10 Gbps system is 8 dBm and 40 Gbps system is 10 dBm. It can also been seen that backward pumped DRA has higher limit of input power of almost 10 dB compared to forward and bi-directional pumped DRA for same crosstalk.

![Variation of XPM and SPM Crosstalk (dB) with signal wavelength (nm) for bidirectional pumped DRA for signal power = 0 dBm, 3 dBm, 5 dBm.](image-url)
Fig. 3.23 Variation of SRS Crosstalk (dB) with input signal power for forward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.

Fig. 3.24 Variation of SRS Crosstalk (dB) with input signal power for backward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.
Fig. 3.25 Variation of SRS Crosstalk (dB) with input signal power for bidirectional pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.

Fig. 3.26 Variation of XPM and SPM Crosstalk (dB) with input signal power for forward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.
Fig. 3.27 Variation of XPM and SPM Crosstalk (dB) with input signal power for backward pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.

Fig. 3.28 Variation of XPM and SPM Crosstalk (dB) with input signal power for bidirectional pumped DRA for bitrate = 40 Gbps, 10 Gbps, 2.5 Gbps.
3.4. SUMMARY

Closed form formulae have been derived for crosstalk due to SRS, XPM and SPM in WDM system employing DRA. It is found that backward pumped DRA suffers minimum crosstalk among the three pumping scheme i.e. forward, backward and bi-directional. With the increase in interchannel separation, keeping wavelength range fixed, crosstalk suffered by the signals decreases. With the increase in interchannel separation, without keeping wavelength range fixed SRS crosstalk suffered by the signals increases, whereas XPM and SPM crosstalk decreases. With the increase in input power of the system, crosstalk increases. With the increase in bit rate of the system, crosstalk decreases. System bound for backward pumped DRA is almost 10 dB higher than forward pumped and bidirectional pumped DRA. Moreover with the increase in bit rate of the system, limits of input power increases.