Chapter 5

A Path analogue of the Hamilton - Waterloo Problem

The Hamilton - Waterloo Problem (HWP) states that for all \( r, s \geq 1 \), \( K_n \) (resp. \( K_n - I \)) has a 2-factorization in which \( r \) of the 2-factors are isomorphic to \( H_1 \) and the remaining \( s \) 2-factors are isomorphic to \( H_2 \) such that \( r H_1 \oplus s H_2 = K_n \) (resp. \( K_n - I \)), where \( H_1 \) and \( H_2 \) are given 2-factors of \( K_n \) (resp. \( K_n - I \)). We denote such factorization as \((H_1, H_2)_{r,s} - \) factorization of \( K_n \) (resp. \( K_n - I \)).

Adams et. al. [5], gave the first substantial result on HWP when \((H_1, H_2) = (C_l, C_k)\), for all \((l, k) \in \{(4, 6), (4, 8), (4, 16), (8, 16), (3, 5), (3, 15), (5, 15)\}\) and for all possible cycle lengths when \(|K_n|\) or \(|K_n - I| \leq 17\). For more results on the HWP when \((H_1, H_2) = (C_3, C_n)\), we refer [53, 54, 88, 108]. The HWP has been completely settled by Danziger et. al. [52] and Kamin [94], when \((H_1, H_2) = (C_3, C_n)\).
\((C_3, C_4)\) and \((H_1, H_2) = (C_3, C_9)\) respectively. Fu and Huang [73] have shown the existence of HWP when \((H_1, H_2) = (C_t, C_{2t})\), for even \(t \geq 4\) and \((H_1, H_2) = (C_4, C_{2t})\) for \(t \geq 3\). Lei et. al. [107] have solved the HWP when \((H_1, H_2) = (C_t, C_2t)\), for even \(t \geq 4\) and \((H_1, H_2) = (C_4, C_{2t})\) for \(t \geq 3\). M. Keranen and S."Ozkan [98] have solved the Hamilton-Waterloo problem with 4-cycles and a single factor of \(n\)-cycles. Further, Bryant and Danziger [39] have settled the \((H_1, H_2)_{r,s}\) - factorization of \(K_{2n} - I\), where \(H_1\) and \(H_2\) are any bipartite 2-factors of \(K_{2n} - I\), except when \(r \in \{1, 2n - 2\}\). Bryant et. al. [40] have proved that \((H_1, H_2)_{r,s}\) - factorization of \(K_{2n} - I\) exists, when \(H_1\) and \(H_2\) are bipartite 2-factors and \(H_1\) is a refinement of \(H_2\). Recently, Sangeetha and Muthusamy [147] proved the existence of \((C_t, C_n)_{r,s}\) - factorization of \(K_n\), for odd \(l \geq 3\). Additional results on HWP can be found in [146], where \(H_1\) and \(H_2\) are bipartite 2-factors.

Bermond et. al. [27], have proved the existence of \((P_2, P_k)_{r,s}\) - factorization of \(K_n(\lambda)\) for all \(k > 2\). The results of Bermond et. al. [27] leads to raise a Path analogue of Hamilton-Waterloo Problem (PHWP) as follows:

For given non isomorphic path factors \(H_1\) and \(H_2\) of \(K_{m,n}(\lambda)\), PHWP asks for a factorization of \(K_{m,n}(\lambda)\) into \(r\) copies of \(H_1\) and \(s\) copies of \(H_2\) such that \(rH_1 \oplus sH_2 = K_{m,n}(\lambda)\), for all \(r, s \geq 1\). We denote such existence as \((H_1, H_2)_{r,s}\) - factorization of \(K_{m,n}(\lambda)\).

In this Chapter, we show that PHWP has solution for \(K_{n,n}(\lambda)\), when \((H_1, H_2) = (P_2, P_k)\), for all \(k > 2\) and \((H_1, H_2) = (P_l, P_k)\), for all even \(l, k > 2\).
Notation: Let $V(K_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, u_3, ..., u_n\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$.

Definition 5.1. For $1 \leq i, j \leq n$, we define $\beta_i$ in $K_{n,n}$, as follows:

$$\beta_i = \begin{cases} \cup_{j=1}^{n} \{u_jv_{2j+i-1}, u_{2j+i+n-2}\}, & \text{if } n \text{ is even;} \\ \cup_{j=1}^{n} \{u_jv_{2j+i-1}\}, & \text{if } n \text{ is odd.} \end{cases}$$

where the additions in the subscript are taken modulo $n$, with residues $1, 2, \ldots, n$.

Clearly, each $\beta_i, 1 \leq i \leq n$, is a 1-factor of $K_{n,n}$ and $\bigoplus_{i=1}^{n} \beta_i = K_{n,n}$.

Example: The $\beta_1$ of $K_{8,8}$ and $K_{9,9}$ are shown in Figure 5.1.

![Graphs](image)

(a): $\beta_1$ of $K_{8,8}$  (b): $\beta_1$ of $K_{9,9}$

Fig.5.1. The graph $\beta_1$ of $K_{8,8}$ and $K_{9,9}$.

Definition 5.2. For $l \mid n$, the set $E_{i,j}^a$, $0 \leq a \leq l - 1$ in $K_{n,n}$ denotes the set of edges of $\beta_j$ incident with vertices $v_i, i \equiv a \pmod{l}$.

Example 5.3. Consider the graph $K_{8,8}$. For $0 \leq a \leq 3$, $E_{4,1}^a$’s are as follows:

$E_{4,1}^0 = \{\{v_1, u_2\}, \{v_8, u_4\}\}; E_{4,1}^1 = \{\{v_1, u_5\}, \{v_5, u_7\}\}; E_{4,1}^2 = \{\{v_2, u_1\}, \{v_6, u_3\}\}$

and $E_{4,1}^3 = \{\{v_3, u_6\}, \{v_7, u_8\}\}$. The dark edges of Fig.5.1(a) gives $E_{4,1}^0$. 
Example 5.4. Consider the graph $K_{9,9}$. For $0 \leq a \leq 2$, $E_{3,1}^a$'s are as follows: $E_{3,1}^0 = \{\{v_3, u_6\}, \{v_6, u_3\}, \{v_9, u_9\}\}$; $E_{3,1}^1 = \{\{v_1, u_5\}, \{v_4, u_2\}, \{v_7, u_8\}\}$ and $E_{3,1}^2 = \{\{v_2, u_1\}, \{v_5, u_7\}, \{v_8, u_4\}\}$. The dark edges of Fig. 5.1(b) gives $E_{3,1}^0$.

Remark 5.5. We define $\beta'_j$ of $K_{n,n}$ as $\beta'_j = (\beta_j \setminus E_{2,j}^0) \oplus E_{2,j-\frac{1}{2}}^0$, $1 \leq j \leq n$. It is easy to observe that the vertices incident with the set of edges $E_{2,j}^0$ and $E_{2,j-\frac{1}{2}}^0$ are same. Hence $\beta'_j$ is also a 1 - factor of $K_{n,n}$.

Example 5.6. The graph $\beta_4$ and $\beta'_4$ of $K_{9,9}$ are given for $l = 6$.

i.e, $\beta'_4 = (\beta_4 \setminus E_{3,4}^0) \oplus E_{3,1}^0$. From Fig. 5.2, it is clear that the end vertices of $E_{3,4}^0$ and $E_{3,1}^0$ are same and hence $\beta'_4$ is also a 1- factor of $K_{9,9}$.

![Diagram](a): $\beta_4$

![Diagram](b): $\beta'_4 = \beta_4 \setminus E_{3,4}^0 \oplus E_{3,1}^0$.

Fig 5.2. $\beta_4$ and $\beta'_4$ of $K_{9,9}$. 
Lemma 5.7. When \( m \) is even, any \((m - 1)\) consecutive \( \beta \)'s of \( K_{n,n}(\lambda) \) can be factorized into \( \frac{m}{2} P_m \) factors.

Proof. Let \( \beta_{d+1}, \beta_{d+2}, \ldots, \beta_{d+m-2}, \beta_{d+m-1} \) be any \((m-1)\) consecutive \( \beta \)'s of \( K_{n,n}(\lambda) \), \( d \geq 0 \).

For \( 0 \leq c \leq \frac{m}{2} - 1 \), we define \( P^c_d \), as follows:

\[
P^c_d = E_{\frac{m}{2},d+1}^{c} + E_{\frac{m}{2},d+2}^{c+1} + E_{\frac{m}{2},d+3}^{c+2} + \cdots + E_{\frac{m}{2},d+m-2}^{c+m-3} + E_{\frac{m}{2},d+m-1}^{c+m-2} + E_{\frac{m}{2},d+m}^{c+m-1} + \cdots + E_{\frac{m}{2},d+\frac{m}{2}+1}^{c+\frac{m}{2}-1} + \cdots + E_{\frac{m}{2},d+\frac{m}{2}+2}^{c+\frac{m}{2}+1} + \cdots + E_{\frac{m}{2},d+\frac{m}{2}+3}^{c+\frac{m}{2}+2} + \cdots + E_{\frac{m}{2},d+\frac{m}{2}+\frac{m}{2}-1}^{c+\frac{m}{2}}, \quad \text{if} \ m \equiv (2 \mod 4)
\]

\[
P^c_d = E_{\frac{m}{2},d+1}^{1+c} + E_{\frac{m}{2},d+2}^{1+c+1} + E_{\frac{m}{2},d+3}^{2+c} + E_{\frac{m}{2},d+4}^{2+c+1} + \cdots + E_{\frac{m}{2},d+m-3}^{c+m-1+c} + E_{\frac{m}{2},d+m-2}^{c+m-1+c} + \cdots + E_{\frac{m}{2},d+\frac{m}{2}+1}^{c+\frac{m}{2}}, \quad \text{if} \ m \equiv (0 \mod 4).
\]

where the additions in the superscript are taken \( \mod \frac{m}{2} \), with residues 0, 1, 2, \ldots, \( \frac{m}{2} \).
1 and the additions in the subscript are taken modulo \( n \), with residues 1, 2, \ldots, \( n \).

Clearly, each \( P_d^{r,s} \) is a \( P_m \)-factor, see Figures 5.3 and 5.4. When \( c \) varies, \( \cup_c P_d^{r,s} \) gives the required \( \frac{m}{2} P_m \)-factors.

**Corollary 5.8.** When \( m \) is even, any \( 2(m - 1) \) consecutive \( \beta' \)'s of \( K_{n,n}(\lambda) \) can be factorized into \( m \) \( P_m \)-factors.

**Proof.** Follows from Lemma 5.7.

**Remark 5.9.** By the definition of \( \beta'_j \) in Remark 5.5, replacement of \( \beta_j \) by \( \beta'_j \) will never affect the construction of \( P_m \)-factors.

## 5.1 \((P_2, P_k)_{r,s} - Factorization of \( K_{n,n}(\lambda)\))

In this section, we show that \((P_2, P_k)_{r,s} - factorization of \( K_{n,n}(\lambda) \) exists, for all \( k > 2 \).

**Theorem 5.10.** For \( n, k > 2 \), \( K_{n,n}(\lambda) \) has a \((P_2, P_k)_{r,s} - factorization if and only if

\[(i) \ 2n \equiv 0 \ (mod \ k), \]
\[(ii) \ | E(K_{n,n}(\lambda)) | = nr + \frac{2n}{k}(k - 1)s, \ r, s \geq 1, \]
\[(iii) \ s \equiv 0 \ (mod \ \frac{k}{2}), \ when \ k \ is \ even \ and \]
\[(iv) \ s \equiv 0 \ (mod \ k), \ when \ k \ is \ odd. \]

**Proof.** **Necessity:** By counting the number of edges and vertices of a \((P_2, P_k) - \)
factor, conditions \((i)\) and \((ii)\) hold. From \((ii)\), we have

\[
\lambda n^2 = nr + \frac{2n}{k}(k-1)s. \tag{5.1}
\]

Equation (5.1) holds only if \(s \equiv 0 \pmod{\frac{k}{2}}\), when \(k\) is even and \(s \equiv 0 \pmod{k}\), when \(k\) is odd. This proves \((iii)\) and \((iv)\).

**Sufficiency:**

**Case 1: \(k\) even**

By the hypothesis, when \(k\) is even, let \(s = \frac{k}{2}t\), for some \(t \geq 1\). Since \(r \geq 1\), then

\[
t \in \{1, ..., \left\lfloor \frac{\lambda n}{k-1} \right\rfloor \}, \quad \text{when } \lambda n \not\equiv 0 \pmod{(k-1)} \quad \text{or} \quad t \in \{1, ..., \left\lfloor \frac{\lambda n}{k-1} \right\rfloor - 1\}, \quad \text{when } \lambda n \equiv 0 \pmod{(k-1)}.
\]

Substituting, \(s = \frac{k}{2}t\) in \(\lambda nk = rk + 2s(k-1)\), we get

\(r = \lambda n - t(k-1)\). Hence the possible choices of \((r, s) = (\lambda n - t(k-1), \frac{k}{2}t)\).

Now we construct \(r P_2\) - factors and \(s P_k\) - factors as follows:

Using Lemma 5.7, for \(t\) times, we get \(\frac{k}{2}t P_k\) - factors from \(t(k-1)\) consecutive \(\beta_i'\)'s. Hence the remaining \((\lambda n - t(k-1))\) \(\beta_i'\)'s, gives the required \((\lambda n - t(k-1))\) \(P_2\) - factors.

**Case 2: \(k\) odd**

By the hypothesis, when \(k\) is odd, let \(s = kt\), for some \(t \geq 1\). Since \(r \geq 1\), then

\[
t \in \{1, ..., \left\lfloor \frac{\lambda n}{2(k-1)} \right\rfloor \}, \quad \text{when } \lambda n \not\equiv 0 \pmod{2(k-1)} \quad \text{or} \quad t \in \{1, ..., \left\lfloor \frac{\lambda n}{2(k-1)} \right\rfloor - 1\}, \quad \text{when } \lambda n \equiv 0 \pmod{2(k-1)}.
\]

Substituting, \(s = kt\) in \(\lambda nk = rk + 2s(k-1)\), we get

\(r = \lambda n - 2(k-1)t\). Hence the possible choices of \((r, s) = (\lambda n - 2(k-1)t, kt)\).

We now construct \(r P_2\) - factors and \(s P_k\) - factors as follows. Let \(b = \frac{k+1}{2}\). For
1 \leq j \leq t \text{ and } 0 \leq c \leq k - 1, \text{ we define } P^c_j, \text{ as follows:} \\

\[ P^c_j = E_{k,j}^c \oplus E_{k,j+1}^{c+b} \oplus E_{k,j+2}^{c+1+b} \oplus E_{k,j+3}^{c+1+b} \oplus E_{k,j+4}^{c+2} \oplus E_{k,j+5}^{c+2+b} \oplus \cdots \oplus \]

\[ E_{k,j+k-3}^{c+\frac{k-3}{2}+b} \oplus E_{k,j+k-2}^{c+\frac{k-2}{2}+b} \oplus \cdots \oplus E_{k,j+k-3}^{c+\frac{k-3}{2}+b} \oplus E_{k,j+k-2}^{c+\frac{k-2}{2}+b}, \]

where the additions in the superscript are taken modulo \( k \), with residues 0, 1, ..., \( k-1 \) and the additions in the subscript are taken modulo \( n \), with residues 1, 2, ..., \( n \). Clearly, each \( P^c_j \) is a \( P_k \) - factor (see Figure 5.5) and \( \cup_j(\cup_c P^c_j) \) gives the required \( kt \ P_k \) - factors, for all \( t \in \{1, ..., \lfloor \frac{\lambda n}{2(k-1)} \rfloor \} \), when \( \lambda n \not\equiv 0 \pmod{2(k-1)} \) (or) \( t \in \{1, ..., \lfloor \frac{\lambda n}{2(k-1)} \rfloor - 1 \} \), when \( \lambda n \equiv 0 \pmod{2(k-1)} \) respectively.

\[ P^0_1 \text{ of } K_{9,9} = \]

Fig.5.5. \( P_9 \) - factor of \( K_{9,9} \).

Further, it is easy to observe that for every \( j, 1 \leq j \leq t \), \( \cup_c P^c_j = \cup_{i=0}^{2k-2} \beta_{j+i} \setminus \beta_{j+k-1} \) provides \( k \ P_k \) - factors using \( (2k-2) \ \beta_i \)'s. Hence in the above construction, we have used \( 2(k-1)t \ \beta_i \)'s to construct the required \( kt \ P_k \) - factors. The remaining \( (\lambda n - 2t(k-1)) \ \beta_i \)'s provides the required \( (\lambda n - 2t(k-1)) \ P_2 \) - factors.

\[ \square \]

**Remark 5.11.** The case \( t = \frac{\lambda n}{2(k-1)} \), when \( k \) is odd and \( \lambda n \equiv 0 \pmod{2(k-1)} \) (or) the case \( t = \frac{\lambda n}{(k-1)} \), when \( k \) is even and \( \lambda n \equiv 0 \pmod{(k-1)} \) of Theorem
5.10. deduce the result of M.L. Yu [191] on $P_k$ - factorization of complete bipartite multigraphs.

5.2 $(P_l, P_k)_{r,s} -$ Factorization of $K_{n,n}(\lambda)$

Theorem 5.12. For $n > 2$ and even $l, k > 2$, $K_{n,n}(\lambda)$ has a $(P_l, P_k)_{r,s} -$ factorization, if and only if

(i) $2n \equiv 0 \pmod{l}$, $2n \equiv 0 \pmod{k}$,

(ii) $|E(K_{n,n})(\lambda)| = \frac{2n}{l}(l - 1)r + \frac{2n}{k}(k - 1)s, r, s \geq 1,$

(iii) $r \equiv 0 \pmod{l}$ and $s \equiv 0 \pmod{k}$, when $\lambda n \equiv 0 \pmod{2}$ and

(iv) $r \equiv 0 \pmod{\frac{l}{2}}$ and $s \equiv 0 \pmod{\frac{k}{2}}$, when $\lambda n \equiv 1 \pmod{2}$.

Proof. Necessity:

By counting the number of edges and vertices of a $(P_l, P_k)$ - factor, conditions (i) and (ii) hold. From (ii), we have

$$\lambda n^2 = \frac{2n}{l}(l - 1)r + \frac{2n}{k}(k - 1)s. \quad (5.2)$$

Equation (5.2) holds only if $r \equiv 0 \pmod{l}$ and $s \equiv 0 \pmod{k}$, when $\lambda n \equiv 0 \pmod{2}$, ( respectively $r \equiv 0 \pmod{\frac{l}{2}}$ and $s \equiv 0 \pmod{\frac{k}{2}}$, when $\lambda n \equiv 1 \pmod{2}$). This proves (iv) and (v).
5.2. \((P_l, P_k)_{r,s}\) - Factorization of \(K_{n,n}(\lambda)\)

Sufficiency:

**Case 1:** \(\lambda n \equiv 0 \pmod{2}\)

By the hypothesis, when \(\lambda n \equiv 0 \pmod{2}\), let \(r = lt\), for some \(t \geq 1\). Since \(s \geq 1\), then \(t \in \{1, \ldots, \left\lfloor \frac{\lambda n}{2(l-1)} \right\rfloor \}\) when \(\lambda n \not\equiv 0 \pmod{2(l-1)}\) (or) \(t \in \{1, \ldots, \left\lfloor \frac{\lambda n}{2(l-1)} \right\rfloor - 1\}\), when \(\lambda n \equiv 0 \pmod{2(l-1)}\). Substituting, \(r = lt\in \lambda n^2 = \frac{2n}{l}(l-1)r + \frac{2n}{k}(k-1)s\), we get \(s = k\left(\frac{\lambda n-l(l-1)k}{2(k-1)}\right)\). Hence the possible choices of \((r, s) = (lt, k\left(\frac{\lambda n-l(l-1)k}{2(k-1)}\right))\).

Now we construct \(r P_l\) - factors and \(s P_k\) - factors as follows:

By using Corollary 5.8, for \(t\) times, we get \(lt P_l\)-factors, from \((l-1) t\) consecutive \(\beta's\). Hence, after constructing \(lt P_l\)-factors, we have \(\lambda n - 2(l-1)t\) unused \(\beta's\) of \(K_{n,n}(\lambda)\). Since \(2(k-1) | (\lambda n - 2(l-1)t)\), we have \(\left(\frac{\lambda n-l(l-1)k}{2(k-1)}\right) k P_k\) - factors, by Corollary 5.8. Thus we factorized \(K_{n,n}(\lambda)\) into \(r P_l\) - factors and \(s P_k\) - factors.

**Case 2:** \(\lambda n \equiv 1 \pmod{2}\)

By the hypothesis, when \(\lambda n \equiv 1 \pmod{2}\), let \(r = \frac{l}{2} t\), for \(t \geq 1\). Since \(s \geq 1\), then \(t \in \{1, \ldots, \left\lfloor \frac{\lambda n}{l(l-1)} \right\rfloor \}\) when \(\lambda n \not\equiv 0 \pmod{(l-1)}\) (or) \(t \in \{1, \ldots, \left\lfloor \frac{\lambda n}{l(l-1)} \right\rfloor - 1\}\), when \(\lambda n \equiv 0 \pmod{(l-1)}\). Substituting, \(r = \frac{l}{2} t\in \lambda n^2 = \frac{2n}{l}(l-1)r + \frac{2n}{k}(k-1)s\), we get \(s = \frac{k}{2}\left(\frac{\lambda n-l(l-1)k}{(k-1)}\right)\). Hence the possible choices of \((r, s) = (\frac{lt}{2}, k\left(\frac{\lambda n-l(l-1)k}{(k-1)}\right))\).

Now we construct \(r P_l\) - factors and \(s P_k\) - factors as follows:

By using Lemma 5.7 for \(t\) times, we get \(\frac{1}{2} t P_l\) - factors, from \((l-1) t\) consecutive \(\beta's\). Hence after constructing \(\frac{1}{2} t P_l\) - factors, we have \(\lambda n - t(l-1)\) unused \(\beta's\) of \(K_{n,n}(\lambda)\). Since \((k-1) | (\lambda n - (l-1)t)\), we have \(\left(\frac{\lambda n-(l-1)t}{k(k-1)}\right) k P_k\) - factors, by Lemma 5.7. Thus we factorized \(K_{n,n}(\lambda)\) into \(r P_l\) - factors and \(s P_k\) - factors. □
Remark 5.13. The results of Bermond et. al. [27] together with the results given in Section 5.1 settled the existence of $(P_2, P_k)_{r,s}$ - factorization of $K_n(\lambda)$ and $K_{n,n}(\lambda)$ for $k > 2$. The results of Section 5.2 settled the existence of $(P_l, P_k)_{r,s}$ - factorization of $K_{n,n}(\lambda)$ for all even $l, k > 2$. The remaining cases are still open.