Chapter 2

$(H_1, H_2)$ - Multidecomposition of Multigraphs

In this chapter, we discuss the existence of $(H_1, H_2)$ - multidecomposition of $G(\lambda)$, when $G = K_{n+1}$ and $G = K_{n,n}$. We recall that, $(H_1, H_2)$ - multidecomposition of $G(\lambda)$ is the decomposition of $G(\lambda)$ into $r$ $H_1$ and $s$ $H_2$, such that $r$ $H_1 \oplus s$ $H_2 = G(\lambda)$, for some integers $r, s \geq 1$. Atif Abueida and M.Daven [1, 2] introduced the study of $(H_1, H_2)$ - multidecomposition of multigraphs. Later, Atif Abueida and Theresa O’ Neil [4] have settled the existence of $(H_1, H_2)$ - multidecomposition of $K_n(\lambda)$ when $(H_1, H_2) = (K_{1,m-1}, C_m)$ for $m = 3, 4, 5$ and $n \geq m$ and they conjectured that “For any integer $n \geq m \geq 3$, there is a $(H_1, H_2)$ - multidecomposition of $K_n(\lambda)$, where $(H_1, H_2) = (K_{1,m-1}, C_m)$”. In support of the above conjecture, we show that $(H_1, H_2)$ - multidecompose-
2.1. \((H_1, H_2)\) - Multidecomposition of \(K_{n+1}(\lambda)\)

Multidecomposition of \(K_{n+1}(\lambda)\) exists when \((H_1, H_2) = (K_{1,n-1}, P_n)\). Further, we give necessary and sufficient conditions for the existence of \((H_1, H_2)\) - multidecomposition of \(K_{n,n}(\lambda)\) when (i) \((H_1, H_2) = (K_{1,n-1}, P_n)\), (ii) \((H_1, H_2) = (K_{1,n-1}, C_n)\) and (iii) \((H_1, H_2) = (P_n, C_n)\).

2.1 \((H_1, H_2)\) - Multidecomposition of \(K_{n+1}(\lambda)\)

To prove our results we require the following.

**Theorem 2.1.** (Walecki’s Construction) [10]: \(K_{2n+1}\) has a Hamilton cycle decomposition.

**Corollary 2.2.** [10]: \(K_{2n}\) has a Hamilton path decomposition.

**Lemma 2.3.** \(K_{1,n}(\lambda)\) has a \(K_{1,n-1}\) - decomposition when \(\lambda \equiv 0 \mod (n-1)\).

**Proof.** Let \(V(K_{1,n}) = \{\{a\}, \{b_1, b_2, b_3, \ldots, b_n\}\}\). Now we construct \(K_{1,n-1}\) - decomposition of \(K_{1,n}(\lambda)\) as follows: Let \(S_i = (a: b_i, b_{i+1}, \ldots, b_{i+n-2}), i = 1, 2, \ldots, n\). Clearly each \(S_i\) is a \(K_{1,n-1}\) and variation of \(i\) gives a \(K_{1,n-1}\)-decomposition of \(K_{1,n}(\lambda)\). Thus \(K_{1,n-1} \mid K_{1,n}(\lambda)\), see, Fig. 2.1.

**Example:** \(K_{1,4}\) decomposition of \(K_{1,5}(4)\) is shown in Fig. 2.1.
Fig 2.1. $K_{1,4}$ decomposition of $K_{1,5}(4)$.

Lemma 2.4. There exists a $(K_{1,n-1}, P_n)$ - multidecomposition in $K_{n+1}(\lambda)$ for all
2.1. \((H_1, H_2)\) - Multidecomposition of \(K_{n+1}(\lambda)\)

\[\lambda \equiv 0 \mod (n - 1)\]

**Proof.** Case 1: \(n\) is odd.

Consider the graph \(K_{n+1}(\lambda)\). Let \(V(K_{n+1}) = \{1, 2, 3, \ldots, n, n + 1\}\). Place the vertices of \(K_{n+1}(\lambda)\) in the circumference of a circle. Let \(\lambda = n - 1\). We denote \(K_{1,n-1}\) with center at \(i\) as \((i : i + 1, i + 2, \ldots, i + n - 1)\), \(i = 1, 2, 3, \ldots, n + 1\) and the additions are taken modulo \(n + 1\). When \(i\) varies, we get \((n + 1) K_{1,n-1}\).

Repeating this procedure for \(\binom{n-1}{2}\) times, we get \(\binom{n-1}{2} K_{1,n-1}\) from \(K_{n+1}(\lambda)\).

The remaining edges form a graph isomorphic to \(C_{n+1} \left(\frac{n-1}{2}\right)\) which lies in the circumference of the circle and can be easily decomposed into \(\frac{n+1}{2} P_n\).

Case 2: \(n\) is even.

The graph \(K_{n+1}(\lambda)\) can be written as \(K_{n+1}(\lambda) = K_n(\lambda) \oplus (\overline{K_n} \lor K_1)(\lambda)\). As \(n\) is even, \(K_n\) and hence \(K_n(\lambda)\) is Hamilton path decomposable by Corollary 2.2.

The remaining graph \((\overline{K_n} \lor K_1)(\lambda) \cong K_{1,n}(\lambda)\) and has a \(K_{1,n-1}\) - decomposition by Lemma 2.3. Thus \((K_{1,n-1}, P_n) | K_{n+1}(\lambda)\) for all \(\lambda \equiv 0 \mod (n - 1)\). \(\square\)

**Theorem 2.5.** For \(n \geq 3\), \(K_{n+1}(\lambda)\) has a \((K_{1,n-1}, P_n)\) - multidecomposition if and only if

(i) \(|E(K_{n+1}(\lambda))| = r(n - 1) + s(n - 1)\), where \(r, s \geq 1\).

(ii) \(\lambda \equiv 0 \mod (n - 1)\).

(iii) one of \(n\) or \(\lambda\) should be even.

**Proof.** Necessity:

Assume that \(K_{n+1}(\lambda)\) has a \((K_{1,n-1}, P_n)\) - multidecomposition. Then there exists
2.2. \((H_1, H_2) - \text{Multidecomposition of } K_{n,n}(\lambda)\)

Integers \(r, s \geq 1\), such that \(K_{n+1}(\lambda) = rK_{1,n-1} \oplus sP_n\). Hence \(|E(K_{n+1}(\lambda))| = r(n - 1) + s(n - 1)\) for some \(r, s \geq 1\). Thus \((i)\) holds. From \((i)\) and counting the number of edges, we have

\[
|E(K_{n+1}(\lambda))| = r(n - 1) + s(n - 1) = \frac{\lambda n(n + 1)}{2} = \frac{\lambda}{2} n(n - 1) + \lambda n \quad (2.1)
\]

From the equation \((2.1)\), it is clear that one of \(\lambda\) or \(n\) should be even and \(\lambda \equiv 0 \mod (n - 1)\).

Sufficiency:

Follows from Lemma 2.4. \(\Box\)

2.2 \((H_1, H_2) - \text{Multidecomposition of } K_{n,n}(\lambda)\)

In this section, we prove the existence of \((H_1, H_2) - \text{multidecomposition of } K_{n,n}(\lambda)\)
when \((i)\) \((H_1, H_2) = (K_{1,n-1}, P_n)\) \((ii)\) \((H_1, H_2) = (C_n, K_{1,n-1})\) and \((iii)\) \((H_1, H_2) = (C_n, P_n)\).

To prove our results, we require the following:

Theorem 2.6. [160] For \(m, n \geq k\), \(K_{m,n}\) can be decomposed into \(C_{2k}\) if and only if \(m\) and \(n\) are even and \(2k\) divides \(mn\).

Theorem 2.7. [191] Let \(F\) be any 1 - factor of \(K_{k,k}\). Then the following holds.

\((a)\) if \(k\) is even, then \(K_{k,k} - F\) has a \(P_k\) - factorization.

\((b)\) if \(k\) is odd, then \(K_{k,k}(2) - F(2)\) has a \(P_k\) - factorization.
Lemma 2.8. If $K_{n,n}(\lambda)$ has a $(H_1, H_2)$-multidecomposition, then so does $K_{ns,ns}(\lambda)$.

Proof. We define a new graph from $K_{ns,ns}(\lambda)$ by identifying disjoint $n$-subsets of the partite sets of $K_{ns,ns}(\lambda)$ as vertices such that two of them are adjacent if the corresponding $n$-subsets induce $K_{n,n}(\lambda)$ in $K_{ns,ns}(\lambda)$. The resulting graph is isomorphic to $K_{s,s}$. We know that $K_{s,s}$ is 1-factorable. Corresponding to a 1-factor of $K_{s,s}$, we have a $K_{n,n}(\lambda)$-factor in $K_{ns,ns}(\lambda)$. Hence the 1-factorization of $K_{s,s}$ gives a $K_{n,n}(\lambda)$-factorization of $K_{ns,ns}(\lambda)$. By the hypothesis, $K_{n,n}(\lambda)$ possess $(H_1, H_2)$-multidecomposition and hence $K_{ns,ns}(\lambda)$ has a $(H_1, H_2)$-multidecomposition, see Fig 2.2.

Example: One-factorization of $K_{s,s}$ gives a $K_{n,n}$-factorization of $K_{ns,ns}$.

![1-factor of $K_{s,s}$](image1.png) ![1-factor of $K_{n,n}$-factor of $K_{ns,ns}$](image2.png)

Fig 2.2. 1-factor of $K_{s,s}$ implies $K_{n,n}$-factor of $K_{ns,ns}$.

Notation: Let $V(K_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, u_3, ..., u_n\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$.
\((K_{1,n-1}, P_n)\) - multidecomposition of \(K_{n,n}(\lambda)\).

**Lemma 2.9.** There exists a \((K_{1,n-1}, P_n)\) - multidecomposition in \(K_{n,n}(\lambda)\) for all \(\lambda \equiv 0 \mod (n - 1)\)

**Proof.** For \(\lambda = n - 1\), we decompose \(K_{n,n}(n - 1)\) into \((K_{1,n-1}, P_n)\) as follows.

**Case 1:** \(n\) odd.

We construct \(2n\) \(K_{1,n-1}\) as follows. Let

\[
S^i_1 = (u_n : v_i, v_{i+1}, ..., v_{i+n-2});
S^i_2 = (v_n : u_i, u_{i+1}, ..., u_{i+n-2});
\]

where \(i = 1, 2, 3, ..., n\) and the additions in the subscripts are taken modulo \(n\).

Clearly, each \(S^i_j\), \(j = 1, 2\) is a star \(K_{1,n-1}\). When \(i\) varies, we get \(2n\) \(K_{1,n-1}\) and the remaining edges form a graph \(K_{n-1,n-1}(n-1)\). By Theorem 2.6, \(C_{2(n-1)} | K_{n-1,n-1}\) and hence \(P_n | K_{n-1,n-1}\) as \(C_{2(n-1)} = 2P_n\).

**Case 2:** \(n\) even. We construct \(n(n-1)K_{1,n-1}\) as follows. Let

\[
S^i_1 = (u_i : v_i, v_{i+1}, ..., v_{i+n-2});
S^i_2 = (u_i : v_{i+1}, v_{i+2}, ..., v_{i+n-1});
S^i_3 = (u_i : v_{i+2}, v_{i+3}, ..., v_{i+n});
...
S^i_{n-1} = (u_i : v_{i+n-2}, v_{i+n-1}, ..., v_{i+2n-4});
\]
where \(i = 1, 2, 3, ..., n\) and the additions in the subscripts are taken modulo \(n\).

Clearly, each \(S_j^i\) is a star \(K_{1,n-1}\), \(1 \leq j \leq n - 1\). When \(i\) varies, we get \(n(n-1)K_{1,n-1}\). After decomposing \(n(n-1)K_{1,n-1}\) from \(K_{n,n}(n-1)\), the remaining edges \(\oplus_{i=1}^{n}\{\{u_i, v_{i+n-2}\}, \{u_i, v_{i+n-3}\}, \ldots, \{u_i, v_{i+n-1}\}\}\) form a graph isomorphic to \(K_{n,n} - F = G\), where \(F = \oplus_{i=1}^{n}u_iv_{i+n-2}\) and the additions are taken modulo \(n\).

By Theorem 2.7, \(P_n \mid G\) and hence \((K_{1,n-1}, P_n) \mid K_{n,n}(n-1)\). By repeating the procedure, we get \((K_{1,n-1}, P_n) \mid K_{n,n}(\lambda)\) for all \(\lambda \equiv 0 \mod (n-1)\).

**Theorem 2.10.** \(K_{n,n}(\lambda)\) has a \((K_{1,n-1}, P_n)\) - multidecomposition if and only if

(i) \(|E(K_{n,n})(\lambda)| = r(n-1) + s(n-1), r, s \geq 1\) and

(ii) \(\lambda \equiv 0 \mod (n-1)\).

**Proof.** **Necessity:**

Assume that \(K_{n,n}(\lambda)\) has a \((K_{1,n-1}, P_n)\) - multidecomposition. Then there exist integers \(r, s \geq 1\), such that \(K_{n,n}(\lambda) = rK_{1,n-1} + sP_n\). Hence

\[ |E(K_{n,n})(\lambda)| = r(n-1) + s(n-1) \text{ for some } r, s \geq 1. \]

Thus (i) holds. Now from (i), we have,

\[ |E(K_{n,n})(\lambda)| = r(n-1) + s(n-1) = (n-1)(r + s) = \lambda n^2, \]

where \(r, s \geq 1\). This shows that \(\lambda \equiv 0 \mod (n-1)\), since \(\text{g.c.d. } (n, n-1) = 1\).

**Sufficiency:**

Follows from Lemma 2.9.
From Lemma 2.8 and Theorem 2.10, we have the following:

**Corollary 2.11.** If $K_{n,n}(\lambda)$ has a $(K_{1,n-1}, P_n)$ - multidecomposition, then $K_{ns,ns}(\lambda)$ has a $(K_{1,n-1}, P_n)$ - multidecomposition.

**Theorem 2.12.** $K_{n,n}(\lambda)$ has a $(C_n, K_{1,n-1})$ - multidecomposition if and only if

(i) $n \equiv 0 \pmod{2}$ and $n \geq 4$,

(ii) $|E(K_{n,n}(\lambda))| = rn + s(n - 1)$, $r, s \geq 1$ and

(iii) $\lambda \geq 2$, when $n > 4$.

**Proof. Necessity:**

Since the graph is bipartite, cycles are of even length, so $n \equiv 0 \pmod{2}$ and by the definition of multidecomposition, $n \geq 4$. This proves (i).

(ii) Assume that $K_{n,n}(\lambda)$ has a $(C_n, K_{1,n-1})$ - multidecomposition. Then there exist integers $r, s \geq 1$, $K_{n,n}(\lambda) = rC_n \oplus sK_{1,n-1}$.

Hence $|E(K_{n,n}(\lambda))| = rn + s(n - 1)$ for $r, s \geq 1$. Thus (ii) holds.

(iii) Assume that $\lambda = 1$. From (ii) above, $rn + s(n - 1) = |E(K_{n,n}(\lambda))|$. Let $r = 1, s = n$. Without loss of generality, let $C_n = u_1v_1u_2v_2...v_{\frac{n}{2} - 1}u_{\frac{n}{2}}v_{\frac{n}{2}}u_1$. Then the edges incident with the vertices of $C_n$ in $K_{n,n} - C_n$, are not exhausted by any $K_{1,n-1}$, when $n > 4$, a contradiction to the hypothesis. Hence $\lambda \geq 2$.

**Sufficiency:**

Case 1: $n = 4$. 

We now construct a \((C_4, K_{1,3})\) - multidecomposition of \(K_{4,4}\) as follows:

Let \(V(K_{4,4}) = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3, v_4\}\). Then \(\{u_1v_1u_2v_2u_1, (u_3 : v_1, v_2, v_3), (u_4 : v_1, v_2, v_4), (v_3 : u_1, u_2, u_4), (v_4 : u_1, u_2, u_3)\}\) gives a \((C_4, K_{1,3})\) - multidecomposition of \(K_{4,4}\). Therefore \(K_{4,4}(\lambda)\) has a \((C_4, K_{1,3})\) - multidecomposition, for all \(\lambda \geq 1\).

**Case 2:** \(\lambda = 2\) and \(n > 4\).

We now construct a \((C_n, K_{1,n-1})\) - multidecomposition of \(K_{n,n}(2)\) as follows:

Let \(C_1^1 = u_1v_1u_2v_2\ldots v_{\frac{n}{2}-1}u_{\frac{n}{2}}v_{\frac{n}{2}}u_1, C_1^2 = u_{\frac{n}{2}+1}v_{\frac{n}{2}+1}u_{\frac{n}{2}+2}v_{\frac{n}{2}+2}\ldots v_{n-1}u_nv_nu_{\frac{n}{2}+1}\) be the two edge disjoint cycles of length \(n\) in \(K_{n,n}(2)\). The \(2n\) copies of \(K_{1,n-1}\) are constructed as follows:

\[
\{(v_i : u_i, u_{i+2}, \ldots, u_{i+n-1}), \; i = 1, 2, 3, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 1, \ldots, n - 1; \\
(v_{\frac{n}{2}} : u_2, u_3, \ldots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \ldots, u_n); \\
(v_n : u_1, u_2, \ldots, u_{n-2}, u_{\frac{n}{2}+1}, \ldots, u_n) \text{ and} \\
(v_i : u_{i+1}, u_{i+2}, \ldots, u_{i+n-1}), \; i = 1, 2, 3, \ldots, n\} \text{ where the additions in the subscripts are taken modulo } n. \text{ The above set consisting } 2n \text{ copies of } K_{1,n-1} \text{ decompose } K_{n,n}(2) - \{(C_1^1, C_1^2)\} \text{ and hence the case.}
\]

**Case 3:** \(\lambda \geq 3\) and \(n > 4\).

We write \(K_{n,n}(\lambda) = K_{n,n}(\lambda - 2) + K_{n,n}(2)\).

By Case 2, \((K_{1,n-1}, C_n) \mid K_{n,n}(2)\) and \(K_{n,n}(\lambda - 2)\) has a \(C_n\) - decomposition by Theorem 2.6.

From Lemma 2.8 and Theorem 2.12, we have the following:

**Corollary 2.13.** If \(K_{n,n}(\lambda)\) has a \((K_{1,n-1}, C_n)\) - multidecomposition, then \(K_{n,n}(\lambda)\)
2.2. \((H_1, H_2)\) - Multidecomposition of \(K_{n,n}(\lambda)\)

has a \((K_{1,n-1}, C_n)\) - multidecomposition.

\((C_n, P_n)\) - multidecomposition of \(K_{n,n}(\lambda)\).

**Definition 2.14.** A 1-factor of distance \(t\), denoted by \(\alpha_t(V_1, V_2)\), in the complete bipartite graph \(K_{n,n}\) with partite sets \(V_1 = \{u_1, u_2, \ldots, u_n\}\) and \(V_2 = \{v_1, v_2, \ldots, v_n\}\), is defined as \(\alpha_t(V_1, V_2) = \{(u_i, v_{i+t-1}) : 1 \leq i \leq n\}\), \(1 \leq t \leq n\), and the addition in the subscript is taken modulo \(n\).

**Theorem 2.15.** \(K_{n,n}(\lambda)\) has a \((C_n, P_n)\) - multidecomposition if and only if

(i) \(n \equiv 0 \pmod{2}\) and

(ii) \(|E(K_{n,n}(\lambda))| = rn + s(n - 1), r, s \geq 1\).

**Proof.** Necessity:

Since \(K_{n,n}(\lambda)\) is bipartite, the cycles must be of even length and hence \(n \equiv 0 \pmod{2}\).
This proves (i).

(ii) Assume that \(K_{n,n}(\lambda)\) has a \((C_n, P_n)\) - multidecomposition. Then there exist integers \(r, s \geq 1\), such that \(K_{n,n}(\lambda) = rC_n \oplus sP_n\). Hence \(|E(K_{n,n}(\lambda))| = rn + s(n - 1)\) for \(r, s \geq 1\).

**Sufficiency:**

**Case 1:** \(n \equiv 0 \pmod{4}\).

Consider the pairs of distance 1 - factors \((\alpha_2, \alpha_n), (\alpha_1, \alpha_3), (\alpha_4, \alpha_6), \ldots, (\alpha_{n-3}, \alpha_{n-1})\) of \(K_{n,n}\). It is easy to see that each pair induces a 2 - factor consisting 2 cycles of length ‘\(n\)’. Let \(P_n = u_1v_{n-2}u_2v_{n-3}u_3v_{n-4} \ldots u_{\frac{n}{2}}v_{n-\frac{n}{2}-1}\) be a path in \(K_{n,n}(\lambda)\). One
can observe that the edges in $P_n$ are all from distinct distances and also from distinct $(n - 1)$ cycles of the 2-factors obtained above except one cycle from the 2-factor induced by the pair $(\alpha_{n-3}, \alpha_{n-1})$. Hence $(C_n, P_n) \mid K_{n,n}(\lambda)$.

**Case 2:** $n \equiv 2 \pmod{4}$.

Consider the pairs of distance 1-factors $(\alpha_2, \alpha_n), (\alpha_1, \alpha_3), (\alpha_4, \alpha_6), \ldots, (\alpha_{n-5}, \alpha_{n-3})$ of $K_{n,n}$. It is clear that each pair induces a 2-factor consisting 2 cycles of length ‘$n$’ and hence $n - 2$ cycles in total. Further the pair $(\alpha_{n-2}, \alpha_{n-1})$, not in the above list, induces a ‘$C_{2n}$’. Let $P_n = u_1v_{n-2}u_2v_{n-3}u_3v_{n-4} \ldots u_{\frac{n}{2}}v_{n-\frac{n}{2}-1}$ be a path in $K_{n,n}(\lambda)$. Observe that the edges in $P_n$ are all from distinct distances and also from distinct $(n - 3)$ cycles of the 2-factors obtained above and ‘2’ edges from $C_{2n}$. Now $C_{2n} - \{e_1, e_2\} = 2P_n$. Note that one cycle from the 2-factor induced by $(\alpha_1, \alpha_3)$ is not disturbed in the formation of $P_n$ above. Hence $(C_n, P_n) \mid K_{n,n}(\lambda)$.

From Lemma 2.8 and Theorem 2.15, we have the following:

**Corollary 2.16.** If $K_{n,n}(\lambda)$ has a $(C_n, P_n)$-multidecomposition, then $K_{ns,ns}(\lambda)$ has a $(C_n, P_n)$-multidecomposition.

**Remark 2.17.** The results of Section 2.1 and 2.2 are respectively published in the Proceedings of the National Conference on Algebra and Graph Theory and in the Bulletin of the Institute of Combinatorics and its Applications.