Chapter 1

Introduction and Preliminaries

1.1 Introduction

Hausdorff [15] was the first mathematician who studied continuity properties in topological spaces using the notion of openset as primitive. Till then, topological space was known to possess a lattice of open subsets, though the lattice-theoretic principles were not used there. The application of lattice-theoretic ideas to study about topological spaces began with the work of the American Mathematician Marshall Stone on the topological representation of Boolean algebras [45] [46] and distributive lattices [47]. Two revolutionary ideas originated in his paper. The first one proved the importance of ideals in lattice theory through the result that Boolean algebras is a certain type of ring called Boolean ring. The next one was
the link between topology and lattice theory through the following famous theorem.

**Stone’s Representation Theorem**

*Every Boolean algebra is isomorphic to the Boolean algebra of open-closed sets of a totally disconnected compact Hausdorff space.*

Thus Stone’s representation theorem revealed that topological spaces can be constructed from purely algebraic things such as a Boolean algebra. His work gave the motivation for employing lattice theory to solve geometrical problems.

The first person who employed this idea was Henry Wallman [49], an American mathematician, where he used lattice-theoretic ideas to construct “Wallman compactification” of a $T_1$ topological space. A few years later an American logician McKinsey and a Polish mathematician Tarski [28] [29] made a study of the “algebra of topology”. The first text book which presented topology from the lattice-theoretic viewpoint was written by the German Mathematician Nöbeling [30]. Charles Ehresmann [13] and his student Jean Bénabou [4] made remarkable changes in the study of topological spaces from lattice-theoretic viewpoint. He remarked that a lattice with right distributive property (finite meets distribute over arbitrary joins) should be studied as a “generalized topological space” irrespective of it being the openset lattice of some topological space [14]. These “generalized topological space” are called “local lattices” by them.

The works of Dona and Seymour Papert [31] [32] at about the same time seconded the same aspect of study of topological spaces.

The term *frame* was the contribution of C.H.Dowker. Frame theory is lattice theory applied to topology. This approach takes the lattice of opensets as the basic notion. In other words, it is a pointfree topology where one investigates typical properties of lattices of opensets that can be expressed
without reference to points. C.H. Dowker and Dona Papert Strauss extended many results in topology to these “generalized spaces” [8] to [12]. One may think of frames as “generalized spaces”. According to Isbell, the word “generalized” is imprecise because arbitrary spaces are not determined by their lattices of open sets. In 1972, through the paper [18], J.R.Isbell pointed out the need for a separate terminology for the dual category of frames. The objects of this dual category are named as “locales” by him and they are actually the “generalized spaces”. The letters $X, Y, Z, \ldots$ are usually used to represent a locale. The frame corresponding to the locale $X$ is denoted by $\Omega X$.

The notion of sublocales (quotient frames) have been studied by Dowker and Papert and Isbell. The term sublocale was introduced by Isbell. Sublocales of a given locale $X$ correspond to quotient frames of $\Omega X$. Compactness and connectedness are notions traditionally defined in terms of properties of the lattice of open sets of a space. Hence the task of defining them in frames is easy. But those definitions depending on points of a space cannot be carried out to frames as they are free of points. Hence the classical $T_1$ axiom for spaces cannot be adopted since it mentions points. But there are various alternative definitions in use. The “unorderedness axiom” is an example.

For localic version of the Hausdorff axiom there are many candidates. Of these, the most accepted is the one due to Isbell. He defined a locale $L$ as Hausdorff if $L$ can be regarded as a closed sublocale of the localic product $L \oplus L$. The only drawback of this axiom is that it is not equivalent to the classical Hausdorff axiom for spaces because a space $X$ may be closed in the localic product $X \oplus X$ without being closed in the topological product $X \times X$. For this reason Isbell called locales satisfying
this axiom strongly Hausdorff locales. C.H.Dowker and D.Strauss[10], H. Simmons[44], P.T.Johnstone and S.H.Sun[24] and J.Paseka[35] are other mathematicians who defined alternatives for Hausdorff axiom for locales. They have the advantage that they coincide with the classical Hausdorff axiom on space, but are not satisfactory in other respects.

The regularity axiom for spaces, though involve points, is frame theoretic. This is because, in topological spaces the regularity axiom says that each open set is a union of open sets whose closures it contains. The same is defined for locales and it implies all other separation axioms considered. The stronger separation axioms complete regularity and normality are straightforward for locales.

Frame theory has the advantage that many results in topology requiring Axiom of Choice or some of its variants can be proved without its use. Examples are Tychonoff theorem[21], the construction of Stone-Cech compactification[1] or the construction of Samuel compactification[3]. Sometimes the frame situation differs from the classical one. For example, co-products of paracompact locales are paracompact[18] while products of paracompact spaces are not necessarily paracompact. Another example is that coproducts of regular frames preserve the Lindelöf property[12] while product of regular spaces do not.

We wish to give a brief description of five important problems settled in topology by eminent mathematicians for which the frame counterpart we discuss in this thesis.

The concept of simple extension of a topology was studied by Norman
Levine [26]. If \((X, \tau)\) is a topological space, then

\[
\tau(A) = \{ O \cup (O' \cap A) : O, O' \in \tau \}
\]

where \(A \notin \tau\) is called a simple extension of \(\tau\). He studied the conditions under which the simple extension of a topological space with a specified topological property also holds that property. In his paper, it is proved that the simple extension \((X, \tau(A))\) of a compact topological space \((X, \tau)\) is compact if \(A^c\) is compact in \((X, \tau)\). Also if \((X, \tau)\) is regular(completely regular or normal), then \((X, \tau(A))\) is regular(completely regular or normal) provided \(A^c \in \tau\).

A topological space \((X, \tau)\) is said to be maximal compact if it is compact and there is no strictly stronger topology on \(X\) which is compact. In 1948, A.Ramanathan[42] proved that a topological space is maximal compact if and only if its compact subsets are precisely the closed sets. Also, E. Hewitt[16] proved that a compact Hausdorff space is maximal compact as well as minimal Hausdorff. Topological spaces in which closed subspaces coincide with compact subspaces was studied by N.Levine [27]. He called such spaces C-C spaces. In this paper, it is proved that if the product topology is C-C, then each component is C-C. The converse of this need not be true. Here he proved that \(X \times X\) is C-C if and only if \(X\) is C-C and Hausdorff. It is also proved that a C-C space is necessarily compact and \(T_1\).

The concept of minimal topologies was introduced by A.S.Parhomeko [33]. A topological space \((X, \tau)\) is said to be minimal Hausdorff if it is Hausdorff and there is no strictly weaker topology on \(X\) that is Hausdorff. As remarked in the above paragraph, compact Hausdorff spaces are always
minimal Hausdorff. The following characterization for minimal Hausdorff topological spaces was given in [6] in terms of convergence of filters.

A necessary and sufficient condition that a Hausdorff space \((X, \tau)\) be minimal Hausdorff is that \(\tau\) satisfies the following property:

1. Every open filter-base has an adherent point;

2. If an open filter-base has a unique adherent point, then it converges to this point.

It is also shown that a Hausdorff space which satisfies condition (2) also satisfies condition (1) and such a space is minimal Hausdorff. Also a compact Hausdorff space is minimal Hausdorff[5] and the converse need not be true.

The concept of reversibility in spaces was studied by M.Rajagopalan and A.Wilanski [38]. A topological space \((X, \tau)\) is called reversible if it has no strictly stronger topology \(\tau^*\) such that \((X, \tau)\) and \((X, \tau^*)\) are homeomorphic. Equivalently, it has no strictly weaker topology \(\tau^*\) such that \((X, \tau)\) and \((X, \tau^*)\) are homeomorphic. Then it is proved that a space is reversible if and only if each continuous bijection of the space onto itself is a homeomorphism. It is also proved that the finite product of reversible spaces is reversible if and only if each component is reversible. The concept of reversibility has also been extended to fuzzy topological space[19] by T.P.Johnson and to partially ordered sets[25] by Michal Kukiela.

De Groot[7] proved that any group is isomorphic to the group of homeomorphisms of topological space. A related problem is to determine the
subgroups of the group of permutations of a fixed set $X$ which can be group of homeomorphisms of $(X, \tau)$ for some topology $\tau$ on $X$. This problem was solved by P.T.Ramachandran\[39\] in topology and by T.P.Johnson\[20\] in fuzzy topology. It is proved in topology that the subgroup of the group of permutations on $X$ containing two elements can represent the group of homeomorphisms of a topological space for some topology. But for a finite set $X$ with $|X| \geq 3$ has no topology for which the group of homeomorphisms is the alternating group of permutations on $X$. Also it is proved that for a set $X$ with $|X| \geq 3$ there is no nontrivial proper normal subgroup of the group of permutations on $X$ which is the group of homeomorphisms for some topology.

One can do research in pointfree topology in two ways. The first is the contravariant way where research is done in the category $\text{Frm}$ but the ultimate objective is to obtain results in $\text{Loc}$. The other way is the covariant way to carry out research in the category $\text{Loc}$ itself directly. According to Johnstone [23], “frame theory is lattice theory applied to topology whereas locale theory is topology itself”. The most part of this thesis is written according to the first view. In this thesis, we make an attempt to study about
1. the frame counterparts of maximal compactness, minimal Hausdorffness and reversibility,
2. the automorphism groups of a finite frame and its relation with the subgroups of the permutation group on the generator set of the frame.

Chapter 1 contains a quick review of the preliminary materials required to read and understand this thesis.

The concept of singly generated extension of a frame was introduced by
B.Banaschewski [2]. In chapter 2, we study some problems concerned with the singly generated extension of a frame. As the first step, we conducted an analogous study on singly generated extension of a frame, following N.Levine. We obtained the conditions under which the singly generated extension of a frame possessing a specified frame isomorphic property also holds that property. The frame isomorphic properties studied are compactness, regularity, complete regularity and normality.

In chapter 3, we introduce the concept of CCE frames - frames in which closed sublocales are exactly the compact sublocales- analogous to that in topology. As an application of the theorem that gives the condition for preserving compactness under singly generated extension of a frame, we have characterized CCE frames as maximal compact frames. We have also discussed some properties and characterizations of such frames in this chapter.

In chapter 4, we introduce the concept of minimal Hausdorff frames and obtained a partial characterization for them in terms of convergence of filters in frames. Some other properties of minimal Hausdorff frames are also discussed.

In chapter 5, we proceed to introduce reversibility in frames. The association between reversible spatial frames and the corresponding topological spaces is also studied here. A characterization for reversible frames is proved. Also, it is proved that a frame which is maximal or minimal with respect to some frame isomorphic property is reversible and conversely. Reversibility in frames can be used as a tool for solving some problems related to reversible topological spaces. As an application, we solved a problem put forward by M.Rajagopalan and A.Wilansky in [38]
1.2. Categorical concepts

Certain concepts in one branch of Mathematics have remarkable resemblance to those from other branches in Mathematics. For example, the concept of a homeomorphism in Topology has resemblance with the concept of an isomorphism in Groups or a bijection in Set Theory. The theory of Categories seeks to isolate what is common to these various branches of Mathematics. The theory is useful because it puts construction in one branch into a broader perspective and inspires similar constructions in other branches.

Definition 1.2.1. A *category* consists of the following data:

- Objects: $A, B, C, \ldots,$
- Arrows: $f, g, h, \ldots,$
- For each arrow $f$ there are given objects $\text{dom}(f)$ called the *domain*
and \( \text{codom}(f) \) called the codomain. We write \( f : A \to B \) to indicate that \( A = \text{dom}(f) \) and \( B = \text{codom}(f) \),

- Given arrows \( f : A \to B \) and \( g : B \to C \) with \( \text{codom}(f) = \text{dom}(g) \) there is given an arrow \( g \circ f : A \to C \) called the composite of \( f \) and \( g \),

- For each object \( A \) there is given an arrow \( 1_A : A \to A \) called the identity arrow of \( A \).

These data are required to satisfy the following laws:

- Associativity: \( h \circ (g \circ f) = (h \circ g) \circ f \) for all \( f : A \to B, g : B \to C, h : C \to D \).

- Unit: \( f \circ 1_A = f = 1_B \circ f \) for all \( f : A \to B \).

A category is anything that satisfies this definition.

**Definition 1.2.2.** The opposite or dual category \( C^{\text{op}} \) of a category \( C \) has the same objects as \( C \) and an arrow \( f : C \to D \) is an arrow \( f : D \to C \). That is \( C^{\text{op}} \) is just \( C \) with all of the arrows formally turned around.

**Definition 1.2.3.** In any category \( C \), an arrow \( f : A \to B \) is called a monomorphism if given any \( g, h : C \to A \), we have \( f \circ g = f \circ h \) implies \( g = h \).

**Definition 1.2.4.** In any category \( C \), an arrow \( f : A \to B \) is called an epimorphism if given any \( i, j : B \to D \), we have \( i \circ f = j \circ f \) implies \( i = j \).
Functors are means of passing from one category to another and they resemble functions in many respects. Often in Mathematics, it happens that to each object of a category, we can associate an object of another category which reflects the properties of the original object. The advantage of such an association is that information about one category can lead to information about another category. The basic features of a category are compositions and identity arrows. Thus it is natural to require that they must be preserved under transaction from one category to another.

**Definition 1.2.5.** A *covariant functor* between two categories \( C \) and \( D \) is a mapping \( F : C \to D \) of objects to objects and arrows to arrows in such a way that

1. \( F(f : A \to B) = F(f) : F(A) \to F(B) \).
2. \( F(g \circ f) = F(g) \circ F(f) \).
3. \( F(1_A) = 1_{F(A)} \).

**Definition 1.2.6.** A *contravariant functor* between two categories \( C \) and \( D \) is a mapping \( F : C \to D \) of objects to objects and arrows to arrows in such a way that

1. \( F(f : A \to B) = F(f) : F(B) \to F(A) \).
2. \( F(g \circ f) = F(f) \circ F(g) \).
3. \( F(1_A) = 1_{F(A)} \).

**Definition 1.2.7.** Let \( F, G : A \to B \) be any two covariant functors. A natural transformation \( \tau \) from \( F \) to \( G \) denoted by \( \tau : F \to G \) is a map
that assigns to each object $A$ of $\mathcal{A}$ an arrow $\tau_A : FA \to GA$ in such a way that for each arrow $f : A \to A'$, we have $\tau_{A'} \circ Ff = Gf \circ \tau_A$.

**Definition 1.2.8.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be any two functors. Let $\eta : id_B \to GF$ and $\epsilon : FG \to id_A$ be two natural transformations satisfying

1. $G \eta^G \xrightarrow{G\epsilon} GFG \xrightarrow{Gid} G = G^idG$.

2. $F \epsilon^F \xrightarrow{F\eta} FGF \xrightarrow{Fid} F = F^idF$.

In this situation we say that $F$ is a left adjoint for $G$ and $G$ is a right adjoint for $F$. The natural transformation $\eta : id_B \to GF$ is called the *unit* and the natural transformation $\epsilon : FG \to id_A$ is called the *co-unit*.

**Definition 1.2.9.** A map $f : L \to M$ between partially ordered sets $L$ and $M$ is said to be monotone if for all $a, b \in L, a \leq b \Rightarrow f(a) \leq f(b)$. Let $f : L \to M$ and $g : M \to L$ be monotone maps between partially ordered sets $L$ and $M$. We say that the pair $(f, g)$ is a *Galois connection* if for all $a \in L, b \in M, f(a) \leq b \iff a \leq g(b)$. The above condition is also equivalently given as $fg(b) \leq b$ and $gf(a) \geq a$ for all $a \in L, b \in M$. If such a situation exists $f$ will be the left adjoint of $g$ and $g$ will be the right adjoint of $f$.

**Remark 1.2.1.** The following points are useful.

- $g$ is uniquely determined by $f$ and the other way is also true,

- $f$ preserves existing joins and $g$ preserves existing meets,
• each join preserving map \( f : L \to M \) is a left adjoint and each meet preserving map \( g : M \to L \) is a right adjoint, with \( L \) and \( M \) being complete lattices.

1.3 Order theoretic concepts

**Definition 1.3.1.** A set \( L \) with a binary relation “\( \leq \)” satisfying the following conditions is called a partially ordered set.

1. \( a \leq a \),
2. \( a \leq b, b \leq a \) implies \( a = b \),
3. \( a \leq b, b \leq c \) implies \( a \leq c \).

where \( a, b \) and \( c \in L \)

**Definition 1.3.2.** An element \( x \in A \subseteq L \) is called *minimal* if \( a \in A, a \leq x \) implies \( a = x \). If \( L \) has a unique minimal element, then it is called the *least element* (bottom) of \( L \) denoted by \( 0_L \).

**Definition 1.3.3.** An element \( x \in A \subseteq L \) is called *maximal* if \( a \in A, x \leq a \) implies \( x = a \). If \( L \) has a unique maximal element, then it is called the *greatest element* (top) of \( L \) denoted by \( 1_L \).

**Definition 1.3.4.** An element \( x \in L \) is called an *upperbound* of \( A \subseteq L \), if for all \( a \in A \), we have \( a \leq x \). The least element of the set of all upperbounds of \( A \) in \( L \), if it exists, is called the *least upper bound* (supremum) of \( A \). It is denoted by \( \bigvee A \).
Definition 1.3.5. An element $x \in L$ is called a lowerbound of $A \subseteq L$, if for all $a \in A$, we have $x \leq a$. The greatest element of the set of all lowerbounds of $A$ in $L$, if it exists, is called the greatest lower bound (infimum) of $A$. It is denoted by $\bigwedge A$.

Definition 1.3.6. A partially ordered set $L$ in which for every pair of elements $a$ and $b$ there exists the supremum $a \vee b$ and the infimum $a \wedge b$ is called a lattice. A partially ordered set $L$ for which every set $A \subseteq L$ has the supremum $\bigvee A$ and the infimum $\bigwedge A$ exist in $L$ is called a complete lattice.

Definition 1.3.7. A lattice $L$ is distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ which is equivalent to $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition 1.3.8. A map $f : L \to M$ where $L, M$ are partially ordered sets is called monotone (order preserving) if $a \leq_L b \Rightarrow f(a) \leq_M f(b)$ for all $a, b \in L$. If $f$ is bijective and its inverse $f^{-1}$ is also monotone, then it is called an order isomorphism.

Definition 1.3.9. Let $L$ be any partially ordered set and $A \subseteq L$ be any subset. Then $A$ is called the downset of $L$ generated by $A$ if $\downarrow A = A$ where $\downarrow A = \{x \in L : \text{there exists } a \in A, x \leq a\}$

Definition 1.3.10. Let $L$ be any partially ordered set and $A \subseteq L$ be any subset. Then $A$ is called the upset of $L$ generated by $A$ if $\uparrow A = A$ where $\uparrow A = \{x \in L : \text{there exists } a \in A, x \geq a\}$

Definition 1.3.11. $\uparrow a = \uparrow \{a\}$ and $\downarrow a = \downarrow \{a\}$ are called the principal filter and the principal ideal generated by $a$ respectively.

Definition 1.3.12. Let $L$ be a lattice. Then an element $a \in L$ is an
atom, if $0_L \leq x \leq a$ implies $x = 0_L$ or $x = a$.

**Definition 1.3.13.** Let $L$ be a lattice. Then an element $a \in A$ is a dual atom, if $a \leq x \leq 1_L$ implies $x = a$ or $x = 1_L$.

**Definition 1.3.14.** Let $L$ be a lattice with least element $0_L$ and let $a \in L$. The pseudocomplement of $a$, denoted by $a^*$, is the one satisfying $x \land a = 0_L$ if and only if $x \leq a^*$.

The following remark provides some useful rules on pseudocomplements.

**Remark 1.3.1.** Let $L$ be any lattice with top $1_L$ and bottom $0_L$, then

1. $0_L^* = 1_L$, $1_L^* = 0_L$,

2. $a \leq b$ implies $b^* \leq a^*$,

3. $a \leq a^{**}$,

4. $a^* = a^{***}$.

**Definition 1.3.15.** Let $L$ be a distributive lattice with greatest element $1_L$ and least element $0_L$. The complement $a^c$ of an element $a \in L$ is the one satisfying $a \land a^c = 0_L$ and $a \lor a^c = 1_L$.

**Definition 1.3.16.** A Boolean Algebra is a distributive lattice with $0_L$ and $1_L$ in which every element has a complement.
1.4 Frames and Locales

**Definition 1.4.1.** A frame is a complete lattice $L$ in which the infinite distributive law $a \land \bigvee S = \bigvee \{a \land s: s \in S\}$ holds for all $a \in L, S \subseteq L$.

**Definition 1.4.2.** A map $f: L \to M$ between frames $L, M$ satisfying for every $a_i, a, b \in L$

\[
\begin{align*}
f(\bigvee_i a_i) &= \bigvee_i f(a_i) \quad (1.1) \\
f(a \land b) &= f(a) \land f(b) \quad (1.2)
\end{align*}
\]

is called a frame homomorphism. A bijective frame homomorphism is called a frame isomorphism.

**Definition 1.4.3.** An element $a \in L$ is said to be dense if $a^* = 0_L$.

**Remark 1.4.1.** The category whose objects are frames and morphisms are frame homomorphisms is denoted by Frm. The dual category Frm$^{op}$ is referred to as the category of locales denoted by Loc. The objects of this category are known as locales and as objects they are same as that of frames. These two categories differ only in morphisms. The morphisms in Loc, called localic maps (continuous maps) are frame homomorphisms when considered in the opposite direction.

**Remark 1.4.2.** The category of topological spaces and continuous maps is denoted by Sp.

**Definition 1.4.4.** The functor $\Omega: Sp \to Frm$ maps objects and arrows as below
1.5. Subframes and Sublocales

(1) a topological space \((X, \Omega X)\) is mapped to its frame of open sets \(\Omega X\),
(2) for an arrow \(f : X \to Y\), the corresponding arrow in \(\text{Frm}\) is given by
\(\Omega(f) : \Omega Y \to \Omega X\) where \(\Omega(f)(U) = f^{-1}(U)\) where \(U \in \Omega Y\).

**Theorem 1.4.1.** The functor \(\Omega : \text{Sp} \to \text{Frm}\) is a contravariant functor.

A point of a frame \(L\) is a frame homomorphism \(h : L \to 2\), where 2 is
the two element boolean algebra. We denote by \(\Sigma L\) the set of all points
of \(L\). For \(a \in L\), set \(\Sigma_a = \{h : L \to 2 : h(a) = 1\}\) and \(\tau = \{\Sigma_a : a \in L\}\).

**Theorem 1.4.2.** \((\Sigma L, \tau)\) is a topological space.

**Theorem 1.4.3.** Let \(L\) be any frame. Let \(\Sigma L\) be the set of all points
of \(L\). For each object \(L\) in \(\text{Frm}\), \(\Sigma\) maps that object to the topological
space \((\Sigma L, \tau)\). For a frame homomorphism \(h : L \to M\), define the mapping \(\Sigma h : \Sigma M \to \Sigma L\) by \((\Sigma h)(\alpha) = \alpha \circ h\). Then \(\Sigma : \text{Frm} \to \text{Sp}\) is a
contravariant functor.

**Theorem 1.4.4.** \(\Sigma : \text{Frm} \to \text{Sp}\) is right adjoint to \(\Omega : \text{Sp} \to \text{Frm}\)

**Definition 1.4.5.** A frame \(L\) is said to be spatial if it is isomorphic
to \(\Omega X\) for some set \(X\).

1.5 Subframes and Sublocales

**Definition 1.5.1.** A subset of a frame which is closed under the same
finite meets and arbitrary joins in that frame is called a subframe. Thus
it is clear that every subframe includes the top and bottom of the frame of which it is a subframe.

**Definition 1.5.2.** A *nucleus* on a locale $A$ is a map $j : A \to A$ satisfying the following conditions

1. $j(a \land b) = j(a) \land j(b)$,
2. $a \leq j(a)$,
3. $j(j(a)) \leq j(a)$.

for all $a, b \in A$.

**Remark 1.5.1.** Define $A_j = \{a \in A : j(a) = a\}$. Then $A_j$ is a frame and $j : A \to A_j$ is a frame homomorphism.

**Definition 1.5.3.** A *sublocale* of the frame $L$ is an onto frame homomorphism $h : L \to M$ where $M$ is a frame.

A sublocale can also be defined in another way as follows.

**Definition 1.5.4.** A *sublocale* of a locale $A$ is a subset of the form $A_j$, for some nucleus $j$. The infima in the sublocale coincide with those of $A$ and the suprema given by $\bigvee' x_i = j(\bigvee x_i)$.

**Definition 1.5.5.** The sublocale given by the nucleus $j_1 : A \to \uparrow a$ defined by $x \to a \lor x$ for any $a \in A$ is called a *closed sublocale* and is denoted by $c(a)$.

**Definition 1.5.6.** The sublocale given by the nucleus $j_2 : A \to \downarrow a$
defined by $x \rightarrow a \land x$ for any $a \in A$ is called an open sublocale and is denoted by $o(a)$.

An open sublocale $o(b)$ can also be defined by $o(b) = \{x \in A : b \rightarrow x\} = \{x \in A : b \rightarrow x = x\}$.

**Definition 1.5.7.** A sublocale $A_j$ of $A$ is said to be dense if it contains $0_A$.

The following remark is given as an exercise in page 51[22].

**Remark 1.5.2.** For any sublocale $A_j$ of $A$, there is a unique $A_k$ called the closure of $A_j$, such that $A_j$ is a dense sublocale of $A_k$ and $A_k$ is a closed sublocale of $A$.

By the above remark, the closure of a sublocale is defined as follows.

**Definition 1.5.8.** The closure of a sublocale $K$ of $L$ is the unique sublocale $\overline{K}$ of $L$ satisfying,

1. $\overline{K}$ is a closed sublocale of $L$,

2. $K$ is dense in $\overline{K}$

**Definition 1.5.9.** A lattice $A$ is a Heyting Algebra if and only if for every $a, b \in A$ there is an element $a \rightarrow b$ satisfying $c \leq a \rightarrow b$ if and only if $c \land a \leq b$.

**Definition 1.5.10.** A cover in a frame $L$ is a subset $S$ of $L$ with $\bigvee S = 1_L$. 
1.6 Coproducts in frames

The construction of coproducts in frames was first presented in [12].

**Definition 1.6.1.** Let \( R \subseteq A \times A \) be an arbitrary binary relation on a frame \( A \). An element \( s \in A \) is \( R \)-saturated if

\[
aRb \Rightarrow (a \land c \leq s \iff b \land c \leq s) \text{ for all } a, b, c \in A
\]

**Remark 1.6.1.** Let \( A/R \) denotes the set of all saturated elements of \( A \). Define \( \nu : A \to A/R \) by \( \nu(a) = \nu_R(a) = \bigwedge \{ s \in A : a \leq s \} \) where \( s \) is saturated. Then \( \nu \) is a surjective frame homomorphism.

For a semilattice \( A \), define \( \mathcal{D}(A) = \{ U \subseteq A : \phi \neq U = \downarrow U \} \). Then \( (\mathcal{D}(A), \subseteq) \) is a frame. Define \( \lambda_A : A \to \mathcal{D}(A) \) by \( \lambda_A(a) = \downarrow a \) which is a semilattice homomorphism between them. Let \( A_i, i \in I \) be frames. Set \( \prod'_{i \in I} A_i = \{ (a_i)_{i \in I} \in \prod_{i \in I} A_i : a_i = 1 \text{ for all but finitely many } i \} \cup \{ (0)_{i \in I} \} \). Define \( \gamma_j : A_j \to \prod'_{i \in I} A_i \) by setting

\[
(\gamma_j(a))_i = \begin{cases} 
a & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}
\]

Consider the frame \( \mathcal{D}(A) \) where \( A = \prod'_{i \in I} A_i \).

\[
R = \{ (\lambda_A \gamma_j(\vee_{m \in M} a_m), \vee_{m \in M} \lambda_A \gamma_j(a_m)) : j \in I, a_m \in A_j \} \quad \text{where } M \text{ is any set is a relation.}
\]

**Definition 1.6.2.** The frame \( \bigoplus_{i \in I} A_i = \mathcal{D}(\prod'_{i \in I} A_i)/R \), containing all \( R \)-saturated elements of the frame \( \mathcal{D}(\prod'_{i \in I} A_i) \) is called the frame coproduct.
Definition 1.6.3. The mapping \( \nu : D(\prod'_{i \in I} A_i) \to \bigoplus_{i \in I} A_i \) be as defined above. The maps \( p_j = \nu \circ \lambda \circ \gamma_j : A_j \to \bigoplus_{i \in I} A_i \) are frame homomorphisms, called coproduct injections. Also \( \bigwedge_{i \in I} p_i(a_i) = \oplus(a_i)_{i \in I} \). Thus the set of all elements of the form \( \oplus(a_i)_{i \in I} \) is a join basis of \( \bigoplus_{i \in I} A_i \). The right adjoint of \( p_j \) denoted by \( p_j^* \) is called the projection of the locale \( \bigoplus_{i \in I} A_i \) to the locale \( A_j \).

Remark 1.6.2. The set

\[ O = \{(a_i)_{i \in I} \in \prod_{i \in I} A_i : \text{there exists } i, a_i = 0\} \]

is saturated. It is the least element of the frame coproduct \( \bigoplus_{i \in I} A_i \).

Theorem 1.6.1. The set \( \oplus_{i \in I} a_i = \downarrow (a_i)_{i \in I} \cup O \) is saturated for any \( (a_i)_{i \in I} \in \prod_{i \in I} A_i \).

Corollary 1.6.2. If \( \oplus_{i \in I} a_i \leq \oplus_{i \in I} b_i \) and \( a_i \neq 0 \) for all \( i \), then \( a_i \leq b_i \) for all \( i \).

Remark 1.6.3. Also note that \( a \oplus b = \downarrow (a, b) \cup O \) where \( O = \{(x, y) : x = 0 \text{ or } y = 0\} = 0_{L \oplus L} \), is the bottom of \( L \oplus L \).

Theorem 1.6.3. Let \( L_i, i = 1, 2 \) be frames and \( a_i \in L_i \). Then \( \downarrow a_1 \oplus \downarrow a_2 = \downarrow (a_1 \oplus a_2) \).

The following theorem is proved in [36].

Theorem 1.6.4. For each \( U \in \oplus_{i \in I} L_i \), the set \( U = \bigvee \{\oplus_{i \in I} a_i : \oplus_{i \in I} a_i \leq U\} \). That is \( \oplus_{i \in I} a_i \) forms the join basis for \( \bigoplus_{i \in I} L_i \).
We state the following result proved in [22] for proving some results in this chapter.

**Theorem 1.6.5.** Let $X_i, i \in I$ be family of spaces. Then $\oplus_i(\Omega X_i)$ is isomorphic to $\Omega(\oplus X_i)$ if and only if it is a spatial frame.

### 1.7 Special Frames

At first we define the concept of ideals and filters in a frame.

**Definition 1.7.1.** [22] An *ideal* in a frame $L$ is a nonempty subset $I$ with the property that $0 \in I$, $a \leq b \in I$ implies $a \in I$, and $a \lor b \in I$ whenever $a$ and $b$ are in $I$.

**Definition 1.7.2.** [22] A *filter* in a frame $L$ is a nonempty subset $F$ with the property that $0 \notin F$, $a \geq b \in F$ implies $a \in F$, and $a \land b \in F$ whenever $a$ and $b$ are in $F$. An *ultrafilter* is a maximal filter.

We state the following results proved in [22] for proving some results in this chapter.

**Theorem 1.7.1.** Let $I$ be an ideal of a lattice $A$, and $F$ a filter disjoint from $I$. Then there exists an ideal $M$ of $A$ which is maximal amongst those containing $I$ and disjoint from $F$.

**Theorem 1.7.2.** Let $F$ be a filter in a distributive lattice $A$, and $I$ an ideal which is maximal amongst those disjoint from $F$. Then $I$ is prime.
Now we give the definitions for some special frames which we encounter in the coming chapters.

**Definition 1.7.3.** [22] A frame $L$ is said to be compact if each subset $A$ of $L$ with $\bigvee A = 1_L$ has a finite subset $B \subseteq A$ with $\bigvee B = 1_L$.

**Definition 1.7.4.** [34] A frame $L$ is called an almost compact frame if whenever $\bigvee \{x_i : i \in I\} = 1_L$ then there exists a finite subset $K \subseteq I$ of the index set $I$ such that $(\bigvee \{x_i : i \in K\})^{**} = 1_L$ where "*" denotes the pseudo-complementation operator in $L$.

**Theorem 1.7.3.** [34]

1. A compact frame is almost compact.

2. A frame $L$ is not almost compact if and only if an ideal $Q$ in $L$ exists such that $Q \subseteq S_L = \{l \in L : l^* = 0_L\}$ and $\bigvee Q = 1_L$.

**Theorem 1.7.4.** [22] The product of compact locales is compact.

**Definition 1.7.5.** [36] Let $a, b \in L$. A frame $L$ is said to be sub-fit(conjunctive) if $a \not\leq b \Rightarrow$ there exists $c$ such that $a \lor c = 1_L \neq b \lor c$.

**Definition 1.7.6.** [36] Let $a, b \in L$. The relation $a \prec b$ holds if $a^* \lor b = 1_L$ where $*$ denotes the pseudocomplementation operator in $L$.

**Theorem 1.7.5.** [36] Let $L$ be any frame. Then the following rules hold in $L$.

1. $0_L \prec a \prec 1_L$ for any $a \in L$.

2. $a \prec b$ implies $a \leq b$. 
3. $x \leq a \prec b \leq y$ implies $x \prec y$.

4. If $a \prec b$, then $b^* \prec a^*$.

5. If $a \prec b$, then $a^{**} \prec b$.

6. If $a_i \prec b_i$ for $i = 1, 2$, then $a_1 \lor a_2 \prec b_1 \lor b_2$ and $a_1 \land a_2 \prec b_1 \land b_2$.

**Definition 1.7.7.** [36] The relation $a \prec\prec b$ holds if there are $x_r \in L$ for $r$ dyadic rational in the interval $(0,1)$ such that $x_0 = a, x_1 = b$ and $x_r \prec x_s$ for $r < s$.

**Theorem 1.7.6.** [36] Let $L$ be any frame. Then the following rules hold in $L$.

1. $0_L \prec\prec a \prec\prec 1_L$ for any $a \in L$.

2. $a \prec\prec b$ implies $a \leq b$.

3. $x \leq a \prec\prec b \leq y$ implies $x \prec\prec y$.

4. If $a \prec\prec b$, then $b^* \prec\prec a^*$.

5. If $a \prec\prec b$, then $a^{**} \prec\prec b$.

6. If $a_i \prec\prec b_i$ for $i = 1, 2$, then $a_1 \lor a_2 \prec\prec b_1 \lor b_2$ and $a_1 \land a_2 \prec\prec b_1 \land b_2$.

**Definition 1.7.8.** [36] The frame $L$ is said to be a regular frame if $a = \bigvee \{ x \in L / x \prec a \}$ for all $a \in L$.

**Theorem 1.7.7.** [22] A compact regular locale is spatial.

**Theorem 1.7.8.** [22] The product of regular locales is regular.
Definition 1.7.9. [36] The frame $L$ is said to be completely regular if $a = \bigvee \{ x \in L \mid x \preceq a \}$ for all $a \in L$.

Definition 1.7.10. [36] A frame $L$ is normal if whenever $a \lor b = 1_L$ for $a, b \in L$, there exist $u, v \in L$ with $u \land v = 0_L$ such that $u \lor b = 1_L$ and $a \lor v = 1_L$.

We take the definition for Hausdorff frame as given by Isbell [18] throughout this thesis.

Definition 1.7.11. A frame $A$ is called a Hausdorff frame if for any $U \in A \oplus A$, the codiagonal $\nabla : A \oplus A \to A$ defined by

$$\nabla(U) = \bigvee \{ a \land b : (a, b) \in U \}$$

is a closed sublocale.

Remark 1.7.1. For any frame $L$,

1. Set $d_L = \bigvee \{ x \oplus y : x \land y = 0_L \} \in L \oplus L$.

2. Also note that $\nabla (a \oplus b) = a \land b$ where $a, b \in L$.

The following result is proved in [36].

Theorem 1.7.9. A compact Hausdorff locale is regular.

For proof of the following theorem, see [36].

Theorem 1.7.10. A frame $L$ is Hausdorff if and only if for any $a, b \in L$, $a \oplus b \leq ((a \land b) \oplus (a \land b)) \lor d_L$. 
The following theorem tells which of the above concepts are extensions of the classical ones in topology and for proofs refer [36].

**Theorem 1.7.11.** [22] Let \((X, \Omega X)\) be any topological space. Then \((X, \Omega X)\) is

1. compact if and only if \(\Omega X\) is compact,
2. regular if and only if \(\Omega X\) is regular,
3. completely regular if and only if \(\Omega X\) is completely regular,
4. normal if and only if \(\Omega X\) is normal.

In contrary to the above, Hausdorffness is not an extension of the classical Hausdorff axiom in topology. The following theorem gives only a sufficient condition. We end up this section with the following theorem from [22].

**Theorem 1.7.12.** If \(\Omega X\) is Hausdorff, then the \(T_0\) topological space \((X, \Omega X)\) is Hausdorff. The converse need not be true.