Chapter 3

Maximal Compact Frames

3.1 Introduction

In topological spaces, a closed subspace of a compact space is compact and a compact subspace of a Hausdorff space is closed. Thus in a compact Hausdorff space, closed subspaces coincide with compact subspaces. In 1948 A. Ramanathan [42] proved that a topological space is maximal compact if and only if its compact subsets are precisely the closed sets. A topological space in which the closed subspaces are precisely the compact subspaces are called C-C Spaces. N. Levine made a study on such spaces [27] and proved that product of maximal compact spaces are not necessarily maximal compact. In this chapter, we extend some of his results into

Some results of this chapter are included in the following paper.

P. N., Jayaprasad and Johnson, T.P.: On Frames with Closed Sublocales Equivalent to Compact Sublocales, Submitted.
the frame theoretic set up. The following are some of the results proved in [27].

**Theorem 3.1.1.** If \((X, \tau)\) is a compact Hausdorff space, then \(\tau\) is M.R.C.

**Theorem 3.1.2.** Suppose that \((X, \tau)\) is a topological space. Then \((X, \tau)\) is C-C if and only if \(\tau\) is M.R.C.

**Theorem 3.1.3.** Let \((X, \tau)\) be a topological space. If \((X, \tau)\) is C-C, then it is compact and \(T_1\).

**Theorem 3.1.4.** Let \((X, \tau)\) be a topological space and let \((X \times X, \tau^*)\) be the cartesian product of \((X, \tau)\) with itself. Then \((X \times X, \tau^*)\) is C-C if and only if \((X, \tau)\) is C-C and Hausdorff.

## 3.2 Closed and Compact Equivalent Frames

It is known that[22] every closed sublocale of a compact locale is compact and every compact sublocale of a regular locale is closed. Hence in compact regular locales closed sublocales coincide with compact sublocales. We try to answer when does a closed sublocale equivalent to a compact sublocale. This leads to our definition for what is named as CCE frames. We also formulate some characterizations of CCE frames(locales).

**Definition 3.2.1.** A frame \(A\) is called a Closed and Compact Equivalent Frame( CCE Frame) if the closed sublocales of \(A\) coincide with the compact sublocales of \(A\).
3.2. Closed and Compact Equivalent Frames

Any compact regular frame is a CCE frame as closed sublocales of a compact locale are compact and compact sublocales of a regular locale are closed. In topology, compact Hausdorff spaces are maximal compact [27]. We, therefore, examine whether such frames possess this maximality. We introduce the following definition.

**Definition 3.2.2.** A frame $A$ is said to be maximal relative to compactness (M.R.C.) if,

1. $A$ is compact,
2. if $A$ is a proper subframe of the frame $L$, then $L$ is not compact.

**Lemma 3.2.1.** Let the frame $A \subseteq L$ be M.R.C. Then $\uparrow L a$ is compact for $a \in A$ if and only if $\uparrow L a$ contains no element of $L - A$.

**Proof.** Suppose that $\uparrow L a$ is compact
Let $b \in L - A$ and $b \in \uparrow L a$. Consider the singly generated extension $A[b]$ of the frame $A$ by adding the element $b$. We prove that $A[b]$ is compact. Let $S \subseteq A[b]$ with $\bigvee S = 1$. Let $S = \{a_i \lor (a_i' \land b) : a_i, a_i' \in A, a_i \leq a_i', i \in I\}$. Now $1 = \bigvee S = (\bigvee a_i) \lor [(\bigvee a_i') \land b] = (\bigvee a_i') \land [(\bigvee a_i) \lor b] = (\bigvee a_i') \land [\bigvee (a_i \lor b)]$. Thus $\bigvee_{i \in I} a_i' = 1$ and $\bigvee_{i \in I} (a_i \lor b) = 1$. Since $A$ is compact, there exists a finite subset $J_1 \subseteq I$ with $\bigvee_{j_1 \in J_1} a_{j_1}' = 1$. Also $a_i \lor b \geq b \geq a$ and hence $a_i \lor b \in \uparrow L a$. Since $\uparrow L a$ is compact, there exists a finite subset $J_2 \subseteq I$ with $\bigvee_{j_2 \in J_2} (a_{j_2} \lor b) = 1$. Set $J = J_1 \cup J_2$ and $F = \{a_j \lor (a_j' \land b) : j \in J\}$. Clearly $F \subseteq S$ and $F$ is finite. Then $\bigvee F = (\bigvee a_j) \lor [(\bigvee a_j') \land b] = (\bigvee a_j') \land (\bigvee (a_j \lor b)) = 1 \land 1 = 1$, because $J_1 \subseteq J, J_2 \subseteq J$ and $\bigvee_{j_1 \in J_1} a_{j_1}' = 1, \bigvee_{j_2 \in J_2} (a_{j_2} \lor b) = 1$. Hence $A[b]$ is compact. Since $A \subset A[b]$ and $A$ is M.R.C., this leads to a contradiction.
as there is no strictly larger frame that is compact. Thus our assumption that \( b \in L - A \) and \( b \in \uparrow_L a \) is wrong. Thus \( \uparrow_L a \) contains no element of \( L - A \).

Conversely suppose that \( \uparrow_L a \) contains no element of \( L - A \).

Then \( \uparrow_L a = \uparrow_A a \). But \( \uparrow_A a \) is compact as it is a closed sublocale of \( A \) which is compact being M.R.C. Thus \( \uparrow_L a \) is compact for \( a \in A \).

We state the following definition due to J. Paseka and B. Šmarda[34] for proving the next result.

**Definition 3.2.1.** Define \( F_C = \{ a \in L : \uparrow a \text{ is compact in } L \} \). Then the locale generated by the set \( \{(l, 0_L) : l \in L\} \cup \{(a, 1) : a \in F_C\} \) is defined as \( L_{F_C} \). \( L_{F_C} \) is a compact locale called the *one point compactification* [34] of \( L \).

The next theorem is proved in [36]

**Theorem 3.2.2.** An image of a compact sublocale \( S \subseteq L \) under a localic map \( f : L \to M \) is compact.

Now we state and prove the main theorem in connection with CCE Frames.

**Theorem 3.2.3.** Let \( A \) be any frame. Then it is a CCE frame if and only if it is M.R.C.

*Proof.* Assume that \( A \) is a CCE Frame. Suppose \( A \) is not M.R.C. Then there exists a frame \( B \) such that \( A \subset B \) and \( B \) is compact. Let us assume that these two frames are subframes of a boolean frame \( L(\ Oth-
erwise we can consider the isomorphic copies of them in a Boolean frame, according to Corollary 2.1.2 that any frame is isomorphic to a subframe of a complete Boolean algebra). Let \( b \in B - A \) with \( b^c \) exists in \( L \). Consider the singly generated extension \( A[b] \) of the frame \( A \) by adding the element \( b \). Clearly \( A[b] \subseteq B \). Since \( B \) is compact and every subframe of a compact frame is compact, we have \( A[b] \) is compact. Then by Lemma 2.2.1, \( b^c \) is compact relative to \( A[b] \) and hence relative to \( A \). Hence \( \downarrow b^c \) is compact regarding it as a locale itself.

Case 1: Suppose \( b^c \in A \)

Define \( \mathfrak{o}(b^c) = \{ x \in A : b^c \rightarrow x \} = \{ x \in A : b^c \rightarrow x = x \} \). We know that \( \mathfrak{o}(b^c) \) is a sublocale and we claim that it is compact in \( A \).

For, it is the image of \( \downarrow b^c \) regarded as a locale under the localic map obtained as the adjoint of the frame homomorphism \( j : A \rightarrow \downarrow b^c \) defined by \( x \rightarrow b^c \wedge x \) and since \( \downarrow b^c \) is compact as a locale, \( \mathfrak{o}(b^c) \) is compact in \( A \), by Theorem 3.2.2. Now we prove that \( \mathfrak{o}(b^c) \) is not closed in \( A \).

Suppose that \( \mathfrak{o}(b^c) \) is closed in \( A \). Then there exists \( y \in A \) such that \( \mathfrak{o}(b^c) = \uparrow_A y \). Then \( y \in \uparrow_A y = \mathfrak{o}(b^c) \). Since \( 0 \in \mathfrak{o}(b^c) \), we have \( 0 \in \uparrow_A y \) and hence \( y = 0 \). Then \( \mathfrak{o}(b^c) = A \). Thus \( b^c = 1 \) and hence \( b = 0 \). But \( b \in B - A \) and hence \( b \neq 0 \). Thus we get a contradiction and hence \( \mathfrak{o}(b^c) \) is not closed in \( A \).

Case 2: Suppose \( b^c \notin A \)

Let \( p = \wedge \{ x \in A : x \geq b^c \} \). Then \( p \neq 1 \). For, if \( p = 1 \), then the only element \( x \in A \) with \( x \geq b^c \) is 1. Then the filter \( F = \uparrow_A b^c \) is disjoint from the ideal \( I = \{ x \in A : x \leq b^c \} \) in \( L \) as \( b^c \notin A \). Now, by Theorem 1.7.1, there exists a maximal ideal \( M \subseteq A \) containing \( I \) and disjoint from \( F \). Then, by Theorem 1.7.2, \( M \) is a prime ideal. Now \( b^c \wedge b = 0 \in M \). Since \( b^c \notin A \) and \( M \) is a prime ideal, \( b \in A \), which is not true. Hence \( p \neq 1 \).
Consider \( \downarrow_A p \). We prove that \( \downarrow_A p \) is compact but not closed in \( A \). For, it needs to prove that \( p \) is compact relative to \( A \). Let \( \bigvee S = p \) where \( S \subseteq A \). Then \( \bigvee S \geq b^c \) since \( p \geq b^c \). Since \( b^c \) is compact relative to \( A \), there exists a finite subset \( F \subseteq S \) with \( \bigvee F \geq b^c \). Hence \( \bigvee F \in \{ x \in A : x \geq b^c \} \) and we get \( \bigvee F \geq p \). Also \( F \subseteq S \) and hence \( \bigvee F \leq p \). Combining we get \( \bigvee F = p \) where \( F \subseteq S \) is finite. Hence \( p \) is compact relative to \( A \).

Now \( o(p) \) can be proved to be a compact sublocale but not closed, by repeating the proof in case 1 with \( b^c \) replaced by \( p \).

Thus we have a compact but not closed sublocale of \( A \) and this contradict the fact that \( A \) is a CCE Frame. Hence in this case \( A \) must be M.R.C.

Assume that \( A \) is M.R.C. in a frame \( L \). Since every closed sublocale of a compact frame is compact, it needs to prove that every compact sublocale of \( A \) is closed in \( A \). Let \( K \subseteq A \subseteq L \) where \( K \) is a compact sublocale of \( A \). Assume the contrary that \( K \) is not closed in \( A \).

Case 1: \( K \) is closed in \( L \).

Then there exists \( \alpha \in L - A \) such that \( K = \uparrow_L \alpha \). Consider the singly generated extension \( A[\alpha] \) of the frame \( A \) by adding the element \( \alpha \). We prove that \( A[\alpha] \) is compact. Let \( S \subseteq A[\alpha] \) with \( \bigvee S = 1 \).

Let \( S = \{ a_i \vee (a_i' \wedge \alpha) : a_i, a_i' \in A, a_i \leq a_i', i \in I \} \)

\[
\bigvee S = (\bigvee a_i) \vee [(\bigvee a_i') \wedge \alpha] \\
\bigvee S = (\bigvee a_i') \wedge [(\bigvee a_i) \vee \alpha] \\
1 = (\bigvee a_i') \wedge [(\bigvee (a_i \vee \alpha))]
\]

Thus \( \bigvee_{i \in I} a_i' = 1 \) and \( \bigvee_{i \in I} (a_i \vee \alpha) = 1 \). Since \( A \) is compact, there exists
a finite subset $J_1 \subseteq I$ with $\bigvee_{j_1 \in J_1} a_{j_1}' = 1$.

Also $a_i \lor \alpha \geq \alpha$ and hence $a_i \lor \alpha \in \Uparrow L \alpha = K$. Since $\Uparrow L \alpha = K$ is compact, there exists a finite subset $J_2 \subseteq I$ with $\bigvee_{j_2 \in J_2} (a_{j_2} \lor \alpha) = 1$. Set $J = J_1 \cup J_2$ and $F = \{a_j \lor (a_j' \land \alpha) : j \in J\}$. Clearly $F \subseteq S$ and $F$ is finite.

Then

$$\bigvee F = (\bigvee a_j) \lor [(\bigvee a_j') \land \alpha]$$

$$= (\bigvee a_j') \land (\bigvee (a_j \lor \alpha))$$

$$= 1 \land 1$$

$$= 1$$

because $J_1 \subseteq J, J_2 \subseteq J$ and $\bigvee_{j_1 \in J_1} a_{j_1}' = 1, \bigvee_{j_2 \in J_2} (a_{j_2} \lor \alpha) = 1$. Hence $A[\alpha]$ is compact. Since $A \subset A[\alpha]$ and $A$ is M.R.C., this leads to a contradiction as there is no strictly larger frame that is compact. Thus our assumption that $K$ is not closed in $A$ is wrong. Thus $K$ must be closed in $A$ in this case.

Case 2: Assume that $K$ is not closed in $L$.

Let $K$ be the closure of $K$ by Definition 1.5.8, which is a unique closed sublocale of $L$ such that $K$ is dense in it. Since $K$ is closed in $L$, there exists $\beta \in L - A$ such that $K = \Uparrow L \beta$. Consider the singly generated extension $A[\beta]$ of the frame $A$ by adding the element $\beta$. We prove that $A[\beta]$ is compact. Let $S \subseteq A[\beta]$ with $\bigvee S = 1$. 
Let \( S = \{ a_i \lor (a_i' \land \beta) : a_i, a_i' \in A, a_i \leq a_i', i \in I \} \)

\[
\begin{align*}
\bigvee S &= (\bigvee a_i) \lor [(\bigvee a_i') \land \beta] \\
\bigvee S &= (\bigvee a_i') \land [(\bigvee a_i) \lor \beta] \\
1 &= (\bigvee a_i') \land [\bigvee (a_i \lor \beta)]
\end{align*}
\]

Thus \( \bigvee_{i \in I} a_i' = 1 \) and \( \bigvee_{i \in I} (a_i \lor \beta) = 1 \). Since \( A \) is compact, there exists a finite subset \( J_1 \subseteq I \) with \( \bigvee_{j_1 \in J_1} a_j = 1 \). Since \( A \) is M.R.C., \( L \) is not compact. So we consider the one point compactification \( L_{FC} \) of \( L \) in Definition 3.2.1.

**Claim:** \( A \subseteq F_C \)

Let \( a \in A \). Since \( \uparrow_A a \) is a closed sublocale of \( A \), it is compact in \( A \). Since \( A \) is M.R.C., by Lemma 3.2.1, all the compact upsets \( \uparrow_A a \) in \( A \) are same as compact upsets \( \uparrow_L a \) in \( L \). Thus for each \( a \in A \), \( \uparrow_L a \) is compact in \( L \). Hence \( A \subseteq F_C \).

Now \( a_i \lor \beta \in L \) and \( a_i \in F_C \). Hence by definition of \( L_{FC} \), we have \( (a_i \lor \beta, 0) \lor (a_i, 1) = (a_i \lor \beta, 1) \in L_{FC} \). Now

\[
\begin{align*}
\bigvee_{i \in I} (a_i \lor \beta, 1) &= (\bigvee_{i \in I} (a_i \lor \beta), 1) \\
&= (1, 1)
\end{align*}
\]

Since \( L_{FC} \) is compact, there exists a finite subset \( J_2 \subseteq I \) with \( \bigvee_{j_2 \in J_2} (a_{j_2} \lor \beta, 1) = (1, 1) \) and hence \( \bigvee_{j_2 \in J_2} (a_{j_2} \lor \beta) = 1 \). Set \( J = J_1 \cup J_2 \) and \( F = \{ a_j \lor (a_j' \land \beta) : j \in J \} \). Clearly \( F \subseteq S \) and \( F \) is finite. As seen before, \( \bigvee F = 1 \). Hence \( A[\beta] \) is compact. Since \( A \subseteq A[\beta] \) and \( A \) is
M.R.C., this leads to a contradiction as there is no strictly larger frame that is compact. Thus our assumption that $K$ is not closed in $A$ is wrong. Thus $K$ must be closed in $A$ in this case. Thus every compact sublocale of $A$ is closed. Hence $A$ is a CCE Frame.

**Corollary 3.2.4.** Every compact regular frame is maximal relative to compactness.

*Proof.* Every closed sublocale of a compact frame is compact. Also every compact sublocale of a regular frame is closed. Thus every compact regular frame is CCE and hence M.R.C. □

**Corollary 3.2.5.** Let $A$ be any compact frame. Then no subframe of $A$ is regular.

*Proof.* Every subframe of a compact frame is compact. If such a frame becomes regular, then by Corollary 3.2.4, it is M.R.C. which is a contradiction as $A$ is compact. □

**Corollary 3.2.6.** The topological space $(X, \Omega X)$ is a C-C space if and only if $\Omega X$ is a CCE Frame.

*Proof.* Assume that $(X, \Omega X)$ is a C-C space. Then it is M.R.C by Theorem 3.1.1. Then $\Omega X$ is M.R.C. Conversely, if $\Omega X$ is a CCE Frame, then it is M.R.C. by Theorem 3.2.3. Hence $(X, \Omega X)$ is M.R.C and thus a C-C space by Theorem 3.1.1. □

**Corollary 3.2.7.** Let $A$ be a spatial CCE Frame. Then it is compact and subfit.
Proof. Since $A$ is CCE Frame, it is compact. Since $A$ is a CCE Frame, by Corollary 3.2.6, the topological space which corresponds to $A$ will be a C-C space and hence compact and $T_1$ by Theorem 3.1.3. Since frame of opens of a $T_1$ topological space is subfit, we have $A$ is subfit. 

Example 3.2.1. Let $(X, \tau)$ be a cofinite topological space. Then it is compact and $T_1$ but not a C-C space. Then the frame $\tau$ is subfit as the frame of opens of a $T_1$ topological space is subfit. $\tau$ is also compact. Now by Corollary 3.2.6, $\tau$ is not a CCE frame.

Corollary 3.2.8. If $A$ is a compact Hausdorff frame, then it is a CCE frame.

Proof. A compact Hausdorff frame is regular by Theorem 1.7.9. Since a compact regular frame is spatial by Theorem 1.7.7, $A$ is spatial. Also the topological space corresponding to such a frame is Hausdorff. Thus the topological space corresponding to the frame $A$ is compact Hausdorff. Then by Theorem 3.1.1, it is M.R.C and hence a C-C space by Theorem 3.1.2. Now by Corollary 3.2.6, $A$ is a CCE Frame. 

The following is an example of a CCE frame which is compact but not Hausdorff.

Example 3.2.2. Let $(R, \Omega R)$ be the space of rationals with the relative topology and let $(R, \Omega R^*)$ be the one point compactification of $(R, \Omega R)$. Then it is proved in [27] that $(R, \Omega R^*)$ is not Hausdorff but it is a C-C space. Since $(R, \Omega R^*)$ is not Hausdorff, the frame $\Omega R^*$ is not a Hausdorff frame, as the topological space representing a Hausdorff spatial frame is Hausdorff. Again by Corollary 3.2.6, the frame $\Omega R^*$ is a CCE frame.
frame as \((R, \Omega^R)\) is a C-C space.

**Theorem 3.2.9.** Let \(A\) be a non spatial CCE Frame. Then it cannot be subfit.

**Proof.** Since \(A\) is a CCE Frame, it is compact. If \(A\) is subfit, then by **Theorem 2.11** of[18], a compact subfit frame is spatial, which is a contradiction. \(\square\)

### 3.3 CCE Frame and Coproduct

Most of the topological properties are preserved under the act of taking product. In this section, we are seeking whether this is true in the case of CCE Frames.

**Theorem 3.3.1.** Let \(\{A_i : i \in I\}\) be a non empty family of non empty compact frames and let \(A\) be the frame coproduct. If \(C_j\) is a compact sublocale of \(A_j\), then \(p_j^{*}\)(\(C_j\)) is compact in \(A\).

**Proof.** Take \(A_j = C_j\) in \(\bigoplus_{i \in I} A_i\), then we have \(p_j^{*}(\bigoplus_{i \in I} A_i) = C_j\). Thus \(p_j^{*}\)(\(C_j\)) = \(\bigoplus_{i \in I} A_i\) where \(A_j = C_j\). Since \(C_j\) is a compact sublocale and all the other \(A_i\)'s are compact, by Tychonoff theorem for locales, \(\bigoplus_{i \in I} A_i\) is compact. Hence \(p_j^{*}\)(\(C_j\)) is compact in \(A\). \(\square\)

We state the Kuratowski-Mrowka Theorem for locales[36] for proving the next result.
Theorem 3.3.2. A locale $L$ is compact if and only if the product projection $p^* : L \oplus M \to M$ (the coproduct injection $p : M \to L \oplus M$ in frame language) is closed for every locale $M$.

Theorem 3.3.3. Let $\{A_i : i \in I\}$ be a non empty family of non empty compact frames and let $A$ be the frame coproduct. If $A$ is a CCE Frame, then each $A_i$ is a CCE Frame.

Proof. If $C_j$ is closed in $A_j$, then it is compact as a closed sublocale of a compact frame is compact. Now suppose that $C_j$ is a compact sublocale of $A_j$. Then by Theorem 3.3.1, $C = p_j^{*-1}(C_j)$ is compact in $A$ and hence it is closed in $A$ as $A$ is a CCE Frame. Then $p_j^*(C) = C_j$ and since $A$ is compact by Tychonoff theorem for locales $C_j$ is closed as the projections of $A$ to $A_i$ being closed maps by Theorem 3.3.2. Hence each $A_i$ is a CCE Frame. □

The converse of the above result need not be true. We prove this through the next theorem which tells that the coproduct of a compact frame $A$ with itself is a CCE frame if and only if $A$ is a CCE frame that is strongly Hausdorff. Hence, if the condition strong Hausdorffness is dropped, then the coproduct may not be a CCE frame.

Theorem 3.3.4. Let $A$ be any compact frame and let $A \oplus A$ be the coproduct of $A$ with itself. Then $A \oplus A$ is a CCE Frame if and only if $A$ is CCE and Hausdorff.

Proof. Suppose that $A \oplus A$ is a CCE Frame. Then $A$ is a CCE Frame by Theorem 3.3.3. Let $\Delta(U) = \bigvee \{a \land b : (a, b) \in U\}$ where $U \in A \oplus A$. Then $\Delta : A \oplus A \to A$ called the codiagonal is a surjective frame homomorphism.
and hence its right adjoint $\Delta^*$ is a sublocale map by which $A$ is a sublocale of $A \oplus A$. Since $A$ is a CCE Frame, it is compact. Hence $A$ is a compact sublocale of $A \oplus A$. Since $A \oplus A$ is a CCE Frame, $A$ is closed in $A \oplus A$. Thus $A$ is Hausdorff as the diagonal $\Delta$ embeds $A$ as a closed sublocale of $A \oplus A$, by definition of a Hausdorff frame.

Assume that $A$ is a CCE Frame and Hausdorff. A compact Hausdorff frame is regular by Theorem 1.7.9. Since a compact regular frame is spatial by Theorem 1.7.7, $A$ is spatial. Then the topological space corresponding to $A$ is a C-C space by Corollary 3.2.6. Also the space corresponding to a regular frame is regular and hence Hausdorff. Thus $A$ is a C-C space that is Hausdorff and hence by Theorem 3.1.4, the product topological space is C-C. Then by Corollary 3.2.6, the product topology is a CCE Frame. Since $A$ is compact and regular, by Theorem 1.7.4 and Theorem 1.7.8, $A \oplus A$ is compact and regular. Then, by Theorem 1.7.7, $A \oplus A$ is spatial. Now, by Theorem 1.6.5, $A \oplus A$ is isomorphic to the product topology. Hence $A \oplus A$ is a CCE Frame.

We know that every subframe of a compact frame is compact. But every sublocale of a compact locale need not be compact. It happens when the sublocale becomes a closed sublocale. We know that a sublocale is different from a subframe. A sublocale is quotient frame and hence it cannot be regarded as subframes of the frame. Hence a CCE Frame can have a sublocale which in its own respect may become a CCE Frame. In the next theorem, we prove that the above situation occurs when the sublocale is closed.

**Theorem 3.3.5.** Let $A$ be a CCE Frame. A sublocale $K$ of $A$ is CCE if and only if $K$ is closed in $A$. 

Proof. Suppose that the sublocale \( K \) of \( A \) is CCE. Then it is a compact sublocale of \( A \) and hence closed, as \( A \) is a CCE Frame.

Conversely, suppose that \( K \) is a closed sublocale of \( A \). Then \( K \) is a compact sublocale of \( A \) as it is CCE. Thus any closed sublocale of \( K \) is compact. Now assume that \( K_1 \) is a compact sublocale of \( K \). Then it is a compact sublocale of \( A \). Since \( A \) is a CCE Frame \( K_1 \) is closed in \( A \). Therefore \( K_1 = \uparrow_A a \) where \( a \in A \). Since \( K_1 \) is a sublocale of \( K \), we have \( \uparrow_A a = \uparrow_K a \). Hence \( K_1 \) is closed in \( K \). Thus \( K \) is a CCE locale. \qed