Chapter 2

Singly Generated Extension of Frames

2.1 Introduction

The construction of enlarging the topology of a given space by adding a new open set is a familiar type of topological construction. This can be algebraically viewed as extending a given frame by adjoining a new element. The conditions under which the simple extension of a topological space having a specified topological property also holds that property was studied by N. Levine[26]. The same notion called singly generated extension in frames was introduced by B. Banaschewski[2]. In this chapter, we discuss

Some results of this chapter are included in the following paper.

the conditions under which a singly generated extension of the frame $A$ possesses a property that is already owned by the frame $A$.

We start with the definition of *singly generated extension of a frame* due to Banaschewski appeared in his paper[2].

**Definition 2.1.1.** A frame $M$ is called a *singly generated extension* of a frame $A$ if $A$ is a subframe of $M$, and $M$ is generated by $A$ and some $b \in M$. We write $M = A[b]$.

The next theorem and the first remark is due to B. Banaschewski[2].

**Theorem 2.1.1.** Let $L$ be any frame. Let $A$ be a subframe of $L$. Let $b \in L - A$. Then $A[b] = \{ a \lor (a' \land b) : a, a' \in A \}$ where $\lor$ and $\land$ are respectively the join and meet operations in $L$ is a subframe of $L$.

**Remark 2.1.1.** We can take $a \leq a'$ in the above description of $A[b]$ because $a \lor (a' \land b) = a \lor (a \land b) \lor (a' \land b) = a \lor ((a \lor a') \land b)$.

**Remark 2.1.2.** If $x \in A[b]$, then $x \lor b = 1$ because there exists $a_1, a_2 \in A$ such that $x = a_1 \lor (a_2 \land b)$, so that $x \lor b = a_1 \lor (a_2 \land b) \lor b = a_1 \lor b = a_1 \lor (1 \land b) \in A[b]$.

The following corollary is proved in [36]

**Corollary 2.1.2.** Every frame is isomorphic to a subframe of a complete Boolean algebra.

The results proved in this chapter require the existence of complement of the element $b$ added to the frame $A$ which in general need not happen. Let $\phi$ represents the frame isomorphism which makes $A[b]$ isomorphic to a
subframe of a Boolean frame $B$ according to corollary 2.1.2, then $\phi(A[b]) = \{ \phi(a) \lor (\phi(a') \land \phi(b)) : a, a' \in A \}$ will be a subframe of the Boolean frame $B$ which is the singly generated extension of the frame $\phi(A)$ in $B$ on adding the element $\phi(b)$ in $B - \phi(A)$. The complement of $\phi(b)$ exists in $B$ as $B$ is a Boolean frame. Even if the complement of the added element “$b$” does not exist in the given frame “$A$”, we can consider the frame isomorphic copy of “$A$” and the image of the element “$b$” under the frame isomorphism $\phi$ in the embedded Boolean frame $B$. Now the situation is what we discuss here and can determine whether the singly generated extension $\phi(A[b])$ in the Boolean frame preserves the specified frame isomorphic property when $\phi(b)$ is added and if it is so, then definitely the singly generated extension of $A$ on adding $b$ also preserves the frame isomorphic property.

**Theorem 2.1.3.** Let $A$ be any subframe of $L$. Let $b \in L - A$ and let $A[b]$ be the singly generated extension of $A$ in $L$. Then $b^c \in A$ if and only if $b^c \in A[b]$.

*Proof.* If $b^c \in A$, then $b^c = b^c \lor (0 \land b) \in A[b]$, by definition of $A[b]$. Conversely assume that $b^c \in A[b]$. Then, $b^c = a \lor (a' \land b)$ where $a \leq a'$ and $a, a' \in A$.

Now, $1 = b \lor b^c = [a \lor (a' \land b)] \lor b = a \lor [(a' \land b) \lor b] = a \lor b$.

Also $0 = b \land b^c = b \land [a \lor (a' \land b)] = (b \land a) \lor (b \land a')$. Thus $a \land b = 0$.

Hence $b^c = a \in A$. \hfill \Box

In the following sections we proceed to investigate whenever $A$ has a frame isomorphic property $p$, under what conditions $A[b]$ also has the property $p$. 
2.2 Singly Generated Extension and Compactness

In this section, we find the conditions under which a compact subframe \( A \) of the frame \( L \) still remains compact when extended singly to a frame by adding a single element \( b \in L - A \). Every subframe of a compact frame is compact but a sublocale of a compact frame need not be compact. A closed sublocale of a compact frame is compact. A topological space is compact if and only if the frame of opens is compact, by Theorem 1.7.11. Thus compactness in frames is equivalent to compactness in topology when the frame is a spatial frame.

We introduce the following definition for further discussion.

**Definition 2.2.1.** Let \( L \) be any frame and \( A \) be a subframe. An element \( b \in L \) is said to be compact relative to the subframe \( A \) if for every \( S \subseteq A \) with \( \bigvee S \geq b \), there exists \( F \subseteq S \) with \( F \) finite and \( \bigvee F \geq b \).

**Theorem 2.2.1.** Let \( A \) be a subframe of the frame \( L \) and \( b \in L - A \) be complemented in \( L \). Consider the following statements about \( A[b] \).

1. \( A[b] \) is compact.
2. \( b^c \) is compact relative to \( A[b] \).

Then, the following statements hold.

(a) Statement (1) implies statement (2).
(b) If \( A \) is compact, then (1) and (2) are equivalent.

**Proof.** (a) Suppose that (1) holds and let \( S \subseteq A[b] \) with \( \bigvee_{A[b]} S \geq b^c \). Then \( (\bigvee_{A[b]} S) \vee b = 1 \), which implies \( \bigvee_{A[b]} \{ s \vee b : s \in S \} = 1 \). Since \( s \vee b \in A[b] \) by remark 2, for every \( s \in S \), the compactness of \( A[b] \) implies
that there are finitely many elements \( s_1, s_2, \ldots, s_m \) in \( S \) such that \( b \vee s_1 \vee s_2 \vee \ldots \vee s_m = 1 \). This implies that \( s_1 \vee s_2 \vee \ldots \vee s_m \geq b^c \), whence \( b^c \) is compact relative to \( A[b] \).

(b) Assume that \( A \) is compact and \( b^c \) is compact relative to \( A[b] \).

Let \( S = \{ s_i : i \in I \} \) be a cover of \( A[b] \). For each \( i \in I \), let \( s_i = a_i \vee (b_i \wedge b) \) where \( a_i \leq b_i \) with \( a_i, b_i \in A \). Since \( s_i = a_i \vee (b_i \wedge b) \leq b_i \vee (b_i \wedge b) = b_i \) for each \( i \), we have \( \bigvee b_i = 1 \). By compactness of \( A \), there is a finite \( J \subseteq I \) such that \( \bigvee_{A[b]} \{ b_i : i \in J \} = 1 \). Since \( b^c \) is compact relative to \( A[b] \), there is a finite \( K \subseteq I \) such that \( b^c \leq \bigvee_{A[b]} \{ s_i : i \in K \} \). Set \( H = J \cup K \), and note that \( H \) is a finite subset of \( I \). Since \( \bigvee_{A[b]} b_i = 1 \), we have that \( b = \bigvee_{A[b]} \{ b_i \wedge b : i \in H \} \) and hence

\[
1 = b \vee b^c \leq \bigvee_{A[b]} \{ b_i \wedge b : i \in H \} \vee \bigvee_{A[b]} \{ s_i : i \in H \}
= \bigvee_{A[b]} \{ s_i \vee (b_i \wedge b) : i \in H \}
= \bigvee_{A[b]} \{ s_i : i \in H \}
\]

This shows that \( A[b] \) is compact.

We can derive Theorem 6 of [26] as a simple corollary of the above theorem.

Corollary 2.2.2. Let \( (X, \tau) \) be a compact topological space and let \( A \notin \tau \). Then \( (X, \tau(A)) \), the simple extension in the sense of [26], is compact if and only if \( A^c \) is compact in \( (X, \tau) \).

Proof. A topological space is compact if and only if the frame of its open sets is compact, by Theorem 1.7.11. Now the proof follows from
Theorem 2.1.3 and Theorem 2.2.1.

2.3 Singly Generated Extension and Regularity

In this section, we discuss the separation axioms reguarity and complete regularity in connection with singly generated extension of a frame. It is known that a regular frame is always Hausdorff. Also a completely regular frame is always regular. The condition under which the singly generated extension $A[b]$ is Hausdorff provided $A$ is Hausdorff is that there exists $c \in A$ such that $c \vee b, c \wedge b \in A$. This is proved by B. Banaschewski in [2]. We examine the conditions under which a regular(completely regular) frame is again reguar(completely regular) when extended by adding a single element. The following lemma finds application in the proof of the main result in this section.

**Lemma 2.3.1.** Let $A$ be any subframe of the frame $L$. Let $b \in L - A$ be complemented in $L$ and let $b' \in A$. Denote by $\preceq$ and $\preceq\preceq$ the rather below and the completely below relations in $A[b].$ Then the following statements hold.

(a) $x \prec a$ and $y \prec a'$ in $A$ imply $x \lor (y \land b) \preceq a \lor (a' \land b)$.

(b) $x \prec\prec a$ and $y \prec\prec a'$ in $A$ imply $x \lor (y \land b) \preceq\preceq a \lor (a' \land b)$.

**Proof.** Let $p = a \lor (a' \land b)$.

(a) Since $x \prec a, y \prec a'$ we have

$$x^* \lor a = 1, y^* \lor a' = 1 \quad (2.1)$$
where \( x^* \) and \( y^* \) are the pseudocomplements of \( x \) and \( y \) in \( A \) respectively. Let \( \times \) denotes the pseudocomplementation with respect to \( A[b] \). Since \( x^* \leq x^\times, y^* \leq y^\times \), from equation 2.1 we have

\[
x^\times \lor a = 1, y^\times \lor a' = 1 \quad (2.2)
\]

Then \( x^\times \lor p = 1 \) using equation 2.2. Also \( y^\times \lor p = y^\times \lor a \lor (a' \land b) \geq (y^\times \lor a') \land (y^\times \lor b) = y^\times \lor b \) by equation 2.2.

\( A[b] \) satisfies the De Morgan's law \( (\alpha \lor \beta)^\times = \alpha^\times \land \beta^\times \) on pseudocomplements and \( (\alpha \land \beta)^\times \geq \alpha^\times \lor \beta^\times \) because \( A[b] \) is a distributive pseudocomplemented lattice. Now \([x \lor (y \land b)]^\times \lor p \geq (x^\times \lor p) \land (y^\times \lor b^c \lor p) = y^\times \lor b^c \lor p \geq y^\times \lor b^c \lor b = 1\). So \([x \lor (y \land b)]^\times \lor p = 1\). Thus \( x \lor (y \land b) \leq p \).

(b) Take \( x_0 = x, x_1 = a \) and \( y_0 = y, y_1 = a' \). Set \( p_0 = x_0 \lor (y_0 \land b) = x \lor (y \land b) \) and \( p_1 = x_1 \lor (y_1 \land b) = p \). Let \( l, m \) are two dyadic rational numbers in \((0,1)\) with \( l < m \). Then by definition of \( x \ll a \), we have \( x_l, x_m \in A \) with \( x_l \ll x_m \). Similarly \( y \ll a' \), gives \( y_l, y_m \in A \) with \( y_l \ll y_m \). Under the assumption \( b^c \in A \), repeating steps in (a), \( x_l \ll x_m \) and \( y_l \ll y_m \) implies \( x_l \lor (y_l \land b) = p_l \leq p_m = x_m \lor (y_m \land b) \). Thus \( x \lor (y \land b) \leq p \).

**Theorem 2.3.1.** Let \( A \) be a regular subframe of the frame \( L \). Let \( b \in L - A \) with the complement \( b^c \) of \( b \) exists in \( L \). Then \( A[b] \) is regular if \( b^c \in A \).

**Proof.** Let \( p \in A[b] \). Then \( p = c \lor (d \land b) \) where \( c, d \in A \) and of course \( c \leq d \). Since \( A \) is regular, we can write \( c = \bigvee_A \{x \in A : x \ll c\} \) and
\[ d = \bigvee_A \{ x \in A : x \prec d \}. \] Now

\[
p = \left( \bigvee_A \{ x \in A : x \prec c \} \right) \lor \left( \bigvee_A \{ y \in A : y \prec d \} \land b \right)
\]

\[
= \left( \bigvee_{A[b]} \{ x \in A : x \prec c \} \right) \lor \left( \bigvee_{A[b]} \{ y \land b : y \prec d, y \in A \} \right)
\]

\[
= \bigvee_{A[b]} \{ x \lor (y \land b) : x \prec c, y \prec d, x, y \in A \}.
\]

```text
\[
= \bigvee_{A[b]} \{ x \lor (y \land b) : x \lor (y \land b) \preceq p \}
\]
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using Lemma 2.3.1. Hence \( A[b] \) is regular. 

We can derive Theorem 2 of [26] as a simple corollary of the above theorem.

**Corollary 2.3.2.** Let \((X, \tau)\) be a regular topological space and \(A \notin \tau\). Then \((X, \tau(A))\), the simple extension in the sense of [26], is regular if \(A^c \in \tau\).

**Proof.** A topological space is regular if and only if the frame of its open sets is regular. Now the proof follows from Theorem 2.3.1. 

**Theorem 2.3.3.** Let \( A \) be a completely regular subframe of the frame \( L \). Let \( b \in L - A \) with the complement \( b^c \) of \( b \) exists in \( L \). Then \( A[b] \) is completely regular if \( b^c \in A \).

**Proof.** Let \( p \in A[b] \). Then \( p = c \lor (d \land b) \) where \( c, d \in A \) where \( c \leq d \). Since \( A \) is completely regular, we can write \( c = \bigvee_A \{ x \in A : x \ll c \} \) and
\[ d = \bigvee_A \{ x \in A : x \prec c \} \]. Now

\[ p = \left( \bigvee_A \{ x \in A : x \prec d \} \right) \lor \left( \bigvee_A \{ y \in A : y \prec d \} \land b \right) \]

\[ = \left( \bigvee_{A[b]} \{ x \in A : x \prec c \} \right) \lor \left( \bigvee_{A[b]} \{ y \land b : y \prec d, y \in A \} \right) \]

\[ = \bigvee_{A[b]} \{ x \lor (y \land b) : x \prec c, y \prec d, x, y \in A \}. \]

\[ = \bigvee_{A[b]} \{ x \lor (y \land b) : x \lor (y \land b) \preceq p \} \]

using Lemma 2.3.1. Hence \( A[b] \) is completely regular. \( \square \)

We can derive Theorem 4 of [26] as a simple corollary of the above theorem.

**Corollary 2.3.4.** Let \((X, \tau)\) be a completely regular topological space and \( A \notin \tau \). Then \((X, \tau(A))\), the simple extension in the sense of
is completely regular if $A^c \in \tau$.

Proof. A topological space is completely regular if and only if the frame of its open sets is completely regular. Now the proof follows from Theorem 2.3.3.

The following example shows that if the condition in the above theorems is dropped, then the singly generated extension need not be regular.

**Example 2.3.1.** Consider a subframe $A = \{0, a, f, 1\}$ of the frame $L$ in Fig.1. $A$ is completely regular. Now consider the element $b \in L - A$. Then the singly generated extension of the frame $A$ on adding $b$ is $A[b] = \{0, a, b, d, f, 1\}$ where $b^c = e$ is not in $A$. It is easy to see that $A[b]$ is not regular, because the only nonzero element with the property $x \prec d$ in the subframe $A$ is $f$ and hence $d$ cannot be written as $d = \bigvee_A \{x : x < d\}$.

### 2.4 Singly Generated Extension and Normality

**Theorem 2.4.1.** Let $A$ be a normal subframe of the frame $L$. Let $b \in L - A$ with the complement $b^c$ of $b$ exists in $L$. Then $A[b]$ is normal if and only if the open quotient $\downarrow b^c = \{b^c \wedge a : a \in A\}$ is normal.


We show that $\downarrow b^c$ is normal. Let $x, y \in \downarrow b^c$ where $x \lor y = b^c = 1_{b^c}$. Let $x = b^c \wedge x_0, y = b^c \wedge y_0$ where $x_0, y_0 \in A$. Now $b^c = x \lor y =$
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\[(b^c \land x_0) \lor (b^c \land y_0) = b^c \land (x_0 \lor y_0) \leq x_0 \lor y_0.\] Then \((x_0 \lor y_0) \lor (y_0 \lor b) \geq b \lor b^c = 1.\] Thus \((x_0 \lor y_0) \lor (y_0 \lor b) = 1.\] Since \(A[b]\) is normal, there exist \(u, v \in A[b]\) with \(u \land v = 0\) and \(x_0 \lor b \lor v = 1, u \lor y_0 \lor b = 1.\) Let \(u = u_1 \lor (u_2 \land b)\) where \(u_1 \leq u_2, u_1, u_2 \in A\) and let \(v = v_1 \lor (v_2 \land b)\) where \(v_1 \leq v_2, v_1, v_2 \in A.\) Then \(u \land v = 0\) implies \(u_1 \land v_1 = 0.\) Also 
\[1 = x_0 \lor b \lor v = x_0 \lor b \lor [v_1 \lor (v_2 \land b)] = x_0 \lor v_1 \lor [b \lor (v_2 \land b)] = x_0 \lor v_1 \lor b \]
Similarly \(y_0 \lor u_1 \lor b = 1.\) Thus
\[x_0 \lor v_1 \lor b = 1, y_0 \lor u_1 \lor b = 1, u_1 \land v_1 = 0\] (2.3)

Let \(v_1 \land b^c = p, u_1 \land b^c = q.\) Then \(p, q \in \downarrow b^c\) because \(u_1, v_1 \in A.\) Now
\[p \land q = (v_1 \land b^c) \land (u_1 \land b^c) = b^c \land u_1 \land v_1 = 0\] from equation 2.3. Also 
\[x \lor p = x \lor (v_1 \land b^c) = (b^c \land x_0) \lor (b^c \land v_1) = (x_0 \lor v_1) \land b^c = (x_0 \lor v_1 \lor b) \land b^c = 1 \land b^c = b^c = 1_{\downarrow b^c}\] from equation 2.3. Similarly \(y \lor q = 1_{\downarrow b^c}.\) Thus \(\downarrow b^c\) is normal.

Conversely assume that \(\downarrow b^c\) is normal. Let \(x, y \in A[b]\) with \(x \land y = 1.\) Let \(x = c \lor (d \land b)\) where \(c \leq d; c, d \in A\) and \(y = e \lor (f \land b)\) where \(e \leq f; e, f \in A.\) Now
\[1 = x \lor y = [c \lor (d \land b)] \lor [e \lor (f \land b)] = (c \lor e) \lor (d \lor f) \land (c \lor e \lor b) = (d \lor f) \land (c \lor e \lor b)\] which yield
\[d \lor f = 1, c \lor e \lor b = 1\] (2.4)

Since \(A\) is normal, there exist \(u, v \in A\) such that
\[u \land v = 0, d \lor v = 1, u \lor f = 1\] (2.5)
Consider \( c \land b^c, (e \lor b) \land b^c \) in \( b^c \). Now \((c \land b^c) \lor [(e \lor b) \land b^c] = (c \lor e \lor b) \land b^c = b^c = \downarrow_{b^c}\) using equation 2.4. Since \( \downarrow_{b^c}\) is normal there exist \( p, q \in \downarrow_{b^c}\), \( p = p_0 \land b^c, q = q_0 \land b^c \) and \( p_0, q_0 \in A \) with \( p \land q = 0, b^c = (c \land b^c) \lor q = (c \land b^c) \lor (q_0 \land b^c) = (c \lor q_0) \land b^c, b^c = [(e \lor b) \land b^c] \lor p = (e \land b^c) \lor p = (e \land b^c) \lor (p_0 \land b^c) = (e \lor p_0) \land b^c \). Thus we have \( c \lor q_0 \geq b^c, e \lor p_0 \geq b^c \). Hence \( c \lor q_0 \lor b \geq b^c \lor b = 1, e \lor p_0 \lor b \geq b^c \lor b = 1 \). Thus we get

\[
c \lor q_0 \lor b = 1, e \lor p_0 \lor b = 1
\] (2.6)

Set \( \alpha = q \lor (v \land b), \beta = p \lor (u \land b) \). It is clear that \( \alpha, \beta \in A[b] \).

Now \( \alpha \land \beta = [q \lor (v \land b)] \land [p \lor (u \land b)] = (q \land p) \lor (p \land v \land b) \lor (q \land u \land b) \lor (u \land v \land b) = (p_0 \land b^c \land v \land b) \lor (q_0 \land b^c \land u \land b) = 0, \) since \( p \land q = u \land v = 0 \). Also

\[
\begin{align*}
x \lor \alpha &= [c \lor (d \land b)] \lor [q \lor (v \land b)] \\
&= (c \lor q) \lor [(d \lor v) \land b] \\
&= (c \lor q \lor d \lor v) \land (c \lor q \lor b) \\
&= (d \lor q \lor v) \land (c \lor q \lor b) \\
&= (d \lor (q_0 \land b^c) \lor v) \land (c \lor (q_0 \land b^c) \lor b) \\
&= (d \lor v \lor q_0) \land (d \lor v \lor b^c) \land (c \lor q_0 \lor b) \land (c \lor b \lor b^c) \\
&= 1
\end{align*}
\]

using equations 2.5 and 2.6.

Similarly we can show that \( y \lor \beta = 1 \). Hence \( A[b] \) is normal. 

We can derive Theorem 5 of [26] as a simple corollary of the above theorem.
Corollary 2.4.2. Let \((X, \tau)\) be a normal topological space and \(A \notin \tau, A^c \in \tau\). Then \((X, \tau(A))\), the simple extension in the sense of [26], is normal if and only if \((A^c, \tau \cap A^c)\) is normal.

Proof. A topological space is normal if and only if the frame of its open sets is normal. Now the proof follows from Theorem 2.4.1. \qed