Chapter 2

Study of the Nonlinear Schrödinger Equation

In theoretical physics, the nonlinear Schrödinger equation (NLSE) is a nonlinear PDE and acts as a universal nonlinear model to describe many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena. Unlike the Schrödinger equation, it never describes the time evolution of a quantum state. It is an example of an integrable model. In quantum mechanics, it is a special case of the nonlinear Schrödinger field, and when canonically quantized, it describes bosonic point particles with delta-function interactions the particles either repel or attract when they are at the same point. The NLSE is integrable when the particles move in one dimension space. In the limit of infinite strength repulsion, the NLSE bosons are equivalent to one dimensional free fermions.

In a medium with inhomogeneous refractive index, light does not travel in straight lines. This inhomogeneity of the refractive index $n$ may be caused by thermal effect if $n$ depends sensitively on temperature. For a laser beam of finite cross-section propagating in a medium, the intensity of the beam varies in space, resulting in inhomogeneity in temperature and hence inhomogeneity in the refractive index. The beam may then shrink in size—a phenomenon called self-focusing.

To derive NLSE, consider a medium with inversion symmetry for which the polarization is given by

$$P = \alpha E + \alpha_{NL}|E|^2E.$$  

(2.1)
Three of the Maxwell equations for a neutral, homogeneous medium are given by

\[ \nabla \times E = -c^{-1} \partial B / \partial t, \]  
\[ \nabla \times B = c^{-1} \partial D / \partial t, \]  
\[ \nabla \cdot E = 0, \]  

(2.2) (2.3) (2.4)

where \( D = E + 4\pi P \). Combining these three equations and using the identity, \( \nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - \nabla^2 E \), one obtains

\[ \nabla^2 E - c^{-2} \partial^2 E / \partial t^2 = (4\pi/c^2)\partial^2 P / \partial t^2. \]  

(2.5)

Putting Eq.(2.1) into Eq.(2.5) one has

\[ (\partial^2 - c^{-2} \partial^2 / \partial t^2) E = (4\pi\alpha/c^2)\partial^2 E / \partial t^2 + (4\pi\alpha_{NL}/c^2)\partial^2 (|E|^2 E) / \partial t^2. \]  

(2.6)

Let \( E = u(x, y, z, t) \exp[i(\omega t - k z)] + cc \), where \( z \) is the propagation direction of the beam and \( cc \) denotes complex conjugate. The complex amplitude \( u \) is assumed to be slowly varying.

From Eq.(2.6), we have

\[ (\omega^2/c^2 - k^2 + 4\pi\alpha\omega^2/c^2)u - 2iku_z + (2i\omega/c^2)(1 + 4\pi\alpha)u_t + u_{xx} + u_{yy} \]
\[ + 12\pi(\omega/c^2)\alpha_{NL}|u|^2u = 0. \]  

(2.7)

In the linear regime, the cubic term \(|u|^2u\) is dropped and \( u = const \); Eq.(2.7) gives the linear dispersion relation \( \omega^2/c^2 - k^2 + 4\pi\alpha\omega^2/c^2 = 0 \). Note that in Eq.(2.7), only first harmonics are kept. Now in the weakly nonlinear regime, Eq.(2.7) simplifies to

\[ u_{xx} + u_{yy} + 12\pi(\omega/c^2)\alpha_{NL}|u|^2u - 2iku_z + (2i\omega/c^2)(1 + 4\pi\alpha)u_t = 0. \]  

(2.8)

In the steady state, \( u_t = 0 \) and after rescaling, Eq.(2.8) reduces to

\[ u_{xx} + u_{yy} + 2|u|^2u - iu_z = 0. \]  

(2.9)

This is the equation that describes optical self-focusing, which is equivalent to the NLSE if one dimensional spatial variation of \( u \) is assumed. In such a case \( u_{yy} = 0 \), say) one has

\[ -iu_z + u_{xx} + 2|u|^2u = 0, \]  

(2.10)

which is the NLSE with \( z \) playing the role of time.

Optical solitons have been the subject of extensive theoretical and experimental studies in recent
years because of their promising potential to become principal information carriers in telecommunication due to their capability of propagating long distance without attenuation and changing their shapes. These special type of optical wave packets, appearing as a result of interplay between dispersion and nonlinearity, are good information carriers for high-bit-rate optical transmission systems [54]. The waveguides used in the picosecond optical pulse propagation in nonlinear optical communication systems are usually of Kerr type and consequently the dynamics of light pulses are described by nonlinear Schrödinger family of equations with cubic nonlinear terms. The propagation of picosecond optical solitons in a monomode optical fiber is well described by the celebrated NLSE

\[ iE_x + a_1 E_{tt} + a_2 |E|^2 E = 0, \]  

(2.11)

where \( E(x, t) \) is a complex envelope of electrical field in a comoving frame, \( t \) is the retarded time, \( x \) represents the distance along the direction of propagation, \( a_1 \) is group velocity dispersion (GVD) parameter and \( a_2 \) specifies the strength of Kerr nonlinearity [55]. Depending on the sign of GVD, the NLSE has two distinct localized solutions, bright and dark soliton solutions which are, respectively, existent in the anomalous and normal dispersion regimes [56].

Optical solitons have been regarded as the next generation technology for high-capacity optical communications, mainly because of their promise to transmit signals over long distances while resisting chromatic dispersion [57] and high frequency of optical carrier makes possible high bit rate transmission and to increase the bit rate further it is desirable to use shorter femtosecond pulses. Generalization of the NLSE is necessitated to take into account higher order dispersion, self-steepening of the pulse due to the dependence of the slowly varying part of the non-linear polarization on time and the delayed effect of Raman response for describing optical pulse propagation in the femtosecond domain [58]. For higher order nonlinear Schrödinger (HNLS) equations, to our knowledge, only a few kinds of these equations satisfy certain proportion relations between model coefficients are completely integrable by inverse scattering-like methods. A thorough discussion on these situations has been given in [59].

In recent years, many authors have analyzed the HNLS equation from different points of view and some interesting results have also been obtained [60, 61, 62, 63]. In this work, we analytically derive both bright and dark solitary wave solutions of HNLS equation under some parametric conditions. To date, most applications are designed for bright solitons because they are relatively easy to generate in low-loss and low dispersion optical fibers, however, it is well known that dark solitons have some advantages over their bright counterparts. Compared with
the bright solitons, they have better stability against various perturbations such as fiber loss, mutual interaction between neighboring pulses, the Raman effect and the superposition of noise emitted from optical amplifiers.

2.1 Bright and dark soliton solutions of HNLS equation

The HNLS equation describes the propagation of femtosecond optical pulses in optical fibers and can be written as [55, 64]

\[ iE_x + a_1 E_{tt} + a_2 |E|^2 E + i[a_3 E_{ttt} + a_4 (|E|^2 E)_t + a_5 E (|E|^2)_t] = 0. \] (2.12)

Here \( E \) represents the complex envelope of the electric field, \( x \) is the normalized distance along the fiber, \( t \) is the normalized time with the frame of the reference moving along the fiber at the group velocity. The subscripts \( x \) and \( t \) denotes the spatial and temporal partial derivatives respectively and the coefficients \( a_i (i = 1, 2, ..., 5) \), particularly, \( (a_1 = \beta_2 / 2, a_2 = \gamma_1, a_3 = \beta_3 / 6, a_4 = -\gamma_1 / \omega_0 \) and \( a_5 = \gamma_1 T_R \) are the real parameters related to GVD, self phase modulation (SPM), third order dispersion (TOD), self-steepening and self-frequency shift arising from stimulated Raman scattering (SRS). Here \( \beta_j = (d^2 \beta / d\omega^2)_{\omega=\omega_0} \) are the dispersion coefficients evaluated at the carrier frequency \( \omega_0 \), with \( \beta_1 \), the inverse of group velocity, \( \beta_2 \), the group velocity dispersion parameter, \( \beta_3 \), the third order dispersion parameter and so on. \( \beta \) is propagation constant. More specifically, \( \gamma_1 \) is coefficient of cubic nonlinearity, which results from the intensity dependent refractive index. The term related to \( \gamma_1 / \omega_0 \) results from the intensity dependence of the group velocity and causes self-steepening and shock formation at the pulse edge. The last term related to \( a_5 = \gamma_1 T_R \) incorporates the intrapulse Raman scattering and originates from the delayed Raman response, which cause a self-frequency shift, where \( T_R \), is called Raman time constant, can be estimated from the slope of the Raman gain (SRS). The characteristic Raman time constant \( T_R \) is defined as the first moment of the nonlinear response function. This model, unlike the NLSE, is not integrable in general. When the last three terms are omitted, this propagation equation for the slowly varying envelope of the electric field, \( E \), reduces to the NLSE which is completely integrable by the inverse scattering transform method [54]. A few integrable cases have been identified; these are known as (i) the Sasa-Satsuma case \( [a_3 : a_4 : (a_4 + a_5) = 1 : 6 : 3] [65] \), (ii) the Hirota case \( [a_3 : a_4 : (a_4 + a_5) = 1 : 6 : 0] [66] \) and (iii) derivative NLSE of type I and II [67]. For picosecond light pulses, the third order dispersion, the self-steepening, and the self-frequency shift can be omitted. But the effect of
third-order dispersion is significant for femtosecond light pulses when the GVD is close to zero. It is negligible for optical pulses whose width is of the order of 100 fs or more, having power of the order of 1 W and GVD far away from zero. However, in this case self-steepening as well as self-frequency shift are still dominant and should be retained. The effect of these higher-order terms on pulse propagation have been studied numerically quite extensively [55, 68] and some special solutions to this system are also known.

Now, we derive analytic fundamental bright and dark solitary wave solutions for Eq.(2.12) by using the solitary wave ansatz method [52, 69]. In order to solve Eq.(2.12), consider a soliton ansatz of the form

\[ E(x, t) = P(x, t) \exp[i\phi(x, t)], \]

where \( P(x, t) \) is the amplitude portion of the soliton, while the phase portion of the soliton is given by

\[ \phi(x, t) = -kx + \omega t + \theta, \]

where \( k \) is the soliton frequency, \( \omega \) is the soliton wave number and \( \theta \) is the soliton phase constant. Thus from Eqs.(2.13) and (2.14), we obtain

\[ E_t = (P_t + i\omega P)e^{i\phi}, \quad E_{tt} = (P_{tt} + 2i\omega P_t - \omega^2 P)e^{i\phi}, \]
\[ E_{ttt} = (P_{ttt} + 3i\omega P_{tt} - 3\omega^2 P_t - i\omega^3 P)e^{i\phi}, \quad E_x = (P_x - ik P)e^{i\phi}. \]

Substituting Eq.(2.15) into Eq.(2.12) and decomposing the resultant expression into real and imaginary parts, yields respectively

\[ (k - a_1 \omega^2 + a_3 \omega^3)P + (a_1 - 3a_3 \omega)P_{tt} + (a_2 - a_3 \omega)P^3 = 0, \]
\[ P_x + (2a_1 \omega - 3a_3 \omega^2)P_t + a_2 P_{ttt} + (3a_4 + 2a_5)P^2 P_t = 0. \]

This pair of equations will be analyzed further depending on the type of soliton solution is being fetched. The analysis is now carried out in the following two subsections.

### 2.1.1 Bright soliton solutions

Bright solitons are also known as bell-shaped solitons or non-topological solitons and are modeled by the sech function. Therefore, the function \( P(x, t) \) is considered as

\[ P(x, t) = \frac{A}{\cosh^p[B(x - vt)]}, \]
where constants $A$ and $B$ are amplitude and inverse width of soliton respectively. The index $p$ is unknown at this point and its value will be derived during the course of the derivation of the solution of Eq. (2.12). Thus, after using Eq. (2.18), the Eqs. (2.16) and (2.17) reduce to

\[
(k - a_1\omega^2 + a_3\omega^3)\frac{A}{\cosh^p \tau} + a_1 \left[ \frac{Ap^2 v^2 B^2}{\cosh^p \tau} - \frac{p(p + 1)A v^2 B^2}{\cosh^{p+2} \tau} \right] + (a_2 - a_4\omega) \frac{A^3}{\cosh^{3p} \tau} - 3a_3\omega \left[ \frac{Ap^2 v^2 B^2}{\cosh^p \tau} - \frac{p(p + 1)A v^2 B^2}{\cosh^{p+2} \tau} \right] = 0,
\]

(2.19)

\[-pAB\frac{\tanh \tau}{\cosh^p \tau} + ApvB(2a_1\omega - 3a_3\omega^2)\frac{\tanh \tau}{\cosh^p \tau} + A^3pvB(3a_4 + 2a_5)\frac{\tanh \tau}{\cosh^p \tau} + a_3 \left[ \frac{Ap^2 v^2 B^3}{\cosh^p \tau} - p(p + 1)(p + 2)A v^2 B^3\frac{\tan \tau}{\cosh^{p+2} \tau} \right] = 0,
\]

(2.20)

where $\tau = B(x - vt)$. From Eq. (2.19), equating exponents $3p$ and $p + 2$ gives $p = 1$.

Now from Eq. (2.19), noting that $1/\cosh^{p+j} \tau$ are linearly independent functions for $j = 0, 2$, its coefficients must be respectively set to zero. This leads to

\[
B = \pm \frac{1}{v} \sqrt{\frac{k - a_1\omega^2 + a_3\omega^3}{3a_3\omega - a_1}},
\]

(2.21)

\[
A = \pm \sqrt{\frac{2(k - a_1\omega^2 + a_3\omega^3)}{(a_4\omega - a_2)}}.
\]

(2.22)

Now from Eq. (2.20), setting the coefficients of the linearly independent functions $\frac{\tanh \tau}{\cosh^p \tau}$ for $j = 0, 2$ to zero yields

\[
\omega = a_1 v \pm \sqrt{\frac{a_1^2 v^2 + 3a_3^2 v^3 B - 3a_3 v}{3a_3 v}},
\]

(2.23)

and a constraining condition on parameters is given by $2a_5 + 3a_4 = 0$, which is one of the relation between the parameters for an exact bright soliton solution to exist. From Eq. (2.21) one can note that for the bright solitons to exist, it is necessary to have $(3a_3\omega - a_1)(k - a_1\omega^2 + a_3\omega^3) > 0$. Thus, the bright chiral solitons are given by

\[
E(x, t) = \frac{A}{\cosh[B(x - vt)]} e^{i(-kx + \omega t + \theta)},
\]

(2.24)

and the corresponding intensity of the bright soliton solution takes the form

\[
|E(x, t)|^2 = \left( \frac{2(k - a_1\omega^2 + a_3\omega^3)}{(a_4\omega - a_2)} \right) \cosh^{-2} \left[ \pm \frac{1}{v} \sqrt{\frac{k - a_1\omega^2 + a_3\omega^3}{3a_3\omega - a_1}} (x - vt) \right].
\]

(2.25)
Fig. 2.1 shows the numerical simulation of the solution with the choice of parameter as $a_1 = 0$, $a_2 = 1.0$, $a_3 = 1.0$, $a_4 = 4.0$, $k = 1.0$, $v = 1.0$, $\omega = 0.2904351808$, $B = 1.084352594$ and $A = 3.559271105$. This set of parameter values, which satisfies the constraint relations are chosen in order to perform numerical simulation.

### 2.1.2 Dark soliton solutions

The dark solitons are also known as topological solitons or simply topological defects. To start off, the hypothesis in this case is given by

$$P(x,t) = A \tanh[p(B(x-vt))], \quad (2.26)$$

where $A$ and $B$ are free parameters respectively, whose values are to be determined, while $v$ is the velocity of the wave and $p > 0$ for soliton to exist. Thus, on using Eq.(2.26), Eqs.(2.16) and (2.17) respectively reduce to

$$
(k - a_1 \omega^2 + a_3 \omega^3)A \tanh^p \tau + a_1 A B^2 v^2[p(p-1) \tanh^{p-2} \tau + p(p+1) \tanh^{p+2} \tau]
+ (a_2 - a_4 \omega)A^3 \tanh^{3p} \tau - 3 a_3 \omega A B^2 v^2[p(p-1) \tanh^{p-2} \tau + p(p+1) \tanh^{p+2} \tau]
- 2 A B^2 v^2 p^2 (a_1 - 3 a_3 \omega) \tanh^p \tau = 0, \quad (2.27)
$$
3 a_3 A p^2 B^3 v^3 \left[ \tanh^{p-3} \tau + \tanh^{p+3} \tau - \tanh^{p-1} \tau - \tanh^{p+1} \tau \right] \\
+ a_3 A p^3 B^3 v^3 \left[ \tanh^{p+3} \tau - \tanh^{p-3} \tau \right] + A p B \left[ \tanh^{p-1} \tau - \tanh^{p+1} \tau \right] \\
+ 2 a_5 A^4 p B v \left[ \tanh^{3 p+1} \tau - \tanh^{3 p-1} \tau \right] + 3 a_4 A^3 p B v \left[ \tanh^{3 p+1} \tau - \tanh^{3 p-1} \tau \right] \\
+ 3 a_3 A p^3 B^3 v^3 \left[ \tanh^{p-1} \tau - \tanh^{p+1} \tau \right] + 3 a_2 w^2 A p B v \left[ \tanh^{p-1} \tau - \tanh^{p+1} \tau \right] \\
+ 2 a_3 A p^3 B^3 v^3 \left[ \tanh^{p-1} \tau - \tanh^{p+1} \tau + \tanh^{p+3} \tau - \tanh^{p-3} \tau \right] \\
+ 2 a_1 w A p B v \left[ \tanh^{p+1} \tau - \tanh^{p-1} \tau \right] = 0, \tag{2.28}

where \( \tau = B(x - v t) \). From Eq.(2.27), setting the exponents \( 3 p \) and \( p + 2 \) leads \( p = 1 \). Now setting the coefficients of the linearly independent functions \( \tanh^{p+j} \) to zero for \( j = 0, \pm 2 \) leads to

\[ B = \pm \sqrt{-2 a_3 v (3 a_3 w^2 v - 2 a_1 w v + 1)} \frac{2 a_3 v^2}{2 a_3 v^2}, \tag{2.29} \]

\[ A = \pm \sqrt{3} \sqrt{\frac{3 a_3 w^2 v - 2 a_1 \omega v + 1}{(2 a_5 + 3 a_4) v}}. \tag{2.30} \]

Equation (2.29) shows that the soliton will exist if \( 2 a_3 v^3 (3 a_3 w^2 v - 2 a_1 \omega v + 1) < 0 \). It is to be noted that the coefficients of the linearly independent function \( \tanh^{p-2} \tau \) in Eq.(2.27) are automatically zero for \( p = 1 \). Also in Eq.(2.28), the linearly independent functions are \( \tanh^{p+j} \), where \( j = \pm 1, \pm 3 \). This provides

\[ \omega = a_1 v \pm \sqrt{v (a_1^2 v - 3 a_3)} \frac{3 a_3 v}{3 a_3 v}. \tag{2.31} \]

Hence, the one topological soliton solution of Eq.(2.12) is given by

\[ E(x, t) = A \tanh[B(x - v t)] e^{i(-k x + \omega t + \theta)}, \tag{2.32} \]

and the corresponding intensity of the dark soliton solution becomes

\[ |E(x, t)|^2 = \left( \frac{3(3a_3 \omega^2 v - 2 a_1 \omega v + 1)}{(2 a_5 + 3 a_4) v} \right) \tanh^2 \left[ \pm \sqrt{\frac{2a_1 \omega v - 3 a_3 \omega^2 v - 1}{2a_3 v^3}} (x - v t) \right]. \tag{2.33} \]

Fig. 2.2 shows the intensity profile of the dark solitary wave solution (2.33) for different model coefficients which satisfy the constraint condition. The choice of parameter for the numerical simulation of the dark soliton solution is as \( a_1 = 0.01, a_3 = -1.0, a_4 = 1.0, a_5 = -1.0, v = 1.0, \omega = -0.5806932250, B = 1.005790170 \) and \( A = 2.463672964. \)
It is worth noting that the existence of bright and dark soliton solutions given by Eqs.(2.24) and (2.32) depends on the specific nonlinear and dispersive features of the medium, which have to satisfy the parametric constraint conditions. From the expressions of soliton amplitude and inverse width, we see that as the higher-order dispersive terms, i.e. \(a_1\) and \(a_3\) increase, the amplitude of the solitary wave increases, while the pulse width gets narrower and with increase in the value of self-frequency and stimulated Raman scattering i.e. increase in the value of \(a_4\) and \(a_5\), the amplitude of the solitary wave decreases, while the pulse width increases. From the existence conditions of bright and dark solitons, we note that if the GVD and TOD are both neglected i.e. \(a_1 = a_3 = 0\), then both the bright and dark solitary waves disappear, but if TOD exists i.e. \(a_1 = 0\) and \(a_3 \neq 0\), then the bright and dark solitary waves do not disappear. 

Fig. 2.3(a) shows the intensity profile of the bright solitary wave solution of HNLS equation at different positions, \(x = -1, 0, 1\) for \(a_1 = 0, a_2 = 1.0, a_3 = 1.0, a_4 = 4.0, k = 1.0, v = 1.0\) under some parametric conditions and the corresponding contour plot is depicted in Fig.2.3(b). 

Fig.2.4(a) shows the intensity \(|E|^2\) of dark solitary wave solution and the corresponding contour plot is depicted in Fig.2.4(b) for the parameters \(a_1 = 0.01, a_3 = -1.0, a_4 = 1.0, a_5 = -1.0, v = 1.0\). The numerical simulation shows that these solitary wave solutions remain stable under small perturbations in parameters.
2.2 Integrals of motion of HNLS System

An intrinsic property of HNLS equation is that, in absence of higher-order terms ($a_3 = a_4 = a_5 = 0$), it possesses an infinite number of conserved quantities also known as integrals of motion. But in presence of higher order effects, NLSE does not hold conservation laws except for energy conservation, unless the higher order-terms are of unique type. Here, we computed, the first three conserved quantities of the HNLS equation for Hirota and Sasa-Satsuma cases [70].

(i) Hirota case [$a_3 : a_4 : (a_4 + a_5) = 1 : 6 : 0$]

\[ Q^1_H = \int_{-\infty}^{\infty} |E|^2 dt = \frac{2A^2}{Bv} = 4\frac{\sqrt{(k-a_1\omega^2 + a_3\omega^3)(3a_3\omega - a_1)}}{a_4\omega - a_2}, \]  \hspace{1cm} (2.34)

\[ Q^2_H = i \int_{-\infty}^{\infty} (E^* \partial E - \partial E^* E) dt = -\frac{4A^2\omega}{Bv} = -\frac{8\omega\sqrt{(k-a_1\omega^2 + a_3\omega^3)(3a_3\omega - a_1)}}{a_4\omega - a_2}, \]  \hspace{1cm} (2.35)

\[ Q^3_H = \int_{-\infty}^{\infty} [\partial E^* \partial E - (E^* E)^2] dt = \frac{2A^2}{3Bv}(B^2v^2 + 3\omega^2 - 2A^2) \]

\[ = -\frac{4}{3} \frac{\sqrt{(k-a_1\omega^2 + a_3\omega^3)}}{(a_4\omega - a_2)^2\sqrt{(3a_3\omega - a_1)}}(p\omega^4 + q\omega^3 + r\omega^2 + s\omega + z), \]  \hspace{1cm} (2.36)

where $p = 12a_3^2 - 10a_3a_4$, $q = (4a_4 - 16a_3)a_1 + 10a_2a_3$, $r = 4a_1(a_1 - a_2)$, $s = 12k(a_3 - a_4/12)$ and $z = 4k(a_1 - a_2/4)$. Note that for Hirota HNLS case the conserved charges turn out to be the same as the charges for the NLSE [1].

Figure 2.3: (a) The intensity plot of bright solitary wave $|E|^2$ at different positions $x = -1, 0, 1$ for parameter values $a_1 = 0, a_2 = 1.0, a_4 = 4.0, k = 1.0, v = 1.0$ (b) The corresponding contour plot in $x - t$ plane with the same parameters of (a).
Figure 2.4: (a) The intensity distribution plot of dark solitary wave for different model parameters at $x = 0$. For red curve 1 ($a_1 = 0.01, a_3 = -1.0, a_4 = 1.0, a_5 = -1.0, v = 1.0$), green curve 2 ($a_1 = 0.04, a_3 = -3.0, a_4 = 2.0, a_5 = -2.0, v = 1.0$), and blue curve 3 ($a_1 = 0.02, a_3 = -2.0, a_4 = 2.0, a_5 = -2.0, v = 1.03$.) (b) The corresponding contour plot in $x - t$ plane with the same parameters of curve 1.

(ii) Sasa-Satsuma case $[a_3 : a_4 : (a_4 + a_5) = 1 : 6 : 3]$

\[
Q_1^S = \int_{-\infty}^{\infty} |E|^2 dt = \frac{2A^2}{Bv} = \frac{4\sqrt{(k - a_1\omega^2 + a_3\omega^3)(3a_3\omega - a_1)}}{a_4\omega - a_2},
\]

(2.37)

\[
Q_2^S = 0,
\]

(2.38)

\[
Q_3^S = \int_{-\infty}^{\infty} [3\partial E^* \partial E - 6(E^*E)^2 - i(E^* \partial E - \partial E^*E)] dt
\]

\[
= \frac{2A^2}{Bv} (B^2v^2 + 3\omega^2 - 4A^2 + 2\omega)
\]

\[
= \frac{4\sqrt{(k - a_1\omega^2 + a_3\omega^3)}}{(a_4\omega - a_2)^2} (e\omega^4 + f\omega^3 + g\omega^2 + h\omega - m),
\]

(2.39)

where $e = (24a_2^2 - 10a_3a_4), f = (4a_4 - 32a_3)a_4 + 10(a_2 - 3a_4/5)a_3, g = 8a_1^2 + (2a_4 - 4a_2)a_1 + 6a_2a_3, h = 24k(a_3 - a_4/24) - 2a_1a_2$ and $m = 8k(a_1 - a_2/8)$.

The first conserved quantity is known as the energy or wave power while mathematically, it is known as the $L_2$ norm. The quantity $Q_1^H = Q_1^S$ corresponding to energy is conserved for all values of coefficients while $Q_2^H$ and $Q_3^H$ are conserved only for the specific Hirota and Sasa-Satsuma cases, respectively. In order to compute the conserved quantities, the 1-soliton solution given by Eq.(2.24) is used.
2.3 Periodic solutions of HNLS equation

To develop the periodic traveling wave solution of Eq.(2.12), we use gauge transformation

\[ E(x, t) = A(\xi) \exp[i(kx - \omega t)], \tag{2.40} \]

where \( \xi = \beta t - \lambda x + \xi_0 \) and \( \beta, k, \omega, \lambda \) are constants to be determined later. Substituting Eq.(2.40) into Eq.(2.13) which yields a set of coupled equations.

\[ \beta^2 (a_1 - 3a_3\omega) A_{\xi\xi} + (a_3\omega^3 - a_1\omega^2 - k) A + (a_2 - a_4\omega)A^3 = 0, \tag{2.41} \]

\[ \beta^3 a_3 A_{\xi\xi\xi} + (2\beta a_1\omega - 3\beta a_3\omega^2 + \lambda) A_{\xi} + \beta (3a_4 + 2a_5) A^2 A_{\xi} = 0. \tag{2.42} \]

Note that Eq.(2.42) has only first and third order derivatives. It is possible to integrate Eq.(2.42) and resulting integrated equation can be written as

\[ A_{\xi\xi} + \frac{2\beta a_1\omega - 3\beta a_3\omega^2 + \lambda}{\beta^3 a_3} A + \frac{3a_4 + 2a_5}{3\beta^2 a_3} A^3 = 0, \tag{2.43} \]

Comparing Eqs.(2.41) and (2.43), the free parameters \( k \) and \( \omega \) can be evaluated in the form:

\[ k = \frac{1}{a_3} \left[ \frac{1}{\beta} (3a_3\omega - 1)\lambda - 2\omega(a_1 - 2a_3\omega)^2 \right] \]

\[ = \frac{(3a_2a_3 - 3a_1a_4 - 2a_1a_5)(3a_3a_4 + a_1a_5)^2}{21a_3^2(a_4 + a_5)^3} + \frac{(a_1a_4 - 3a_2a_3)\lambda}{2\beta a_3(a_4 + a_5)}, \tag{2.44} \]

\[ \omega = \frac{3a_1a_4 + 2a_1a_5 - 3a_2a_3}{6a_3(a_4 + a_5)}. \tag{2.45} \]

Eqs.(2.41) and (2.43) are similar to the equation of motion of an anharmonic oscillator with potential

\[ U(A) = \frac{k + a_1\omega^2 - a_3\omega^3}{2\beta^2(a_1 - 3a_3\omega)} A^2 + \frac{a_4\omega - a_2}{4\beta^2(a_1 - 3a_3\omega)} A^4. \tag{2.46} \]

We can take energy integral of Eq.(2.12) and get the traveling wave equation

\[ (A_\xi)^2 = \frac{k + a_1\omega^2 - a_3\omega^3}{\beta^2(a_1 - 3a_3\omega)} A^2 + \frac{a_4\omega - a_2}{2\beta^2(a_1 - 3a_3\omega)} A^4 + 2C, \tag{2.47} \]

where \( C \) is an arbitrary constant of integration, which coincides with the energy value of the quartic anharmonic oscillator. Integrating Eq.(2.47) for different values of \( C \), we get the amplitude function \( A(\xi) \). The elliptical integral Eq.(2.47) admits 12 Jacobi doubly periodic wave solutions.

In order to construct cnoidal wave soliton solution, we choose the integration constant as
One can also derive other periodic solutions like
\[ C_{\text{if we take}} \]
\[ \text{Similarly if we choose} \quad C \]
\[ \text{where} \quad m \]
\[ \text{In order to construct the explicit dark soliton solutions with arbitrary nonlinear parameters, we} \]
\[ \text{The contributions from the nonlinear terms increase, wave period goes to infinity and wave profiles are well labeled by hyperbolic functions. Therefore, when} \quad m \rightarrow 1, \quad \text{JEFs degenerate into hyperbolic functions, i.e.,} \]
\[ \text{In order to construct the explicit dark soliton solutions with arbitrary nonlinear parameters, we} \]
\[ \text{The modulus parameter} \quad m \]
\[ \text{the integral} \]
\[ \text{One can also derive other periodic solutions like} \quad dn, \quad ns, \quad ds \quad \text{and} \quad cs \quad \text{functions. The parameter} \quad m \]
\[ \text{localized cnoidal waves. Period of the elliptical functions is given by} \quad 2K(m) \quad \text{for} \]
\[ \text{the modulus parameter} \quad m \]
\[ \text{Period of the elliptical functions is given by} \quad 2K(m) \quad \text{for} \]
\[ \text{With increase of localization parameter, the contribution from the nonlinear terms increases, wave period goes to infinity and wave profiles are well labeled by hyperbolic functions. Therefore, when} \quad m \rightarrow 1, \quad \text{JEFs degenerate into hyperbolic functions, i.e.,} \]
\[ \text{In order to construct the explicit dark soliton solutions with arbitrary nonlinear parameters, we} \]
\[ \text{One can also derive other periodic solutions like} \quad dn, \quad ns, \quad ds \quad \text{and} \quad cs \quad \text{functions. The parameter} \quad m \]
\[ \text{Similarly if we choose} \quad C \]
\[ \text{where} \quad m \]
\[ \text{In order to construct the explicit dark soliton solutions with arbitrary nonlinear parameters, we} \]
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\[ \text{Similarly if we choose} \quad C \]
\[ \text{One can also derive other periodic solutions like} \quad dn, \quad ns, \quad ds \quad \text{and} \quad cs \quad \text{functions. The parameter} \quad m \]
\[ \text{Similarly if we choose} \quad C \]
\[ \text{Similarly if we choose} \quad C \]
choose $m = 1$ and obtain the dark solitary wave solutions from Eq. (2.48) as

$$E(x, t) = \sqrt{\frac{(k + a_1 \omega^2 - a_3 \omega^3)}{(a_4 \omega - a_2)}} \tanh \left( \sqrt{\frac{(k + a_1 \omega^2 - a_3 \omega^3)}{2\beta^2(3a_3 \omega - a_1)}} \xi \right) \exp[i(kx - \omega t)]. \quad (2.51)$$

From this soliton solution we also calculate the peak power and pulse width as $P_0 = \frac{(k + a_1 \omega^2 - a_3 \omega^3)}{(a_4 \omega - a_2)}$ and $T_0 = \sqrt{\frac{2\beta^2(3a_3 \omega - a_1)}{(k + a_1 \omega^2 - a_3 \omega^3)}}$. The formation condition of the dark solitary wave is $\frac{k + a_1 \omega^2 - a_3 \omega^3}{\beta^2(a_1 - 3a_3 \omega)} < 0$ and $\frac{a_4 \omega - a_2}{2\beta^2(a_1 - 3a_3 \omega)} > 0$.

For $m = 1$, we obtain the bright solitary wave solutions from Eq. (2.49) as

$$E(x, t) = \sqrt{\frac{2(k + a_1 \omega^2 - a_3 \omega^3)}{(a_2 - a_4 \omega)}} \sech \left( \sqrt{\frac{(k + a_1 \omega^2 - a_3 \omega^3)}{\beta^2(a_1 - 3a_3 \omega)}} \xi \right) \exp[i(kx - \omega t)]. \quad (2.52)$$

The corresponding peak power and the pulse width are given as $P_0 = \frac{2(k + a_1 \omega^2 - a_3 \omega^3)}{(a_2 - a_4 \omega)}$ and $T_0 = \sqrt{\frac{\beta^2(a_1 - 3a_3 \omega)}{(k + a_1 \omega^2 - a_3 \omega^3)}}$. The formation condition of the bright solitary wave is $\frac{k + a_1 \omega^2 - a_3 \omega^3}{\beta^2(a_1 - 3a_3 \omega)} > 0$ and $\frac{a_4 \omega - a_2}{2\beta^2(a_1 - 3a_3 \omega)} < 0$. Note that the formation conditions of the bright and dark solitary waves are opposite to each other. These obtained dark and bright solitary waves solutions propagate in the normal ($GVD > 0$) as well as in anomalous dispersion regime ($GVD < 0$).

Now in order to find some more periodic solutions of HNLS equation, here we have used the projective Riccati equation method. In 1992, Conte et al. presented an indirect method to find more new solitary wave solutions of nonlinear PDEs that can be expressed as a polynomial in two elementary functions which satisfy a projective Riccati equation [71]. In [50] Yan developed Conte’s method and presented the general projective Riccati equation method. Several authors have used Yan’s technique to solve many nonlinear evolution equations [72, 41] and obtained some new solitary wave solutions to these equations. In order to obtain some more general exact solutions, we assume that the solutions of the nonlinear PDE is given by

$$u(x, t) = u(\xi) = \sum_{i=0}^{n} a_i (f(\xi))^i + \sum_{j=1}^{n} b_j (f(\xi))^{j-1}, \quad (2.53)$$

where the coefficients $a_i (i = 0, 1, 2, ..., n)$ and $b_j (j = 1, 2, ..., n)$ are constants to be determined. The functions $f$ and $g$ satisfy the following coupled Riccati equations

$$f'(\xi) = -f(\xi)g(\xi), \quad g'(\xi) = 1 - rf(\xi) - g^2(\xi), \quad (2.54a)$$

$$f'(\xi) = -f(\xi)g(\xi), \quad g'(\xi) = -1 + rf(\xi) - g^2(\xi), \quad (2.54b)$$

respectively. Also, we have obtained the first integrals of above Riccati equations as

$$g^2(\xi) = 1 - 2rf(\xi) + (b^2 - a^2 + r^2)f^2(\xi), \quad (2.55a)$$

$$g^2(\xi) = -1 + 2rf(\xi) + (b^2 + a^2 + r^2)f^2(\xi), \quad (2.55b)$$
respectively. The positive integer \( n \) can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE. Substituting Eq.(2.53) into ODE and making use of Eqs.(2.54a) and (2.55b), eliminating any derivative of \((f, g)\) and any power of \( g \) higher than one and setting the coefficients of the different powers of \( f \) and \( g \) to zero, we obtain a set of nonlinear algebraic equations with all parameters which are to be determined. The solution of the set of nonlinear algebraic equations provides all the constants \( a_i (i = 0, 1, 2, \ldots, n), b_j (j = 1, 2, \ldots, n), k, \omega \). Note that the ODEs (2.54a) and (2.54b) have the following special kind of solutions

\[
\begin{align*}
  f(\xi) &= \frac{1}{a \cosh(\xi) + b \sinh(\xi) + r}, \quad g(\xi) = \frac{a \sinh(\xi) + b \cosh(\xi)}{a \cosh(\xi) + b \sinh(\xi) + r}, \quad (2.56a) \\
  f(\xi) &= \frac{1}{a \cos(\xi) + b \sin(\xi) + r}, \quad g(\xi) = \frac{b \cos(\xi) - a \sin(\xi)}{a \cos(\xi) + b \sin(\xi) + r}, \quad (2.56b)
\end{align*}
\]

respectively. So we also obtain the multiple exact special solutions of HNLS equation by combining Eqs.(2.54a) and (2.55b) with Eq.(2.53). Let \( l = 2\beta a_4 - 3\beta a_3 \omega + \lambda \), \( m = \frac{3a_4 + 2a_5}{3\beta a_3} \) thus Eq.(2.43) becomes the Liénard equation \( A''(\xi) + IA(\xi) + mA^3(\xi) = 0 \). Now we suppose that the Liénard equation has a solution of the form (2.53) with \( f \) and \( g \) satisfying the coupled Riccati Eq.(2.54a) and the first integral (2.55a). The balance constant \( n = 1 \) is determined by the leading order term analysis. So we assume that

\[
A(\xi) = c_0 + c_1 f(\xi) + c_2 g(\xi), \quad (2.57)
\]

where the coefficients \( c_0, c_1, c_2 \) are constants to be determined and satisfy \( c_1^2 + c_2^2 \neq 0 \). Substituting Eq.(2.57) into the Liénard equation and making use of Eqs.(2.54a) and (2.55a), we obtain a set of nonlinear algebraic equations with \( c_0, c_1, c_2, r, m, \lambda \). On solving system of algebraic equations with MAPLE, we get the following three sets of solutions.

(i) When \( l = 1/2 \) and \( a, b, r \) are arbitrary constants satisfy \( r^2 + b^2 - a^2 > 0 \).

\[
c_0 = 0, \quad c_1 = \pm \sqrt{\frac{a^2 - b^2 - r^2}{2m}}, \quad c_2 = \pm \sqrt{\frac{-1}{2m}}. \quad (2.58)
\]

(ii) When \( l = -1 \) and \( a, b, m \) satisfy \((a^2 - b^2)m > 0\).

\[
c_0 = 0, \quad c_1 = \pm \sqrt{\frac{2(a^2 - b^2)}{m}}, \quad c_2 = 0, \quad r = 0. \quad (2.59)
\]

(iii) When \( l = 2 \) and \( m < 0 \).

\[
c_0 = 0, \quad c_1 = 0, \quad c_2 = \pm \sqrt{\frac{-2}{m}}, \quad r = 0. \quad (2.60)
\]
Incorporating these solutions back into Eq.(2.53) and using Eq.(2.40), we have obtained solitonic and periodic wave solutions of HNLS equation respectively as

\[ E_1(x, t) = \pm \sqrt{\frac{3^2 a_3 (a^2 - b^2 - r^2)}{2(3a_4 + 2a_5)}} \frac{1}{a \cosh(\xi) + b \sinh(\xi) + r} \]

\[ \pm \sqrt{\frac{-3^2 a_3}{2(3a_4 + 2a_5)}} \frac{a \sinh(\xi) + b \cosh(\xi)}{a \cosh(\xi) + b \sinh(\xi) + r} e^{i(kx - \omega t)}, \]  

(2.61)

where \( k, \omega, r, a, b \) are arbitrary constants such that \( \beta^2 a_3 (3a_4 + 2a_5) < 0, r^2 + b^2 - a^2 > 0, \)

\[ 2\beta a_1 \omega - 3\beta a_3 \omega^2 - \beta^3 a_3 + \lambda = 0. \]

\[ E_2(x, t) = \pm \sqrt{\frac{6^2 a_3 (a^2 - b^2)}{(3a_4 + 2a_5)}} \frac{1}{a \cosh(\xi) + b \sinh(\xi) + r} e^{i(kx - \omega t)}, \]  

(2.62)

where \( k, \omega, r, a, b \) are arbitrary constants such that \( \frac{\beta^2 a_3 (a^2 - b^2)}{(3a_4 + 2a_5)} > 0, 2\beta a_1 \omega - 3\beta a_3 \omega^2 + \beta^3 a_3 + \lambda = 0. \)

\[ E_3(x, t) = \pm \sqrt{\frac{-2^2 a_3}{(3a_4 + 2a_5)}} \frac{a \sinh(\xi) + b \cosh(\xi)}{a \cosh(\xi) + b \sinh(\xi) + r} e^{i(kx - \omega t)}, \]  

(2.63)

where \( k, \omega, r, a, b \) are arbitrary constants such that \( \frac{3a_4 + 2a_5}{\beta^2 a_3} < 0, 2\beta a_1 \omega - 3\beta a_3 \omega^2 - 2\beta^3 a_3 + \lambda = 0. \) In Fig. 2.5, we represent the intensity distribution of solitary wave solutions (2.62) and (2.63) for different choices of parameter values.

Figure 2.5: (a) The intensity distribution \(|E_2|^2\) for parameter values \( a = 2, b = 1, r = \lambda = 1, a_3 = 1, a_4 = 1, a_5 = 1, \beta = 0.1. \) (b) The intensity distribution \(|E_3|^2\) for parameter values \( a = 2, b = 1, r = \lambda = 1, a_3 = -1, a_4 = 1, a_5 = 1, \beta = 0.1 \)

for different choices of parameter values.

Finally, we assume that the Liénard equation has a solution of the form (2.57) with \( f \) and \( g \) satisfying the coupled Riccati equation (2.54b) and the first integral (2.55b). Substituting
When equations, we have $k, \omega, r, a, b$, respectively written as

c_0 = 0, c_1 = \pm \sqrt{-\frac{b^2 + a^2 - r^2}{2m}}, c_2 = \pm \frac{-1}{2m}. \quad (2.64)

(ii) When $l = 1$ and $m < 0$, where $a, b$ are arbitrary constants.
\[ c_0 = 0, c_1 = \pm \sqrt{-\frac{2(a^2 + b^2)}{m}}, c_2 = 0, r = 0. \quad (2.65) \]

(iii) When $l = -2$ and $m < 0$
\[ c_0 = 0, c_1 = 0, c_2 = \pm \sqrt{-\frac{2}{m}}, r = 0. \quad (2.66) \]

Thus the explicit and exact periodic wave solutions of HNLS equation for the above three cases respectively written as

\[ E_4(x, t) = \left[ \pm \sqrt{\frac{-3\beta^2 a_3(b^2 + a^2 - r^2)}{2(3a_4 + 2a_5)}} \right] \frac{1}{a \cos(\xi) + b \sin(\xi) + r} \]
\[ \quad \pm \sqrt{\frac{-3\beta^2 a_3}{2(3a_4 + 2a_5)}} \frac{b \cos(\xi) - a \sin(\xi)}{a \cos(\xi) + b \sin(\xi) + r} \right] e^{i(kx - \omega t)}, \quad (2.67) \]

where $k, \omega, r, a, b$ are arbitrary constants such that $\beta^2 a_3(3a_4 + 2a_5) < 0$, $b^2 + a^2 - r^2 > 0$, $2\beta a_1 \omega - 3\beta a_3 \omega^2 + \beta^3 a_3 + \lambda = 0$.

\[ E_5(x, t) = \left[ \pm \sqrt{\frac{-6\beta^2 a_3(a^2 + b^2)}{(3a_4 + 2a_5)}} \right] \frac{1}{a \cos(\xi) + b \sin(\xi) + r} \]
\[ \quad \frac{b \cos(\xi) - a \sin(\xi)}{a \cos(\xi) + b \sin(\xi) + r} \right] e^{i(kx - \omega t)}, \quad (2.68) \]

where $k, \omega, r, a, b$ are arbitrary constants such that $\frac{\beta^2 a_3(a^2 + b^2)}{(3a_4 + 2a_5)} < 0$, $2\beta a_1 \omega - 3\beta a_3 \omega^2 - \beta^3 a_3 + \lambda = 0$.

\[ E_6(x, t) = \left[ \pm \sqrt{\frac{-2\beta^2 a_3}{(3a_4 + 2a_5)}} \right] \frac{b \cos(\xi) - a \sin(\xi)}{a \cos(\xi) + b \sin(\xi) + r} \]
\[ \quad \frac{b \cos(\xi) - a \sin(\xi)}{a \cos(\xi) + b \sin(\xi) + r} \right] e^{i(kx - \omega t)}, \quad (2.69) \]

where $k, \omega, r, a, b$ are arbitrary constants such that $\frac{(3a_4 + 2a_5)}{\beta^2 a_3} < 0$, $2\beta a_1 \omega - 3\beta a_3 \omega^2 + 2\beta^3 a_3 + \lambda = 0$.

Further Eq.(2.43) can be reduced to elliptic like equation $f'' - \sigma f + \rho f^3 = 0$ with $\sigma = \frac{\lambda - 2\beta a_1 \omega + 3\beta a_3 \omega^2}{\beta^3 a_3}$ and $\rho = \frac{\lambda a_4 + 2a_5}{\beta^2 a_3}$, where $f(\xi) = A(\xi)$ and $f''(\xi) = \partial^2 A/\partial \xi^2$. Using the He’s semi-inverse method [73], the following variational formula has been established.

\[ J = \int_0^\infty \left[ \frac{1}{2}(f')^2 + \frac{\sigma}{2} f^2 - \frac{1}{4} \rho f^4 \right] d\xi. \quad (2.70) \]
Using Ritz method, we look for a bright solitary wave solution in the form $f = p \text{sech}(q\xi)$, where $p$ and $q$ are constants to be further determined. Substituting the value of $f$ into Eq.(2.70) results in

$$J = \int_{0}^{\infty} \left[ \frac{1}{2} p^2 q^2 \text{sech}^2(q\xi) \tan^2(q\xi) + \frac{\sigma}{2} (p \text{sech}(q\xi))^2 - \frac{1}{4} \rho (p \text{sech}(q\xi))^4 \right] d\xi,$$

$$= \frac{1}{2} p^2 q^2 \int_{0}^{\infty} \text{sech}^2(q\xi) \tan^2(q\xi) d\xi + \frac{\sigma}{2} p^2 \int_{0}^{\infty} \text{sech}^2(q\xi) d\xi - \frac{\rho}{4} p^4 \int_{0}^{\infty} \text{sech}^4(q\xi) d\xi,$$

$$= -\frac{\rho p^4}{6q} + \frac{\sigma p^2}{2q} + \frac{p^2 q}{6}.$$  (2.71)

Making $J$ stationary with respect to $p$ and $q$ results in

$$\frac{\partial J}{\partial p} = -\frac{2\rho p^3}{3q} + \frac{\sigma p}{q} + \frac{pq}{3} = 0,$$  (2.72)

$$\frac{\partial J}{\partial q} = \frac{\rho p^4}{6q^2} - \frac{\alpha p^2}{2q^2} + \frac{p^2}{6} = 0.$$  (2.73)

From these equations, we can easily obtain the following relations:

$$p = \left( \frac{2\sigma}{\rho} \right)^{1/2}, \quad q = \sqrt{\sigma}$$  (2.74)

By using Eq.(2.40), the bright solitary wave solution of HNLS equation is approximated as

$$E(x, t) = \left[ \left( \frac{2\sigma}{\rho} \right)^{1/2} \text{sech}(\sqrt{\sigma}(\beta t - \lambda x + \xi_0)) \right] \exp[i(kx - \omega t)]$$

$$= \sqrt{\frac{6\beta a_3 \omega^2 - 2\beta a_1 \omega - \lambda}{\beta(3a_4 + 2a_5)}} \text{sech} \left( \sqrt{\frac{3\beta a_3 \omega^2 - 2\beta a_1 \omega - \lambda}{\beta^3 a_3}} (\beta t - \lambda x + \xi_0) \right) \times \exp[i(kx - \omega t)],$$

which is the exact bright solitary wave solution of HNLS equation.

### 2.4 1- soliton solutions of the HNLS equation with time-dependent coefficients

The optical solitons in a Kerr law media is an important area of study which is governed by the NLSE. It studies the propagation of solitons through optical fibers for trans-continental and tran-oceanic distances. Thus the dynamics of solitons governed by the NLSE is well understood and well known in this context. When inhomogeneities of the media and nonuniformity of the boundaries are taken into account in various real physical situations, the variable-coefficient
NLSE provides more powerful and realistic models than their constant-coefficient counterparts. Recently, some attention is being paid to the NLSE with time-dependent coefficients \[74, 75\]. The governing envelope wave equation for femtosecond optical pulse propagation in inhomogeneous fiber takes the time dependent form

\[
i E_t + a_1(t)E_{xx} + a_2(t)|E|^2 E + i[a_3(t)E_{xxx} + a_4(t)(|E|^2 E)_x + a_5(t)E(|E|^2)_x] = 0. \tag{2.76}
\]

Here, the complex valued function \(E\) represents the wave profile where the independent variables are the spatial \(x\) and time \(t\) and \(a_1, a_2, a_3, a_4, a_5\) are distributed parameters which are all time dependent. In order to solve Eq.(2.76), it is first necessary to write the solution in the phase-amplitude format as

\[
E(x, t) = P(x, t)e^{i\phi(x, t)}, \tag{2.77}
\]

where \(P(x, t)\) is the amplitude portion while \(\phi(x, t)\) is the phase portion of the soliton. So, \(P(x, t)\) represents the pulse shape. The phase \(\phi(x, t)\) is written as

\[
\phi(x, t) = -\kappa(t)x + \omega(t)t + \theta(t), \tag{2.78}
\]

where \(\kappa\) is the frequency of the soliton and \(\omega\) is the wave number of the soliton while \(\theta\) is the phase constant. Since the problem is considered with time-dependent coefficients, it is therefore assumed that these parameters are also all time-dependent. Substituting Eqs. (2.77) and (2.78) into Eq.(2.76) and decomposing resultant equation into imaginary and real parts respectively yields

\[
P_t - 2a_1(t)\kappa P_x + a_3(t)\kappa P_{xxx} - 3a_3(t)\kappa^2 P_x + 3a_4(t)P^2 P_x + a_5(t)P P_{xx} = 0, \tag{2.79}
\]

\[
P[\kappa_t x - \omega - t\omega_t - \theta_t] + a_1(t)\kappa P_{xx} - a_1(t)\kappa^2 P + a_2(t)P^3 + 3a_3(t)\kappa P_{xx} - a_3(t)\kappa^3 P + a_4(t)\kappa P^3 = 0. \tag{2.80}
\]

The starting hypothesis for calculating topological 1-soliton solutions of the HNLS equation is the same as in Eq.(2.26) with all coefficients are time-dependent. For dark optical solitons, the choice for the function \(P(x, t)\) is

\[
P = A \tanh^p \tau, \tag{2.81}
\]

where \(\tau = B(t)[x - v(t)t]\). It needs to be noted that for the case of dark optical solitons the parameters \(A(t)\) and \(B(t)\) are known as free parameters. Thus, Eqs.(2.79) and (2.80) respectively
On setting the coefficient of $tanh$ parameter $A$ where $tanh$ dependent functions $C$, now from Eq.(2.82), equating the exponents of $tanh$ to zero yields

$$3(p - 1) tanh^{p-1} \tau = 0.$$ \hfill (2.82)

Equating the coefficients of linearly independent functions $tanh^{p+1} \tau$ and $tanh^{p+3} \tau$ function gives $3p + 1 = p + 3$ so that $p = 1$. Equating the coefficient of $tanh^{p} \tau$, one gets $\frac{dA}{dt} = 0$ gives $A(t) = A_0$, where $A_0$ is a constant.

On setting the coefficient of $\tau(tanh^{p-1} \tau + tanh^{p+1} \tau)$ to zero yields $\frac{dA}{dt} = 0$, so that the free parameter $B(t) = constant = B_0$. Again, from Eq.(2.82) we determine the soliton parameters by setting the corresponding coefficients of linearly independent functions $tanh^{p+j} \tau$ and $tanh^{3p+j} \tau$ for $j = \pm 3, 0, \pm 1$ to zero, such that

$$B(t) = \pm \frac{1}{6} \left( \frac{\pm a_5(t) + \sqrt{a_5^2(t) - 18a_3(t)a_4(t)}}{a_3(t)} \right) A_0,$$ \hfill (2.84)

where $C_1$ is an integration constant. From Eq.(2.83), equating the coefficients of linearly independent functions $tanh^{p+2} \tau$ and $tanh^{3p} \tau$ gives

$$B(t) = A_0 \sqrt{-\frac{a_2(t) + a_4(t)\kappa}{2a_1(t) + 6a_3(t)\kappa(t)}},$$ \hfill (2.86)
which forces the condition of existence of dark solitons to be \((2a_1(t) + 6a_3(t)\kappa(t))(a_2(t) + a_4(t)\kappa(t)) < 0\). From Eq.(2.83), we determine the wave number as

\[
\omega(t) = \frac{1}{t} \left[ \int (-a_3(t)\kappa(t)^3 + 2a_1(t)B_0^2 - a_1(t)\kappa(t)^2 - 6a_3(t)\kappa(t)B_0^2)dt + C_2 \right], \tag{2.87}
\]

where \(C_2\) is an integration constant which can be determined from initial conditions. Also, from Eq.(2.83) it is easy see that \(\frac{d\kappa}{dt} = 0\), \(\frac{d\theta}{dt} = 0\), so that \(\kappa = \kappa_0\), and \(\theta = \theta_0\) are assumed as constants. Equating the two values of \(B\) from Eqs.(2.84) and (2.86) gives

\[
\left( \pm a_5(t) + \sqrt{a_5^2(t) - 18a_3(t)a_4(t)} \right) = 6\sqrt{-\frac{a_2(t) + a_4(t)\kappa(t)}{2a_1(t) + 6a_3(t)\kappa(t)}}, \tag{2.88}
\]

which is a constraint condition for the dark solitons to exist. Finally, we can write the topological 1-soliton solution to the variable coefficient HNLS equation as follows:

\[
E(x,t) = A_0 \tanh[B_0(x - v(t)t)]e^{i[-\kappa_0 x + w(t)t + \theta_0]} . \tag{2.89}
\]

The intensity of topological soliton takes the form

\[
|E(x,t)|^2 = A_0^2 \tanh^2 \left[ A_0 \sqrt{-\frac{a_2(t) + a_4(t)\kappa_0}{2a_1(t) + 6a_3(t)\kappa_0}}(x - v(t)t) \right], \tag{2.90}
\]

where the time varying soliton parameters are the soliton frequency \(\omega(t)\) shown in Eq.(2.87), the velocity given by Eq.(2.85). Note that this solution exists provided that constraint equation between the model coefficients \(a_1(t), a_2(t), a_3(t), a_4(t)\) and \(a_5(t)\) that is given in Eq.(2.88) is satisfied.

### 2.5 Inhomogeneous NLS equation with time-dependent coefficients

It is well known that NLSE describes numerous nonlinear physical phenomena in the field of nonlinear science such as optical solitons in optical fibres, solitons in the mean-field theory of Bose-Einstein condensates and the rogue waves (RWs) in the nonlinear oceanography etc. The oceanic RWs can be, under the nonlinear theories of ocean waves, modeled by the dimensionless NLS equation

\[
iu_t + \frac{1}{2}u_{xx} + |u|^2u = 0, \tag{2.91}
\]

which describes the two-dimensional quasi-periodic deep-water trains in the lowest order in wave steepness and spectral width. In the present work, we extend the NLS Eq.(2.91) to the
inhomogeneous NLS equation with variable coefficients, including group velocity dispersion $\beta(t)$, linear potential $V(x, t)$, nonlinearity $g(t)$ and the gain/loss term $\gamma(t)$, in the form [76]

$$iu_t + \frac{\beta(t)}{2}u_{xx} + V(x, t)u + g(t)|u|^2u = i\gamma(t)u,$$

(2.92)

and find bright and dark 1-soliton solutions.

### 2.5.1 Bright soliton solutions

In order to solve Eq.(2.92), assume the soliton solution to Eq.(2.92) in the form

$$u(x, t) = \frac{A}{\cosh^p[B(x - vt)]} e^{i(-kx + \omega t + \theta)},$$

(2.93)

where $A$ is the amplitude of the soliton, $B$ is the inverse width and $v$ is the soliton velocity. Also $k$ represents the frequency, $\omega$ is the wave number, while $\theta$ is the phase. It is to be noted that since in Eq.(2.92), the coefficient of dispersion and nonlinearity are time dependent and not constants, the soliton parameters $A$, $B$, $k$, $\omega$ and $\theta$ may be time-dependent.

Substituting Eq.(2.93) into Eq.(2.92) and requiring that the imaginary and real parts of each term be separately equal to zero, we get

$$\frac{dA}{dt} \frac{1}{\cosh^p \tau} - A \frac{\tanh \tau}{\cosh^p \tau} \left\{ \frac{\tau dB}{B dt} - B \left( v + \frac{dv}{dt} \right) - \beta(t)pBk \right\} - \gamma(t) \frac{A}{\cosh^p \tau} = 0,$$

(2.94)

$$\left( \frac{d}{dt} \frac{dk}{\cosh^p \tau} - \omega - \frac{d\omega}{dt} - \frac{d\theta}{dt} \right) \frac{A}{\cosh^p \tau} + \frac{1}{2} \beta(t)(p^2B^2 - k^2) \frac{A}{\cosh^p \tau}$$

$$+ V(x, t) \frac{A}{\cosh^p \tau} - \frac{1}{2} \beta(t)B^2p(p + 1) \frac{A}{\cosh^{p+2} \tau} + g(t) \frac{A^3}{\cosh^{3p} \tau} = 0,$$

(2.95)

where $\tau = B(x - vt)$. From Eq.(2.95), equating the exponents $3p$ and $p + 2$ gives $p = 1$ and from Eq.(2.94) equating the coefficient of $1/\cosh^p \tau$, one gets

$$A(t) = A_0 e^{\int \gamma(t) dt},$$

(2.96)

where $A_0$ is an integration constant related to the initial pulse amplitude. Again from Eq.(2.94), setting the coefficient of $\tanh \tau/\cosh^p \tau$ to zero yields $\frac{dB}{dt} = 0$, so that $B(t) = \text{constant} = B_0$, which shows that the inverse width of the soliton $B(t)$ must be a constant and $B_0$ is the initial inverse width of soliton. Again from Eq.(2.94) setting the coefficient of $\tanh \tau/\cosh^p \tau$ to zero yields

$$v(t) = \frac{1}{t} \left\{ \int \beta(t)k \ dt + C_3 \right\},$$

(2.97)
where $C_3$ is an arbitrary constant. From Eq. (2.95), setting the coefficients of the linearly independent functions $1/\cosh^{p+2} \tau$ and $1/\cosh^{3p} \tau$ to zero yields

$$\beta(t)B^2 = g(t)A^2, \quad (2.98)$$

which, by virtue of Eq. (2.96), leads to

$$B(t) = A_0 e^{\int \gamma(t)dt} \sqrt{\frac{g(t)}{\beta(t)}}. \quad (2.99)$$

Since inverse width $B(t)$ is a constant, Eqs. (2.96) and (2.99) imply that the amplitude $A(t)$ depends on the ratio of nonlinear and dispersion coefficients and also modeled by the gain/loss coefficient $\gamma(t)$. From Eq. (2.99), one needs to have $\beta(t)g(t) > 0$. Now from Eq. (2.99), for $B(t)$ to be a constant, we have

$$g(t)e^{2\int \gamma(t)dt} = c\beta(t), \quad (2.100)$$

where $c \in R$ is a constant. From Eq. (2.95), equating the coefficients of $1/\cosh^p \tau$ to zero gives

$$\omega(t) = \frac{1}{t} \left\{ \int \frac{1}{2} A_0 e^{\int \gamma(t)dt} \left[ (B_0^2 - k^2)\beta(t) + 2V(x,t) \right] dt + C_4 \right\}, \quad (2.101)$$

while the other soliton parameters namely $k$ and $\theta$ remains constants. Thus, finally the 1-soliton solution of the time-dependent NLSE is given by Eq. (2.93), where inverse width $B(t)$ stays constant and the velocity $v(t)$ is given by Eq. (2.97). The other conditions that needs to hold for the solitons to exist is that $\beta(t)$ and $\gamma(t)$ are Riemann integrable and $\beta(t)g(t) > 0$. Fig. 2.6, shows a numerical solution of bright 1-soliton of the NLSE for $\gamma(t) = 0$, $\beta(t) = 1.0$, $g(t) = 2.0$, $k = 1$, $A_0 = 1.0$, $C_3 = 0$.

Figure 2.6: (a) The intensity of bright solitary wave $|u(x,t)|^2$ which satisfies the constraint $\beta(t)g(t) > 0$ for $\gamma(t) = 0$, $\beta(t) = 1.0$, $g(t) = 2.0$, $k = 1$, $A_0 = 1.0$, $C_3 = 0$ (b) The corresponding contour plot in $x − t$ plane with the same parameters as in (a)
2.5.2 Dark soliton solutions

The dark solitons that are also known as topological solitons are also supported by the inhomogeneous NLSE. For dark solitons with time-dependent coefficients the 1-soliton solution is given by

\[ u(x, t) = A \tanh[pB(x - vt)]e^{(kx + \omega t + \theta)}. \tag{2.102} \]

Here, as in the case of bright solitons with time-dependent coefficients, the soliton parameters are all time-dependent.

Inserting Eq.(2.102) into Eq.(2.92) and decomposing the resultant expression into the imaginary and real parts yields

\[
pA\left\{ -\tau \frac{dB}{dt} + B\left( k\beta(t) + v + t \frac{dv}{dt} \right) \right\} \left[ \tanh[p+1] - \tanh[p-1] \right] \\
- \tanh[p] \left( -\frac{dA}{dt} + \gamma(t)A \right) = 0,
\]

(2.103)

\[
A \left[ \frac{1}{2} p\beta(t)B^2 \left\{ (p + 1) \tanh[p+2] + (p - 1) \tanh[p-2] \right\} + g(t)A^2 \right] \tanh[p] \\
- \left\{ \frac{d\omega}{dt} + \omega + \frac{d\theta}{dt} + x \frac{dk}{dt} - V(x, t) + \left( \frac{1}{2} k^2 + p^2 B^2 \right) \beta(t) \right\} \tanh[p] = 0.
\]

(2.104)

By equating the highest exponents of \( \tanh[p+2] \) and \( \tanh[3p] \) terms in Eq.(2.104), one gets \( p = 1 \). From Eq.(2.103), equating the linearly independent functions of \( \tanh[p] \) to zero, we get \( \frac{dA(t)}{dt} = \gamma(t)A(t) \) which gives

\[ A(t) = A_0e^{\int \gamma(t)dt}, \tag{2.105} \]

which shows that unlike their bright counterparts, the amplitude of dark soliton also depends on the nonlinear gain/loss coefficient \( \gamma(t) \). Now setting the coefficients of \( \tanh[p+1] \) in Eq.(2.103) gives

\[ v(t) = \frac{1}{t} \left\{ \int -\beta(t)k \, dt + C_5 \right\}, \tag{2.106} \]

where \( C_5 \) is the integration constant. Equating the coefficient of \( \tau \tanh[p+1] \) in Eq.(2.103) gives \( B(t) = \text{constant} \). Also from Eq.(2.104), setting the coefficients of the linearly independent functions \( \tanh[p+2] \) and \( \tanh[3p] \) to zero yields the amplitude-width relationship given by

\[ B(t) = A_0e^{\int \gamma(t)dt} \sqrt{\frac{g(t)}{\beta(t)}}, \tag{2.107} \]

which shows that it is necessary to have

\[ g(t)\beta(t) < 0, \tag{2.108} \]
for the soliton solutions to exist. Again from Eq.(2.104), equating the coefficient of \( \tanh^p \tau \) we obtain

\[
\omega(t) = -\frac{1}{t} \left\{ \frac{1}{2} A_0 e^{\int \gamma(t) dt} \left[ (2B_0^2 + k^2)\beta(t) + 2V(x, t) \right] dt + C_6 \right\},
\]

(2.109)

under the assumption that \( k \) and \( \theta \) remain constant. Finally, the dark soliton solution to Eq.(2.92) is given by

\[
u(x, t) = A_0 e^{\int \gamma(t) dt} \tanh \left[ A_0 e^{\int \gamma(t) dt} \int \frac{g(t)}{\beta(t)} (x - vt) \right] e^{i(-kx + \omega t + \theta)},
\]

(2.110)

where the relation between the parameters \( A \) and \( B \) is given by Eq.(2.107) and the wave number and the velocity are given by Eqs.(2.109) and (2.106). In this case also, it is seen that the solitons will exist for \( g(t)\beta(t) < 0 \) which is guaranteed from Eq.(2.107). The numerical solution of dark 1-soliton of the NLSE is drawn in Fig.2.7 for \( \gamma(t) = 0, \beta(t) = -1.0, g(t) = 2.0, k = 1, A_0 = 1.0, C_5 = 0 \).

Figure 2.7: (a) The intensity of dark solitary wave \( |u(x, t)|^2 \) which satisfies the constraint \( \beta(t)g(t) < 0 \) for \( \gamma(t) = 0, \beta(t) = -1.0, g(t) = 2.0, k = 1, A_0 = 1.0, C_5 = 0 \) (b) The corresponding contour plot in \( x - t \) plane with the same parameters as in (a)

2.6 Variable coefficient nonlinear Schrödinger equation

The varying dispersion and Kerr nonlinearity are of practical importance in a real optical-fiber transmission system with the consideration of the inhomogeneities resulting from such factors as the variation in the lattice parameters of the fiber media and fluctuation of the fiber’s diameters [77]. When the inhomogeneities of media and nonuniformity of boundaries are taken into account in various real physical situations, the variable-coefficient NLSE provides more powerful and realistic models than their constant-coefficient counterparts. Therefore, investigations on the variable-coefficient NLS-type models for optical fibers have become desirable [78].
The goal of this work is to identify traveling wave solutions of the one dimensional generalized NLSE, by utilizing the homogeneous balance principle and the variable F-expansion technique and to extend the analysis to include the solitary wave solutions.

Here, our emphasis is on the following variable-coefficient NLS model [78]:

\[ i \left[ \Psi_z + \frac{\alpha(z)}{2} \Psi + \sigma(z) \Psi_t \right] - \frac{1}{2} \beta(z) \Psi_{tt} + \gamma(z) |\Psi|^2 \Psi = 0, \] (2.111)

where \( \Psi \) is a complex function of \( z \) and \( t \). The function \( \alpha(z) \) is the linear attenuation coefficient, \( \sigma(z) \), \( \beta(z) \), and \( \gamma(z) \) are the inhomogeneous functions, respectively, related to the intermodal dispersion, group velocity dispersion (GVD), and nonlinear loss or gain. In practical applications, Eq.(2.111) and their various forms are of considerable important for the description of amplification, absorption, compression and broadening of optical solitons in inhomogeneous optical fiber systems and also for the study of stable transmission of solitons [79, 80].

### 2.7 The solution procedure

Let us start our analysis by writing the complex field \( \Psi \) of Eq.(2.111) in terms of its amplitude and phase as

\[ \Psi(z, t) = A(z, t) \exp[i \rho(z, t)]. \] (2.112)

On substituting Eq.(2.112) into Eq.(2.111), the following coupled equations are obtained

\[ [\rho_z + \sigma(z) \rho_t] A + \frac{1}{2} \beta(z) [A_{tt} - \rho_t^2 A] = \gamma(z) A^3, \] (2.113)

\[ A_z + \sigma(z) A_t - \frac{1}{2} \beta(z) [2A_t \rho_t + \rho_{tt}] + \frac{1}{2} \alpha(z) A = 0. \] (2.114)

Now apply the balancing procedure and the F-expansion technique [43], with some modifications to account for the higher order nonlinearities. To search traveling wave solutions to Eqs.(2.113) and (2.114), assume functions \( A \) and \( \rho \) to be of the form

\[ A(z, t) = f_1(z) + f_2(z) F(\xi) + f_3(z) F(\xi)^{-1}, \] (2.115)

\[ \xi = p(z)t + q(z), \] (2.116)

\[ \rho(z, t) = \kappa(z)t^2 + \varphi(z)t + \Omega(z), \] (2.117)

where \( f_1, f_2, f_3, p, q, \kappa, \varphi \) and \( \Omega \) are parameters of function of \( z \) and \( p(z) \) and \( q(z) \) are related to pulse width and group velocity, respectively. \( F \) is a Jacobi elliptic function (JEF), which satisfy
the following general first and second order nonlinear ODEs

\[
\left( \frac{dF}{d\xi} \right)^2 = c_0 + c_2 F^2 + c_4 F^4, \tag{2.118}
\]

\[
\frac{d^2 F}{d\xi^2} = c_2 F + 2c_4 F^3, \tag{2.119}
\]

where \(c_0, c_2\) and \(c_4\) are real constants related to the square of elliptic modulus \(m\) of JEFs (see Appendix I). Also, we assume the phase \(\rho(z,t)\) has a quadratic form i.e. there exists the chirped term \(\kappa(z)\). Substituting Eqs.(2.115)-(2.117) into Eqs.(2.113) and (2.114) and requiring that \(t^s F^n (s = 0, 1, 2; n = 0, 1, 2, 3)\) and \(\sqrt{c_0 + c_2 F^2 + c_4 F^4}\) of each term be separately equal to zero, a system of algebraic and first order ordinary differential equations amongst the parameters becomes as

\[
f_i \left( \frac{d\varphi}{dz} + 2\sigma(z) \kappa - 2\beta(z) \varphi \right) = 0, \tag{2.120a}
\]

\[
f_i \left( \frac{d\kappa}{dz} - 2\beta(z) \kappa^2 \right) = 0, \tag{2.120b}
\]

\[
6\gamma(z) f_i f_j^2 = 0, \tag{2.120c}
\]

\[
f_3 [\gamma(z) f_i^2 - \beta(z) p^2 c_0] = 0, \tag{2.120d}
\]

\[
f_2 [\gamma(z) f_i^2 - \beta(z) p^2 c_4] = 0, \tag{2.120e}
\]

\[
f_1 \left( \frac{d\Omega}{dz} + \sigma(z) \varphi - \frac{\beta(z)}{2} \varphi^2 - 6f_2 f_3 - \gamma(z) f_1^2 \right) = 0, \tag{2.120f}
\]

\[
f_j \left( \frac{d\Omega}{dz} + \sigma(z) \varphi - \frac{\beta(z)}{2} \varphi^2 - 3\gamma(z) f_1^2 - 3\gamma(z) f_2 f_3 + \frac{\beta(z)}{2} p^2 c_2 \right) = 0, \tag{2.120g}
\]

\[
\left( \frac{df_i}{dz} + \frac{\alpha(z)}{2} - \beta(z) \kappa \right) = 0, \tag{2.120h}
\]

\[
f_j \left( \frac{dq}{dz} + \sigma(z) p - \beta(z) \varphi \right) = 0, \tag{2.120i}
\]

\[
f_j \left( \frac{dp}{dz} - 2\beta(z) p \kappa \right) = 0, \tag{2.120j}
\]

where \(i = 1, 2, 3\) and \(j = 2, 3\). By solving Eqs.(2.120a)-(2.120j) self consistently, one may obtain a set of conditions on the coefficients and parameters necessary for Eq.(2.111) to have exact periodic wave solutions.

### 2.8 Various analytic solutions

As such, it is difficult to find the solutions of Eqs.(2.120a)-(2.120j). So, in what follows, we solve these equations under some parametric conditions.
Case (1). When $f_1 = f_3 = \kappa(z) = 0$ and $\sigma(z)$, $\beta(z)$ and $\alpha(z)$ are arbitrary functions of the propagation distance $z$.

For this case, the solutions of Eqs.(2.120a)-(2.120j) are conveniently expressed as

\[
p(z) = p_0, \varphi(z) = \varphi_0, f_2(z) = f_{20} \exp\left( -\frac{1}{2} \int_0^z \alpha(z) dz \right), q(z) = \int_0^z p_0[\beta(z)\varphi_0 - \sigma(z)] dz + q_0, \Omega(z) = \int_0^z [\beta(z)(\varphi_0^2 - p_0^2 c_2) - 2\sigma(z)\varphi_0] dz + \Omega_0, \gamma(z) = \frac{\beta(z)p_0^2 c_4 \exp[f_0^2 \alpha(z) dz]}{f_{20}^2}.
\]

Hence the solution of the generalized NLSE becomes

\[
\Psi(z, t) = f_{20} \exp\left( -\frac{1}{2} \int_0^z \alpha(z) dz \right) F(\xi) e^{i[\varphi_0 t + \Omega(z)]}. \tag{2.121}
\]

Case (2). When $f_1 = \kappa = 0$ and $\sigma(z)$, $\beta(z)$ and $\alpha(z)$ are arbitrary functions of $z$.

Under this restriction, the solutions of Eqs.(2.120a)-(2.120j) are written as

\[
p(z) = p_0, \varphi(z) = \varphi_0, f_2(z) = f_{20} \exp\left( -\frac{1}{2} \int_0^z \alpha(z) dz \right),
\]

\[
f_3(z) = \pm \sqrt{\frac{\alpha_0}{c_4}} f_{20} \exp\left( -\frac{1}{2} \int_0^z \alpha(z) dz \right), q(z) = \int_0^z p_0[\beta(z)\varphi_0 - \sigma(z)] dz + q_0, \Omega(z) = \frac{1}{2} \int_0^z [6\beta(z)p_0^2 \sqrt{\alpha_0 c_4} + (\varphi_0^2 - p_0^2 c_2)\beta(z) - 2\sigma(z)\varphi_0] dz + \Omega_0, 
\]

\[
\gamma(z) = \frac{\beta(z)p_0^2 c_4 \exp[f_0^2 \alpha(z) dz]}{f_{20}^2}, \text{ thus we obtain}
\]

\[
\Psi(z, t) = f_{20} \exp\left( -\frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ F(\xi) \pm \sqrt{\frac{\alpha_0}{c_4} F(\xi)} \right] e^{i[\varphi_0 t + \Omega(z)]}. \tag{2.122}
\]

Case (3). When $f_1 = f_3 = 0$ and $\sigma(z)$, $\kappa(z)$ and $\alpha(z)$ are arbitrary functions of $z$.

For this assumption, the solutions of Eqs.(2.120a)-(2.120j) are expressed as

\[
p(z) = p_0 k(z), \varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z) dz] k(z), f_2(z) = f_{20} \exp\left[ \frac{1}{2} \int_0^z \left( \frac{\alpha_0}{k} - \alpha \right) dz \right],
\]

\[
q(z) = \frac{1}{2} p_0 \int_0^z k(z)[(\varphi_0 - 2 \int_0^z \sigma(z) dz] k(z) - 2\sigma(z)] dz + q_0, \Omega(z) = \frac{1}{4} \int_0^z \left[ 4(f_0^2 \sigma(z) dz)^2 - 4(f_0^2 \sigma(z) dz)\varphi_0 + \varphi_0^2 - p_0^2 c_2 k(z) - 4\sigma(z)(\varphi_0 - 2 \int_0^z \sigma(z) dz] k(z) \right] dz + \Omega_0, \beta(z) = \frac{\kappa_0}{2\sqrt{\beta(z)}}, 
\]

\[
\gamma(z) = \frac{1}{2} \frac{\kappa_0 p_0^2 c_4 \exp[f_0^2 \alpha(z) dz]}{f_{20}^2} \left[ \int_0^z \frac{\alpha_0(z) - \kappa_0(z)}{\kappa(z)} dz \right].
\]

Therefore one obtains

\[
\Psi(z, t) = f_{20} \exp\left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_0}{k} - \alpha \right) dz \right] \left[ F(\xi) \pm \sqrt{\frac{\alpha_0}{c_4} F(\xi)} \right] e^{i[\kappa_0(z) \xi^2 + \varphi(z) t + \Omega(z)]}. \tag{2.123}
\]

Case (4). When $f_1 = 0$ and $\sigma(z)$, $\kappa(z)$ and $\alpha(z)$ are arbitrary functions $z$.

For this case, we obtain

\[
p(z) = p_0 k(z), \varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z) dz] k(z), f_2(z) = f_{20} \exp\left[ \frac{1}{2} \int_0^z \left( \frac{\alpha_0}{k} - \alpha \right) dz \right],
\]

\[
f_3(z) = \pm \sqrt{\frac{\alpha_0}{c_4}} f_{20} \exp\left[ \int_0^z \left( \frac{\alpha_0}{k} - \alpha \right) dz \right], q(z) = \frac{1}{2} p_0 \int_0^z k(z)[(\varphi_0 - 2 \int_0^z \sigma(z) dz] k(z) - 2\sigma(z)] dz + q_0, \Omega(z) = \frac{4\kappa^2(z)}{4\kappa(z)} f_{20} \sigma(z) dz[f_0^2 \sigma(z) dz - \varphi_0 + (\varphi_0^2 - p_0^2 c_2)\kappa(z) - 6\kappa_0 \sqrt{\alpha_0 c_4 + 4\kappa(z)} \Omega_0, 
\]

\[
\beta(z) = \frac{\kappa_0}{2\sqrt{\beta(z)}}, \gamma(z) = \frac{\beta(z) p_0^2 c_4 \exp[f_0^2 \alpha(z) dz]}{f_{20}^2} \left[ \int_0^z \frac{\alpha_0(z) - \kappa_0(z)}{\kappa(z)} dz \right].
\]
Finally we derive the solution as

$$\Psi(z, t) = f_{20}\exp\left(-\frac{1}{2} \int_0^z \alpha(z)dz\right)sn(\xi) e^{i[\varphi_0 t + \int_0^z (\beta(z)\varphi_0 + p_0^2 z + \sigma(z)\varphi_0)|dz + \Omega_0]}$$

(2.124)

where \( p_0, \varphi_0, \Omega_0, g_0 \) are all arbitrary integration constants. From Eqs.(2.121)-(2.124), we can see that chirped term \( \kappa(z) \), which disappears in cases (1) and (2) \( \kappa(z) = 0 \), exists in cases (3) and (4). The sign “±” in resultant expressions means that all possible combination of “+” and “−” can be taken.

In all four cases the nonlinearity coefficient \( \gamma(z) \) is expressed in terms of other coefficients \( \beta(z) \) and \( \alpha(z) \). Note that in cases (3) and (4), the nonlinear gain/loss parameter \( \gamma(z) \) is affected by the presence of chirped term. The form of solutions depends on what JEFs and Weierstrass elliptic functions (WEFs) are utilized. Appendices I and II list some of the JEFs and WEFs that may appear in the solutions. As long as one chooses the constants according to the relations listed in Appendix I and substitutes the appropriate \( F(\xi) \) into Eq.(2.121)-(2.124), one obtains the exact periodic traveling wave solutions to the generalized NLSE. Here the parameter \( m \), describes the degree of the energy localization of cnoidal waves varies between 0 and 1. Thus from the general solutions (2.121)-(2.124), we can develop a large number of families of solutions by choosing \( c_0, c_2, c_4 \) from Appendix I and \( q_2 \) and \( q_3 \) from Appendix II. Some families of such solutions are derived as follows.

**Family 1.** When \( c_0 = 1, c_2 = -(1 + m^2) \) and \( c_4 = m^2 \), then periodic solutions and chirped and chirpless dark solitary wave solutions are obtained from Eqs.(2.121)-(2.124) respectively.

**Case (1.1):**

$$\Psi_{11} = f_{20}\exp\left(-\frac{1}{2} \int_0^z \alpha(z)dz\right)sn(\xi) e^{i[\varphi_0 t + \int_0^z (\beta(z)\varphi_0 + p_0^2 z + \sigma(z)\varphi_0)|dz + \Omega_0]}$$

(2.125)

When \( m \to 1, \text{snn}(\xi) \to \tanh(\xi) \), the chirp-less dark solitary wave solution is given by

$$\Psi'_{11} = f_{20}\exp\left(-\frac{1}{2} \int_0^z \alpha(z)dz\right) \tanh(\xi) e^{i[\varphi_0 t + \int_0^z (\beta(z)\varphi_0 + p_0^2 z + \sigma(z)\varphi_0)|dz + \Omega_0]}$$

(2.126)

where \( \xi = p_0 t + \int_0^z p_0(\beta(z)\varphi_0 - \sigma(z))dz + q_0 \) and \( \gamma(z) = \frac{\beta(z)p_0^2}{f_0} \int_0^z \alpha(z)dz \).

**Case (1.2):**

$$\Psi_{12} = f_{20}\exp\left(-\frac{1}{2} \int_0^z \alpha(z)dz\right) \left[ sn(\xi) \pm \frac{1}{m} ns(\xi) \right] e^{i[\varphi_0 t + \Omega(z)]}$$

(2.127)

When \( m \to 1 \), the solitary wave solution is written as

$$\Psi'_{12} = f_{20}\exp\left(-\frac{1}{2} \int_0^z \alpha(z)dz\right) \left[ \tanh(\xi) \pm \coth(\xi) \right] e^{i[\varphi_0 t + \Omega(z)]}$$

(2.128)
where \( \xi = p_0 t + \int_0^z p_0 [\beta(z) \varphi_0 - \sigma(z)]dz + q_0 \), \( \Omega(z) = \frac{1}{2} \int_0^z [\pm 6 \beta(z)p_0^2 + (\varphi_0^2 + 2p_0^2)\beta(z) - 2\sigma(z)\varphi_0]dz + \Omega_0 \) and \( \gamma(z) = \frac{\beta(z)p_0^2 \exp[\int_0^z \alpha(z)dz]}{f_{20}} \).

**Case (1.3):**

\[
\Psi_{13} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{K_z}{K} - \alpha \right) dz \right] \sin(\xi) e^{i[k(z)t + \varphi(z)t + \Omega(z)]},
\]

(2.129)

When \( m \to 1 \), we obtain the chirped dark solitary wave solution as

\[
\Psi_{13} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{K_z}{K} - \alpha \right) dz \right] \tanh(\xi) e^{i[k(z)t + \varphi(z)t + \Omega(z)]},
\]

(2.130)

where \( \xi = p_0 \kappa(z)t + \frac{1}{2}p_0 \int_0^z \kappa(z)[(\varphi_0 - 2 \int_0^z \sigma(z)dz)K_z - 2\sigma(z)]dz + q_0 \), \( \varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z)dz]K_z, \Omega(z) = \frac{1}{4} \int_0^z \left\{ [4(\int_0^z \sigma(z)dz)^2 - 4(\int_0^z \sigma(z)dz)\varphi_0 + \varphi_0^2 + 2p_0 \int_0^z \sigma(z)dzK(z)]dz + \Omega_0, \beta_2(z) = \frac{K_z}{2\sigma(z)} \right\}
\]

and \( \gamma(z) = \frac{1}{2} \frac{\kappa(z)p_0^2 \exp[\int_0^z \alpha(z)dz]}{f_{20}} \left[ \int_0^z \frac{\alpha(z) - K_z}{\kappa(z)} dz \right]. \)

**Case (1.4):**

\[
\Psi_{14} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{K_z}{K} - \alpha \right) dz \right] \left( \sin(\xi) \pm \frac{1}{m} \cos(\xi) \right) e^{i[k(z)t + \varphi(z)t + \Omega(z)]},
\]

(2.131)

When \( m \to 1 \), the chirped solitary wave solution is

\[
\Psi_{14} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{K_z}{K} - \alpha \right) dz \right] \left( \tanh(\xi) \pm \coth(\xi) \right) e^{i[k(z)t + \varphi(z)t + \Omega(z)]},
\]

(2.132)

where \( \xi = p_0 \kappa(z)t + \frac{1}{2}p_0 \int_0^z \kappa(z)[(\varphi_0 - 2 \int_0^z \sigma(z)dz)K_z - 2\sigma(z)]dz + q_0 \), \( \varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z)dz]K_z, \Omega(z) = \frac{4\kappa^2(z)}{f_{20}^2} \left[ \frac{\int_0^z \sigma(z)dzdz - \varphi_0 + \varphi_0^2 - 2p_0 \int_0^z \sigma(z)dzK(z)]dz + \Omega_0, \beta(z) = \frac{\kappa(z)p_0^2 \exp[\int_0^z \alpha(z)dz]}{f_{20}} \right] \]

and \( \gamma(z) = \frac{\beta(z)\kappa^2(z)p_0^2 \exp[\int_0^z \alpha(z)dz]}{f_{20}} \left[ \int_0^z \frac{\alpha(z) - K_z}{\kappa(z)} dz \right]. \)

**Family 2.** When \( c_0 = 1 - m^2 \), \( c_2 = 2m^2 - 1 \) and \( c_4 = -m^2 \), then cnoidal waves and chirped and chirpless bright solitary wave solutions are obtained as

**Case (2.1):**

\[
\Psi_{21} = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z)dz \right) \sin(\xi) e^{i[p_0 t + \int_0^z (\beta(z)\varphi_0^2 - \sigma(z))dz + q_0].
\]

(2.133)

When \( m \to 1 \), \( \sin(\xi) \to \text{sech}(\xi) \), the chirpless bright solitary wave solution is given by

\[
\Psi_{21} = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z)dz \right) \text{sech}(\xi) e^{i[p_0 t + \int_0^z (\beta(z)\varphi_0^2 - \sigma(z))dz + q_0],
\]

(2.134)

where \( \xi = p_0 t + \int_0^z p_0 [\beta(z)\varphi_0 - \sigma(z)]dz + q_0 \) and \( \gamma(z) = - \frac{\beta(z)p_0^2 \exp[\int_0^z \alpha(z)dz]}{f_{20}} \).

**Case (2.2):**

\[
\Psi_{22} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{K_z}{K} - \alpha \right) dz \right] \sin(\xi) e^{i[k(z)t + \varphi(z)t + \Omega(z)]},
\]

(2.135)
When $m \to 1$, the chirped bright solitary wave solution is written as

$$
\mathcal{\Psi}_2 = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \sech(\xi) e^{i[k(z)t^2 + \varphi(z)t + \Omega(z)]}, \tag{2.136}
$$

where $\xi = p_0\kappa(z)t + \frac{1}{2} p_0 \int_0^z \kappa(z) [(\varphi_0 - 2 \int_0^z \sigma(z)dz) \kappa_z - 2 \sigma(z)]dz + q_0$,

$$
\varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z)dz] \kappa(z), \quad \Omega(z) = \frac{1}{2} \int_0^z \left\{ \left[4(\int_0^z \sigma(z)dz)^2 - 4(\int_0^z \sigma(z)dz)\varphi_0 + \varphi_0^2 \right] k_z - 4\sigma(z)(\varphi_0 - 2 \int_0^z \sigma(z)dz)k(z) \right\}dz + \Omega_0,
$$

and $\gamma(z) = -\frac{1}{2} \frac{k_z}{f_{20}} \exp \left[ \int_0^z \frac{\alpha(z) - \kappa(z)}{\kappa(z)} dz \right]$.

Family 3. When $c_0 = \frac{m^2 c_4^2}{(m^2 + 1)c_4}$, $c_2 < 0$ and $c_4 > 0$, then cnoidal waves solutions are given as Case (3.1):

$$
\mathcal{\Psi}_3 = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z)dz \right) \left[ \sqrt{-\frac{m^2 c_2}{(m^2 + 1)c_4}} \ \text{sn} \left( \sqrt{-\frac{c_2}{c_4}} \xi \right) \right] e^{i(\varphi_0 t + \Omega(z))}, \tag{2.137}
$$

When $m \to 1$, the unchirped dark solitary wave solution is written as

$$
\mathcal{\Psi}_4 = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z)dz \right) \left[ \sqrt{-\frac{c_2}{2c_4}} \ \tanh \left( \sqrt{-\frac{c_2}{2}} \xi \right) \right] e^{i(\varphi_0 t + \Omega(z))}, \tag{2.138}
$$

where $\xi = p_0 + \int_0^z (\beta(z) - \sigma(z))dz + q_0$, $\Omega(z) = \int_0^z [\beta(z)(\varphi_0^2 - p_0^2 c_2) - 2\sigma(z)\varphi_0]dz + \Omega_0$ and $\gamma(z) = \frac{\beta(z)p_0 c_4 \exp[\int_0^z \alpha(z)dz]}{f_{20}}$.

Case (3.2):

$$
\mathcal{\Psi}_3 = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z)dz \right) \left[ \sqrt{-\frac{m^2 c_2}{(m^2 + 1)c_4}} \ \text{sn} \left( \sqrt{-\frac{c_2}{c_4}} \xi \right) \right] + \sqrt{-\frac{m^2 c_2}{(m^2 + 1)c_4}} \ \text{ns} \left( \sqrt{-\frac{c_2}{c_4}} \xi \right) \right] e^{i(\varphi_0 t + \Omega(z))}, \tag{2.139}
$$

When $m \to 1$, the solitary wave solution is given by

$$
\mathcal{\Psi}_4 = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z)dz \right) \left[ \sqrt{-\frac{c_2}{2c_4}} \ \tanh \left( \sqrt{-\frac{c_2}{2}} \xi \right) \right] + \sqrt{-\frac{c_2}{8c_4}} \ \text{coth} \left( \sqrt{-\frac{c_2}{2}} \xi \right) \right] e^{i(\varphi_0 t + \Omega(z))}, \tag{2.140}
$$

where $\xi = p_0 + \int_0^z (\beta(z) - \sigma(z))dz + q_0$, $\Omega(z) = \int_0^z [\beta(z)(\varphi_0^2 - p_0^2 c_2) - 2\sigma(z)\varphi_0]dz + \Omega_0$ and $\gamma(z) = \frac{\beta(z)p_0^2 c_4 \exp[\int_0^z \alpha(z)dz]}{f_{20}}$.

Case (3.3):

$$
\mathcal{\Psi}_3 = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{-\frac{m^2 c_2}{(m^2 + 1)c_4}} \ \text{sn} \left( \sqrt{-\frac{c_2}{c_4}} \xi \right) \right] \times e^{i[k(z)t^2 + \varphi(z)t + \Omega(z)]}. \tag{2.141}
$$
When \( m \to 1 \), the chirped dark solitary wave solution is given by
\[
\Psi_{33} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{-\frac{c_2}{2c_4}} \tanh \left( \sqrt{-\frac{c_2}{2}} \xi \right) \right] e^{i (\kappa(z) t^2 + \varphi(z) t + \Omega(z))},
\]
(2.142)
where \( \xi = p_0 \kappa(z) t + \frac{1}{2} p_0 \int_0^z \kappa(z) [\varphi_0 - 2 \int_0^z \sigma(z) dz] \kappa_z - 2 \sigma(z) dz + q_0 \),
\[
\varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z)dz] \kappa(z), \quad \Omega(z) = \frac{1}{4} \int_0^z \left\{ [4(\int_0^z \sigma(z)dz)^2 - 4(\int_0^z \sigma(z)dz)\varphi_0 + \varphi_0^2 + p_0^2 c_2 k_z - 4 \sigma(z)(\varphi_0 - 2 \int_0^z \sigma(z)dz)k(z) \right\} dz + \Omega_0,
\]
and
\[
\gamma(z) = -\frac{1}{2} \frac{\kappa_0 p_0^2 c_4}{f_{20}^2} \exp \left[ \int_0^z \frac{\kappa(z) - \kappa_0}{\kappa(z)} dz \right].
\]

Case (3.4):
\[
\Psi_{34} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{-\frac{m^2 c_2}{(m^2 + 1)c_4}} \right] \left[ \sqrt{\frac{-c_2}{2}} \xi \right] e^{i (\kappa(z) t^2 + \varphi(z) t + \Omega(z))},
\]
(2.143)
When \( m \to 1 \), the following chirped solitary wave solution is obtained as
\[
\Psi'_{34} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{-\frac{m^2 c_2}{(m^2 + 1)c_4}} \right] \left[ \sqrt{\frac{-c_2}{2}} \xi \right] e^{i (\kappa(z) t^2 + \varphi(z) t + \Omega(z))},
\]
(2.144)
where \( \xi = p_0 \kappa(z) t + \frac{1}{2} p_0 \int_0^z \kappa(z) [\varphi_0 - 2 \int_0^z \sigma(z) dz] \kappa_z - 2 \sigma(z) dz + q_0 \), \( \beta(z) = \frac{\kappa_0}{2\kappa(z)} \),
\[
\varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z)dz] \kappa(z), \quad \Omega(z) = \frac{4\kappa(z)^2}{4c_4 f_{20}^2} \int_0^z \kappa(z) \varphi(z)dz - \frac{\kappa_0 p_0^2 c_2 \kappa(z)^2}{4} + \frac{\kappa_0 p_0^2 c_2}{2} + 4\kappa(z) \Omega_0
\]
and \( \gamma(z) = \frac{\beta(z) p_0^2 c_2}{4c_4 f_{20}^2} \exp \left[ \int_0^z \frac{\kappa_0(z) - \kappa(z)}{\kappa(z)} dz \right] \).

Family 4. When \( c_0 = \frac{m^2(m^2 - 1)c_2^2}{(2m^2 - 1)c_4}, \ c_2 > 0, \) and \( c_4 < 0 \), then cnoidal periodic waves solutions will become

Case (4.1):
\[
\Psi_{41} = f_{20} \exp \left( -\frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \sqrt{-\frac{-m^2 c_2}{(m^2 - 1)c_4}} \right] \left[ \frac{c_2}{(m^2 - 1)} \xi \right] e^{i (\varphi_0 t + \Omega(z))},
\]
(2.145)
When \( m \to 1 \), the unchirped bright solitary wave solution is given by
\[
\Psi'_{41} = f_{20} \exp \left( -\frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \sqrt{-\frac{-c_2}{c_4}} \right] \left[ \frac{c_2}{c_4} \right] \left[ \sqrt{-c_2 \xi} \right] e^{i (\varphi_0 t + \Omega(z))},
\]
(2.146)
where \( \xi = p_0 t + \int_0^z p_0 [\beta(z) \varphi_0 - \sigma(z)] dz + q_0 \), \( \Omega(z) = \int_0^z [\beta(z)(\varphi_0^2 - p_0^2 c_2) - 2\sigma(z)\varphi_0] dz + \Omega_0 \)
and \( \gamma(z) = \frac{\beta(z) p_0^2 c_2}{f_{20}} \exp \left[ \int_0^z \alpha(z) dz \right] \).

Case (4.2):
\[
\Psi_{42} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{-\frac{-m^2 c_2}{(m^2 - 1)c_4}} \right] \left[ \frac{c_2}{(m^2 - 1)} \xi \right] \left[ \frac{c_2}{c_4} \right] \left[ \sqrt{-c_2 \xi} \right] e^{i (\kappa(z) t^2 + \varphi(z) t + \Omega(z))},
\]
(2.147)
When \( m \to 1 \), the chirped bright solitary wave solution is written as

\[
\Psi'_{42} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa z}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{-\frac{c_2}{c_4}} \ \text{sech} \left( \sqrt{c_2} \xi \right) \right] e^{i[\kappa(z)^2 + \varphi(z)(t + \Omega(z))]}, \tag{2.148}
\]

where \( \xi = p_0 \kappa(z) t + \frac{1}{2} p_0 \int_0^z \kappa(z) [(\varphi_0 - 2 \int_0^z \sigma(z) dz) \kappa_z - 2 \sigma(z)] dz + q_0 \), \( \varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z) dz] \kappa(z) \), \( \Omega(z) = \frac{4 \kappa^2(z) \int_0^z \sigma(z) dz [\int_0^z \sigma(z) dz - \varphi_0 + (\varphi_0^2 - p_0^2) \kappa(z) + 4 \kappa(z)] \Omega_0}{4 \kappa(z)} \), \( \beta(z) = \frac{\kappa}{2 \sqrt{c_4(z)}} \) and \( \gamma(z) = 0 \).

Family 5. If \( c_0 = \frac{2 - m^2 - 2 \sqrt{1 - m^2}}{4} \), \( c_2 = \frac{m^2}{2} - 1 - 3\sqrt{1 - m^2} \), \( c_4 = \frac{2 - m^2 - 2\sqrt{1 - m^2}}{4} \), then periodic waves solutions are derived as

Case (5.1):

\[
\Psi_{51} = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \frac{m^2 \text{sn}(\xi) \text{cn}(\xi)}{\text{sn}^2(\xi) + (1 + \sqrt{1 - m^2}) \text{dn}(\xi) - 1 - \sqrt{1 - m^2}} \right] \times e^{i[\kappa(z) + \Omega(z)]}. \tag{2.149}
\]

When \( m \to 1 \), the solitary wave solution is written as

\[
\Psi'_{51} = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \frac{\tanh(\xi)}{1 - \text{sech}(\xi)} \right] e^{i[\kappa(z) + \Omega(z)]}, \tag{2.150}
\]

where \( \xi = p_0 t + \int_0^z \beta(z) \varphi_0 + \sigma(z) dz + q_0 \), \( \Omega(z) = \int_0^z \beta(z) \varphi_0^2 + \frac{p_0^2}{2} - 2 \sigma(z) \varphi_0 dz + \Omega_0 \) and \( \gamma(z) = \frac{\beta(z) p_0^2 \exp[\int_0^z \alpha(z) dz]}{4 f_{20}} \).

Case (5.2):

\[
\Psi_{52} = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \frac{m^2 \text{sn}(\xi) \text{cn}(\xi)}{(dn(\xi) - 1) \sqrt{1 - m^2} + \text{sn}^2(\xi) + dn(\xi) - 1} \right] \times e^{i[\kappa(z) + \Omega(z)]} \tag{2.151}
\]

When \( m \to 1 \), the combined solitary wave solution is given by

\[
\Psi'_{52} = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \frac{\tanh^2(\xi) \pm (1 + \text{sech}^2(\xi) - 2 \text{sech}(\xi))}{\tanh(\xi)(1 - \text{sech}\xi)} \right] e^{i[\kappa(z) + \Omega(z)]}, \tag{2.152}
\]

where \( \xi = p_0 t + \int_0^z \beta(z) \varphi_0 - \sigma(z) dz + q_0 \), \( \Omega(z) = \int_0^z \beta(z) p_0^2 + \frac{p_0^2}{2} \beta(z) - 2 \sigma(z) \varphi_0 dz + \Omega_0 \) and \( \gamma(z) = \frac{\beta(z) p_0^2 \exp[\int_0^z \alpha(z) dz]}{4 f_{20}} \).

Case (5.3):

\[
\Psi_{53} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa z}{\kappa} - \alpha \right) dz \right] \left[ \frac{m^2 \text{sn}(\xi) \text{cn}(\xi)}{\text{sn}^2(\xi) + (1 + \sqrt{1 - m^2}) \text{dn}(\xi) - 1 - \sqrt{1 - m^2}} \right] \tag{2.153}
\]
When \( m \to 1 \), the chirped solitary wave solution is derived as

\[
\Psi_{53}' = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \frac{\tanh(\xi)}{1 - \text{sech}(\xi)} \right] e^{i[(\kappa z^2 + \varphi(z)t + \Omega(z))]}, \tag{2.154}
\]

where \( \xi = p_0 \kappa(z) t + \frac{1}{2} p_0 \int_0^z \kappa(z) [(\varphi_0 - 2 \int_0^z \sigma(z) dz) \kappa_z - 2 \sigma(z)] dz + q_0 \), \( \varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z) dz] \kappa(z), \Omega(z) = \frac{1}{4} \int_0^z \left\{ [4 \left( f_0 \sigma(z) dz \right)^2 - 4 (\int_0^z \sigma(z) dz) \varphi_0 + \varphi^2_0 + \frac{\alpha^2}{2} ] k_z - 4 \sigma(z) (\varphi_0 - 2 \int_0^z \sigma(z) dz) k(z) \right\} dz + \Omega_0, \beta(z) = \frac{\kappa_z}{2 \kappa^2(z)} \), and \( \gamma(z) = \frac{1}{8} \frac{\kappa_z p_0^2}{f_{20}^2} \exp \left[ \int_0^z \frac{\alpha \kappa(z) - \kappa_z(z)}{\kappa(z)} dz \right] \).

Case (5.4):

\[
\Psi_{54} = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \frac{m^2 \text{sn}(\xi) \text{cn}(\xi)}{\sqrt{1 - m^2 + \text{sn}^2(\xi) + \text{dn}(\xi) - 1}} \right] e^{i[(\kappa z^2 + \varphi(z)t + \Omega(z))]}. \tag{2.155}
\]

When \( m \to 1 \), the combined chirped solitary wave solution is obtained as

\[
\Psi_{54}' = f_{20} \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_z}{\kappa} - \alpha \right) dz \right] \left[ \frac{\tan^2(\xi) \pm (1 + \text{sech}^2(\xi) - 2 \text{sech}(\xi))}{\tanh(\xi)(1 - \text{sech}(\xi))} \right] e^{i[(\kappa z^2 + \varphi(z)t + \Omega(z))]}. \tag{2.156}
\]

where \( \xi = p_0 \kappa(z) t + \frac{1}{2} p_0 \int_0^z \kappa(z) [(\varphi_0 - 2 \int_0^z \sigma(z) dz) \kappa_z - 2 \sigma(z)] dz + q_0, \beta(z) = \frac{\kappa_z}{2 \kappa^2(z)}, \varphi(z) = [\varphi_0 - 2 \int_0^z \sigma(z) dz] \kappa(z), \Omega(z) = \frac{4 \kappa^2(z)}{f_{20}^2} \sigma(z) dz [f_0^z \sigma(z) dz - \varphi_0 + [\varphi_0^2 + \frac{\kappa^2}{2}] \sigma^2(z) + \frac{2}{4} \kappa^2(z) \Omega_0 \right] \)

and \( \gamma(z) = \frac{\beta(z) \kappa^2(z) p_0^2}{4 f_{20}^2} \exp \left[ \int_0^z \frac{\alpha \kappa(z) - \kappa_z(z)}{\kappa(z)} dz \right] \). Family 6. If \( c_0 = \frac{1}{4 (C^2 - m^2 + B^2)}, c_2 = \frac{1}{2} - m^2 \) and \( c_4 = \frac{B^2 + C^2 m^2}{4}, \) then the following periodic waves solutions are written as

Case (6.1):

\[
\Psi_{61} = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \frac{\sqrt{B^2 + C^2 m^2}}{B \sin(\xi) + C \sin(\xi)} \right] e^{i[(\varphi_0 t + \Omega(z))]}. \tag{2.157}
\]

When \( m \to 1 \), the unchirped solitary wave solution is given by

\[
\Psi_{61}' = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \frac{\sqrt{B^2 + C^2}}{B \tanh(\xi) + C \tanh(\xi)} \right] e^{i[(\varphi_0 t + \Omega(z))]}, \tag{2.158}
\]

where \( \xi = p_0 t + \int_0^z p_0 (\beta(z) \varphi_0 - \sigma(z)) dz + q_0, \Omega(z) = \int_0^z [\beta(z) \varphi_0^2 + \frac{\alpha^2}{2}] - 2 \sigma(z) \varphi_0] dz + \Omega_0 \) and \( \gamma(z) = \frac{\beta(z) p_0^2 (B^2 + C^2) \exp \left[ \int_0^z \alpha(z) dz \right]}{4 f_{20}^2} \).

When \( m \to 0 \), the unchirped trigonometric function solution is derived as

\[
\Psi_{61}'' = f_{20} \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \frac{\sqrt{B^2 + C^2 m^2}}{B \sin(\xi) + C \sin(\xi)} \right] e^{i[(\varphi_0 t + \Omega(z))]}, \tag{2.159}
\]

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where $\xi = p_0 t + \int_0^z p_0 [\beta (z) \varphi_0 - \sigma (z)] dz + q_0$, $\Omega(z)$ = \int_0^z [\beta(z)(\varphi_0^2 - \frac{t^2}{2}) - 2\sigma(z)\varphi_0] dz + \Omega_0$ and $\gamma(z) = \frac{(\beta(z)p_0^2 B^2 \exp[\int_0^z \alpha(z) dz])}{4f_2}$.

Case (6.2):

$$\Psi_{62} = f_2 \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_0}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{\frac{(C^2\kappa^2 + B^2 - C^2)}{B^2 + C^2 \kappa^2} + \text{cn}(\xi)} \right] e^{i[\kappa(z)t^2 + \varphi(z)t + \Omega(z)]}. \quad (2.160)$$

When $m \rightarrow 1$, the chirped solitary wave solution is written as

$$\Psi_{62} = f_2 \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_0}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{\frac{B^2}{B^2 + C^2 \kappa^2} + \text{sech}(\xi)} \right] e^{i[\kappa(z)t^2 + \varphi(z)t + \Omega(z)]}, \quad (2.161)$$

where $\xi = p_0 \kappa(z)t + \frac{1}{2} p_0 \int_0^z \kappa(z) [\varphi_0 - 2 \int_0^z \sigma(z) dz] \kappa_z - 2\sigma(z) dz + q_0$, $\varphi(z) = \varphi_0 - 2 \int_0^z \sigma(z) dz \kappa(z), \Omega(z) = \frac{1}{4} \int_0^z \left\{ [4(\int_0^z \sigma(z) dz)^2 - 4(\int_0^z \sigma(z) dz) \varphi_0 + \varphi_0^2 + \frac{p_0^2}{2}]k_z - 4\sigma(z)(\varphi_0 - 2 \int_0^z \sigma(z) dz) k(z) \right\} dz + \Omega_0, \beta(z) = \frac{\kappa_z}{2\kappa^2},$ and $\gamma(z) = \frac{1}{8} k_0^2 \int_0^z \frac{\alpha''(z) \kappa_z(z)}{\kappa(z)} dz$.

When $m \rightarrow 0$, the chirped trigonometric function solution is given by

$$\Psi_{62} = f_2 \exp \left[ \frac{1}{2} \int_0^z \left( \frac{\kappa_0}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{\frac{B^2}{B^2} + \cos(\xi)} \right] e^{i[\kappa(z)t^2 + \varphi(z)t + \Omega(z)]}, \quad (2.162)$$

where $\xi = p_0 \kappa(z)t + \frac{1}{2} p_0 \int_0^z \kappa(z) [\varphi_0 - 2 \int_0^z \sigma(z) dz] \kappa_z - 2\sigma(z) dz + q_0$, $\varphi(z) = \varphi_0 - 2 \int_0^z \sigma(z) dz \kappa(z), \Omega(z) = \frac{1}{4} \int_0^z \left\{ [4(\int_0^z \sigma(z) dz)^2 - 4(\int_0^z \sigma(z) dz) \varphi_0 + \varphi_0^2 - \frac{p_0^2}{2}]k_z - 4\sigma(z)(\varphi_0 - 2 \int_0^z \sigma(z) dz) k(z) \right\} dz + \Omega_0, \beta(z) = \frac{\kappa_z}{2\kappa^2}$, and $\gamma(z) = \frac{1}{8} k_0^2 \int_0^z \frac{\alpha''(z) \kappa_z(z)}{\kappa(z)} dz$.

Family 7: If $q_2 = \frac{1}{3}(c_2^2 - 3c_1c_0), q_3 = \frac{4c_2^2}{27}(-2c_2^2 + 9c_0c_4), F(\xi) = \sqrt{\frac{1}{c_4}[\varphi(\xi; q_2, q_3) - \frac{1}{3}c_2]},$ then WEFs solutions are given as

Case (7.1):

$$\Psi_{71} = f_2 \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \sqrt{\frac{1}{c_4}[\varphi(\xi; q_2, q_3) - \frac{1}{3}c_2]} \right] e^{i[\xi t + \Omega(z)]}, \quad (2.163)$$

where $\xi = p_0 t + \int_0^z p_0 [\beta(z) \varphi_0 - \sigma(z)] dz + q_0$, $\Omega(z) = \int_0^z [\beta(z)(\varphi_0^2 - p_0^2 c_2) - 2\sigma(z)\varphi_0] dz + \Omega_0$ and $\gamma(z) = \frac{\beta(z)p_0^2 c_4 \exp[\int_0^z \alpha(z) dz]}{f_2}$.

Case (7.2):

$$\Psi_{72} = f_2 \exp \left( - \frac{1}{2} \int_0^z \alpha(z) dz \right) \left[ \sqrt{\frac{1}{c_4}[\varphi(\xi; q_2, q_3) - \frac{1}{3}c_2]} \pm \sqrt{\frac{c_0}{\varphi(\xi; q_2, q_3) - \frac{1}{3}c_2}} \right] \times e^{i[\xi t + \Omega(z)]}, \quad (2.164)$$

where $\xi = p_0 t + \int_0^z p_0 [\beta(z) \varphi_0 - \sigma(z)] dz + q_0$, $\Omega(z) = \frac{1}{2} \int_0^z [\pm 6(\beta(z)p_0^2 \sqrt{c_0 c_4} + (\varphi_0^2 - p_0^2 c_2) \beta(z) - 2\sigma(z)\varphi_0] dz + \Omega_0$ and $\gamma(z) = \frac{\beta(z)p_0^2 c_4 \exp[\int_0^z \alpha(z) dz]}{f_2}$.  

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Case (7.3):

\[
\Psi_{73} = f_{20} \exp \left[ \frac{1}{2} \int_{0}^{z} \left( \frac{\kappa_{z}}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{\frac{1}{c_{4}} \left[ \varphi(\xi; q_{2}, q_{3}) - \frac{1}{3} c_{2} \right]} e^{i \left[ (\kappa(z) t^{2} + \varphi(z) t + \Omega(z)) \right]} \right],
\]

where \( \xi = p_{0} \kappa(z) t + \frac{1}{2} p_{0} \int_{0}^{z} \kappa(z) \left[ (\varphi_{0} - 2 \int_{0}^{z} \sigma(z) dz) \kappa_{z} - 2 \sigma(z) \right] dz + q_{0}, \)

\( \varphi(z) = [\varphi_{0} - 2 \int_{0}^{z} \sigma(z) dz] \kappa(z), \Omega(z) = \frac{1}{4} \int_{0}^{z} \left[ 4 \left( \int_{0}^{z} \sigma(z) dz \right)^{2} - 4 \left( \int_{0}^{z} \sigma(z) dz \right) \varphi_{0} + \varphi_{0}^{2} - p_{0}^{2} c_{2} \right] \kappa_{z} - 4 \sigma(z) \left( \varphi_{0} - 2 \int_{0}^{z} \sigma(z) dz \right) \kappa(z) \right] dz + \Omega_{0}, \beta(z) = \frac{\kappa_{z}}{2 \kappa(z)}, \)

and \( \gamma(z) = \frac{1}{2} \kappa \frac{\partial \kappa}{\partial \xi} e^{\frac{1}{2} \int_{0}^{z} \left( \frac{\kappa_{z}}{\kappa} - \alpha \right) dz}. \)

Case (7.4):

\[
\Psi_{74} = f_{20} \exp \left[ \frac{1}{2} \int_{0}^{z} \left( \frac{\kappa_{z}}{\kappa} - \alpha \right) dz \right] \left[ \sqrt{\frac{1}{c_{4}} \left[ \varphi(\xi; q_{2}, q_{3}) - \frac{1}{3} c_{2} \right]} \pm \sqrt{\frac{c_{0}}{\varphi(\xi; q_{2}, q_{3}) - \frac{1}{3} c_{2}}} \right] \times e^{i \left[ (\kappa(z) t^{2} + \varphi(z)) t + \Omega(z) \right]},
\]

where \( \xi = p_{0} \kappa(z) t + \frac{1}{2} p_{0} \int_{0}^{z} \kappa(z) \left[ (\varphi_{0} - 2 \int_{0}^{z} \sigma(z) dz) \kappa_{z} - 2 \sigma(z) \right] dz + q_{0}, \beta(z) = \frac{\kappa_{z}}{2 \kappa(z)}, \)

\( \varphi(z) = [\varphi_{0} - 2 \int_{0}^{z} \sigma(z) dz] \kappa(z), \Omega(z) = \frac{4 \kappa_{z}^{2}(z) \int_{0}^{z} \sigma(z) dz \left[ \int_{0}^{z} \sigma(z) dz - \varphi_{0} \right] + \left[ \beta(z) - p_{0}^{2} c_{2} \right] \kappa_{z}^{2}(z) + 6 p_{0}^{2} \sqrt{\kappa(z)} + 4 \kappa(z) \Omega_{0}}{4 \kappa(z)} \)

and \( \gamma(z) = \frac{\beta(z) \kappa_{z}^{2}(z) p_{0}^{2} \varphi_{0}}{\int_{0}^{z} \left[ \varphi(\xi; q_{2}, q_{3}) - \frac{1}{3} c_{2} \right]} \). \)

Similarly, by choosing \( c_{0}, c_{2} \) and \( c_{4} \) from Appendix I and \( q_{2}, q_{3} \) from Appendix II, one can get many other families of solutions in terms of JEFs and WEFs. However, for the limit of length of present work, we omit them here.

Figure 2.8: Intensity distribution of chirpless (\( |\Psi_{21}|^{2} \)) and chirped (\( |\Psi_{23}|^{2} \)) bright solitary wave solutions.

In our calculations, the following parameters values are used \( f_{20} = 2, \alpha(z) = \sin(z), p_{0} = \varphi_{0} = 1, q_{0} = 1, \sigma(z) = 1, \beta(z) = \sin(z) \) and \( \kappa(z) = \cos(z). \)
In our calculations, the following parameters values are used $f_{20} = 2, \alpha(z) = \sin(z), p_0 = \varphi_0 = 1, q_0 = 1, \sigma(z) = 1, \beta(z) = \sin(z)$ and $\kappa(z) = \cos(z)$.

In the above section, we found a number of different solutions to the generalized NLSE under different parametric restrictions. Now we describe the dynamics of a few analytic solutions which may be vital to improve the soliton transmission features in some actual physical situations.

Note that the form of the solitons is controlled by the parameter functions $\sigma(z), \beta(z)$ and $\alpha(z)$ or $\sigma(z), k(z)$ and $\alpha(z)$. Table 2.1 depicts some interesting single-JEF soliton solutions. To display unique behavior of these exact soliton solutions, one can choose the dispersion coefficient $\beta(z)$ and the phase chirp parameter $k(z)$ in terms of trigonometric, hyperbolic functions, some constants and linear functions. Here, we present some examples using the linear, trigonometric and hyperbolic distributed control systems.

From Table 2.1, note that the speed of chirpless bright soliton (BS) or dark soliton (DS) is re-
conditions for chirped and chirp free soliton solutions. In unchirped case, the amplitude and communication systems can be controlled. Similarly, in Table 2.1 we obtained different existence

Figure 2.11: Intensity distribution of chirpless bright and dark solitary wave solutions for the following parameters values \( f_{20} = 2, \alpha(z) = 0, \varphi_0 = \varphi_0 = 1, \sigma(z) = 1, \beta(z) = \sin(z) \).

<table>
<thead>
<tr>
<th>Soliton type</th>
<th>Soliton intensity</th>
<th>( \xi ) expression</th>
<th>Existence condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: Unchirped BS</td>
<td>(</td>
<td>\Psi_20</td>
<td>= f_{20} \exp(-f_0 \alpha(z)dz) \text{sech}^2(\xi) )</td>
</tr>
<tr>
<td>Case 2: Chirped BS</td>
<td>(</td>
<td>\Psi_22</td>
<td>= f_{20} \exp \left[ -f_0 \left( \frac{\alpha(z)}{\kappa(z)} \right)dz \right] \text{sech}^2(\xi) )</td>
</tr>
<tr>
<td>Case 3: Unchirped DS</td>
<td>(</td>
<td>\Psi_1</td>
<td>= f_{20} \exp \left[ -f_0 \alpha(z)dz \right] \text{tanh}^2(\xi) )</td>
</tr>
<tr>
<td>Case 4: Chirped DS</td>
<td>(</td>
<td>\Psi_{13}</td>
<td>= f_{20} \exp \left[ -f_0 \left( \frac{\alpha(z)}{\kappa(z)} \right)dz \right] \text{tanh}^2(\xi) )</td>
</tr>
</tbody>
</table>

lated to \( p_0[\beta(z) - \sigma(z)] \) and phase shift is determined by \( f_0^z [\beta(z) - \sigma(z)]dz - 2 \sigma(z) \varphi_0 dz \).

The wave amplitude of BS is given by \( f_{20} \exp \left( -\frac{1}{2} f_0^z \alpha(z)dz \right) = \sqrt{- \beta(z) p_0^2 c_4 / \gamma(z)} \), where \( \frac{c_4 \beta(z)}{\gamma(z)} < 0 \). So by choosing suitable initial conditions and the intermodal dispersion and GVD parameters, the speed and phase shift of unchirped bright and dark solitons in optical fiber communication systems can be controlled. Similarly, in Table 2.1 we obtained different existence conditions for chirped and chirp free soliton solutions. In unchirped case, the amplitude and solitary wave characteristics can exclusively be controlled by the dispersion coefficients \( \beta(z) \) and the gain/loss coefficients \( \gamma(z) \). But for solitary waves with chirp function \( \kappa(z) \) and the gain/loss coefficient \( \gamma(z) \) will determine the wave propagation characteristics.

In Figs 2.8 and 2.9, we present the intensity distributions of chirped and chirpless bright and dark solitary wave solutions for specific choices of control functions, respectively. From Fig. 2.10, it is to be noted that when linear attenuation coefficient \( \alpha(z) \) is either a constant parameter or varies periodically, the amplitude of solitary waves decreases exponentially with propagation. When \( \beta(z) \) is a trigonometric function and \( \alpha(z) = 0 \), it produces a sign changing nonlinearity, so the intensity distribution presents a snake like appearance and solitary wave width remains unchanged, as shown in Fig. 2.11. In Fig. 2.12, a comparison between chirped and unchirped periodic wave solution is given.

The soliton dynamics, in practical cases, can effectively be controlled by managing dispersion
Figure 2.12: Intensity distribution of chirpless ($|\Psi_{11}|^2$) and chirped ($|\Psi_{13}|^2$) periodic wave solutions. The parameters values are same as in Fig 2.9.

Figure 2.13: Evolution of unchirped bright solition ($|\Psi_{21}|^2$) in a DDF at $t = 0$ for different values of $\sigma_1$. For black, blue,red and green curves the values of $\sigma_1 = 0, -0.2, -0.4$ and $-0.6$ respectively and the other parameters are $f_{20} = 2, \alpha(z) = \sin(z), p_0 = 1, \varphi_0 = 3, q_0 = 1, \sigma(z) = 1, \beta_0 = 1$.

and intensity of solitons. Therefore, one can obtain the optical control system by choosing different forms of parameters for the specific problem, appropriately. To be specific, we consider the propagation of an unchirped solitary wave in a dispersion decreasing fiber (DDF) system with the varying GVD coefficient given in the form $\beta(z) = \beta_0 \exp(-\sigma_1 z)$, where $\beta_0$ is the parameter related to GVD and $\sigma_1 > 0$ corresponds to DDF. For such a system, from Table 2.1, the unchirped bright solitary wave solution is written as

$$
\Psi_{21}' = f_{20} \exp\left(-\frac{1}{2} \int_0^z \alpha(z)dz\right) \text{sech}(\xi) \times \exp\left(i \varphi_0 t + \int_0^z [\beta_0 e^{-\sigma_1 z} \varphi_0^2 + 2p_0^2] - 2\sigma(z)\varphi_0 dz + \Omega_0\right),
$$

(2.167)
where \( \xi = p_0 t + p_0 \int_0^z [\beta_0 e^{-\sigma_1 z} \varphi_0 - \sigma(z)] \, dz + q_0 \) and \( \gamma(z) = -\frac{\sqrt{2} \beta_0 e^{-\sigma_1 z} \int_0^z \alpha(z) \, dz}{\int_0^z} \).

Fig. 2.13 presents the evolution of unchirped bright soliton in a DDF at \( t = 0 \) for different values of \( \sigma_1 \) and it is apparent that the amplitude of BS varies with change in value of DDF parameter \( \sigma_1 \).

The obtained solutions of variable-coefficient NLSE include Jacobi elliptic function solutions, Weierstrass elliptic function solutions, chirped and unchirped dark and bright solitary wave solutions and trigonometric function solutions may be significant to explain some physical phenomena.