Chapter 1

Introduction

Over the last few decades the scientific community witnessed some very significant developments in the growth of science. It was like a revolution, albeit a quiet one. Like a revolution these developments touched the lives of many people and transformed their outlooks of the world. Unlike in revolutions, these changes did not always happen abruptly; but there are some important years, if not dates, that one can quote. It is quiet, because no one called a press conference to proclaim it and so there were no headlines in the newspapers. Here we are talking about an exciting interdisciplinary field which is now called nonlinear science. Nonlinear science is not a new branch of science in the typical sense and does not add a new subject of study. Rather, nonlinear science includes all the existing disciplines in science in both natural and social sciences. Nonlinear science, like quantum mechanics and relativity, delivers a whole set of fundamentally new ideas and astonishing results. Yet, unlike quantum mechanics and relativity, nonlinear science covers systems of every scale and objects moving with any speed. Therefore, nonlinear science is more than qualified to be called a revolution. The fact that nonlinear science delivers within the conventional system sizes and speed limits should not be counted as negative towards its novelty but, on the contrary, in view of its wide applicability, makes it more important and powerful. In particular, nonlinear science can be studied with daily macroscopic systems with ordinary tools, such as a camera or a copying machine, making it accessible to almost everybody. As a consequence of the multidisciplinary nature of it, people working in very different disciplines such as in economics and earthquakes now have some common vocabularies and can communicate with each other. Moreover, the applicability of nonlinear science in a broad range of scales implies that one can study the same phenomenon in very different systems with corresponding experimental tools. For example, one can study fractals on the kitchen table.
by photographing potato chips with an ordinary camera, while someone else with a sophisticated and expensive electron microscope will do it in a clean room with semiconductor chips. And, amazingly, they could be both working in the forefront of research in nonlinear science. In short, nonlinear science can really bring people together. Nonlinear science is a game everyone can play. For informative purposes, one may divide the contents of nonlinear science into six categories, viz., fractals, chaos, solitons, pattern formation, cellular automata and complex systems. The common theme underlying this diversity of subjects is the nonlinearity of the systems under study.

1.1 Linear and nonlinear dynamical systems

A dynamical system is a concept where a set of fixed rules describes the time dependence of a point in a geometrical space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe and the number of fishes each spring time in a lake. At any given time, a dynamical system has a state given by a set of real numbers (a vector) that can be represented by a point in an appropriate state space (a geometrical manifold). Small changes in the state of the system create small changes in the numbers. The evolution rule of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic; in other words, for a given time interval only one future state follows from the current one. Thus dynamical systems theory is an area used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. Differential equations are employed for continuous dynamical systems and difference equations describes discrete dynamical systems. When the time variable runs over a set which is discrete over some intervals and continuous over others or is any arbitrary time-set, such as a Cantor set, then one gets dynamical equations on time scales. Some situations may also be modeled by mixed modes such as differential-difference equations. The change in states of physical systems as a function of time is known as evolution, whose study constitutes the subject of dynamics. The change in state of a system takes place due to various forces acting on the system. Since every natural system is influenced by one or more forces and therefore, the nature of evolution of different physical systems depends upon the nature of acting forces and their initial states. The nature of acting forces may be linear or nonlinear. The former leads to the study of linear dynamical systems while the later one forms a basis of the field of nonlinear dynamics [1].
In mathematical terms, a nonlinear system is one that does not satisfy the superposition principle, or one whose output is not directly proportional to its input; a linear system fulfills these conditions. In other words, a nonlinear system is any problem where the variable(s) to be solved for, cannot be written as a linear combination of independent components. A nonhomogeneous system, which is linear apart from the presence of a function of the independent variables, is nonlinear according to a strict definition, but such systems are usually studied alongside linear systems, because they can be transformed to a linear system of multiple variables.

Nonlinear problems are of interest to engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena such as chaos [1]. Some aspects of the weather (although not the climate) are seen to be chaotic, where simple changes in one part of the system produce complex effects throughout.

The concept of linear and nonlinear systems can further be clarified by considering the following two examples. In classical mechanics, the behavior of a system consisting of a point particle with mass $m$ and subjected to a force $F_x = -kx$ acting in the $x$ direction and constrained to move in only the $x$ direction is given by the Newton’s second law of motion:

$$\ddot{x} = -\omega^2 x.$$  \hspace{1cm} (1.1)

This equation is linear in $x$ and in the second derivative of $x$ and hence, we have a linear system. If the mass is displaced from the equilibrium position and released, it will oscillate about the equilibrium position sinusoidally with an angular frequency $\omega = \sqrt{k/m}$. However, if we consider the force $F_x$ a more complicately $x$ dependent i.e. $F_x = -bx^2$, then the time evolution equation becomes

$$\ddot{x} = -k' x^2; \quad k' = \sqrt{b/m}. $$  \hspace{1cm} (1.2)

This system is now be called nonlinear because the position of the particle appears in Eq.(1.2) squared.

Thus, a system is said to be linear if it holds the following condition: suppose that $g(x, t)$ and $h(x, t)$ are linearly independent solutions of the time evolution equation for the system; then $cg(x, t) + dh(x, t)$ is also a solution, where $c$ and $d$ are any numbers. We can also express the notion of nonlinearity in terms of the response of a system to a stimulus. Suppose $h(x, t)$ gives the response of the system to a particular stimulus $S(t)$. If we now change $S(t)$ to $2S(t)$, a linear system will have the response $2h(x, t)$. For a nonlinear system, the response will be larger or smaller than $2h(x, t)$.
1.2 Partial differential equations

It is well known that most of the phenomena that arise in physics and engineering fields can be described by partial differential equations (PDEs). In physics, for example, the heat flow and the wave propagation phenomena are well described by PDEs [2]. In ecology, most of the population models are governed by PDEs [3]. The dispersion of a chemically reactive material is characterized by PDEs. In addition, most of the physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, hydrodynamics waves and many other fields are controlled within the domain of PDEs [2]. Thus for a long time, PDEs have become a useful tool for modeling natural phenomena of science and engineering. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving PDEs and the ways for implementation of these methods.

A PDE is an equation that contains a dependent variable which is an unknown function of independent variables and its partial derivatives. Unlike in ordinary differential equations (ODEs), where the dependent variables depend only on one independent variable, in PDEs the dependent variable must depend on more than one independent variables. In general, a PDE, of order \( l \), is an equation of the form

\[
f(x_1, x_2, \ldots, x_n, u, \partial_1 u, \ldots, \partial_n u, \partial_1^2 u, \ldots, \partial_1^l u) = 0,
\]

relating a function \( u \) of \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and its partial derivatives of order \( \leq l \).

PDEs are classified as linear and nonlinear. The linear PDEs, such as the diffusion equation and the wave equation, which arise in many areas of scientific applications [4]. In spite of the fact that there has been considerable interest in the linear PDEs and their solutions in various contexts, somehow the similar efforts have not been there in studying the nonlinear class of PDEs. There are several phenomena in nature whose theoretical understanding requires the role of nonlinearity in clear mathematical terms. Therefore, in the past, numerous nonlinear PDEs were formulated and solved in diverse fields. Particularly, after the advancement in high performance computing, the importance of an intrinsic analysis of nonlinear phenomena has been gradually understood and led to two concepts, the strange attractor and soliton. Both are related to astonishing properties of nonlinear systems and they seem to be contradict each other. In what follows, we briefly explain the concept of solitons which are intrinsically associated with nonlinear PDEs.
1.3 Solitons

In the year 1965, the concept of the soliton was introduced into nonlinear dynamics by Zabusky and Kruskal in their famous numerical experiments on the Kortweg-de Vries (KdV) equation. Since then the field has grown almost in an exponential manner and has now entered in a period of stability and high respectability. It has attracted the attention of researchers in all modern areas of mathematics, physics, engineering and biology. On the one hand various mathematical concepts such as prolongation structures, jet bundles, space curves and surfaces, gauge-equivalence, Lie-algebraic properties including Kac-Moody and Virasoro algebras, Lie-Bäcklund symmetries, singularity structures and so on have been attributed to soliton properties, while on the other hand numerous applications have been found in diverse areas like fluid dynamics, lattice theory, plasma physics, condensed matter physics, superconductivity, magnetism, nonlinear optics, particle theory, general relativity, aerodynamics, meteorology and electrical networks [1, 5].

An exciting and extremely active area of research investigation during the past twenty years has been the study of solitons and the related issue of the construction of solutions to a wide class of nonlinear equations. Solitons are mathematical objects which have excited theoreticians because of their wide ranging applications in physics. They appear as solutions of particular nonlinear wave equations which often have a certain universal significance.

Before describing the soliton in detail, it will be interesting to take a historical note on the development of concept of solitons. First of all “solitons” were first observed by J. Scott Russell in 1834 whilst riding on horseback beside the narrow Union canal near Edinburgh, Scotland. There are a number of discussions in the literature describing Russell’s observations. Nevertheless we feel that his point of view is so insightful and relevant that we present it here as well. He described his observations as follows:

“I believe that I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put into motion; it accumulated round the prow of the vessel in a state of agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or
nine miles an hour, preserving its original feature some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears.”

Subsequently, Russell did extensive experiments in a laboratory scale wave tank in order to study this phenomenon more carefully. His results are the following: (1). He observed solitary waves, which are long, shallow water waves of permanent form, hence he deduced that they exist; this is his most significant result. (2). The speed of propagation, \( c \), of a solitary wave in a channel of uniform depth \( h \) is given by \( c^2 = gh(\eta + \eta) \), where \( \eta \) is the amplitude of the wave and \( g \) is the force due to gravity.

In 1895, Diederik Johannes Korteweg together with his Ph.D. student, Gustav de Vries derived analytically a nonlinear PDE, well known now as the KdV equation. The KdV equation, that contains nonlinear and dispersive terms, describes the propagation of long waves of small but finite amplitude in dispersive media. It is a generic model for the study of weakly nonlinear long waves, incorporating leading order nonlinearity and dispersion. This equation had already appeared in a work on water waves by Boussinesq in 1872. In its simplest form, it is written as

\[
    u_t + \alpha uu_x + u_{xxx} = 0.
\]  

(1.4)

The term \( u_t \) describes the time evolution of the wave propagating in one direction. Moreover, this equation incorporates two competing effects: nonlinearity represented by \( uu_x \) that accounts for steepening of the wave and linear dispersion represented by \( u_{xxx} \) that describes the spreading of the wave. Nonlinearity tends to localize the wave while dispersion spreads it out. In other words, in some nonlinear media, such as a layer of shallow water or an optical fiber, the widening of a wave packet due to dispersion could be balanced exactly by the narrowing effects due to nonlinearity of the medium. The balance between these weak nonlinear steepening and dispersion explains the formation of solitons that consist of single humped waves. The stability of solitons stems from the delicate equilibrium between these two effects of nonlinearity and dispersion. This equation gives soliton solutions which characterize solitary waves with particle-like properties that decrease monotonically at infinity.

In 1965, Norman J. Zabusky and Martin D. Kruskal investigated numerically the nonlinear interaction of a large solitary wave overtaking a smaller one and the recurrence of initial states [6]. They discovered that solitary waves undergo nonlinear interaction following the KdV equa-
tion. Further, the waves emerge from this interaction retaining its original shape, amplitude and speed and therefore conserved energy and mass. The only effect of the interaction was a phase shift. The remarkable discovery, that solitary waves retain their identities and that their character resembles particle like behavior, motivated them to call these solitary waves solitons. Thus Zabusky and Kruskal marked the birth of soliton, a name intended to signify particle like quantities such as photon, phonon, proton, etc. The interaction of two solitons emphasized the reality of the preservation of shapes and speeds and of the steady pulse like character of solitons, therefore the collision of KdV solitons is considered elastic. However the name solitary wave is more general. Solitons are special kinds of solitary waves with elastic scattering property.

Solitons are found in many physical phenomena. Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive PDEs describing physical systems [2]. Solitons retain their shapes and speed after colliding with each other. Therefore, a soliton is a wave, also a local maximum in the energy density, which preserves its shape and speed when it moves, exactly as a particle does. It corresponds to a solution of a classical field equation which simultaneously exhibits wave and quasi-particle properties. These are the features that one could expect from a quantum system and not from a classical one. The quantum analogy goes so far that soliton tunneling has been found [7]. As stated before, the KdV equation is the pioneer model that gives rise to solitons. Solitons appear either in the sech$^2$ bell shape or in the form of a kink. Soliton possesses particle-like character and retains its identities in a collision. A precise definition of a soliton is not easy to find. However, Drazin et.al [8] defined a soliton as any solution of a nonlinear equation (or a system) which: (i) is a solitary wave of permanent form; (ii) is localized, so that it decays or approaches a constant at infinity; (iii) can interact strongly with other solitons and retain its identity; (iv) is caused by a delicate balance between nonlinear and dispersive effects.

In the physics literature, the difference between solitary waves and solitons has become blurred. Solitary waves may be defined as soliton-like solutions of nonlinear evolution equations describing wave processes in dispersive and dissipative media. It is usually referred to a single soliton solution as a solitary wave [4], but when more than one solitary waves appear in a solution they are called solitons. For equations other than the KdV equation, the solitary wave solution may not be a sech$^2$ function; but may be a sech or arctan($e^{αx}$).

Nonlinear PDEs having soliton solutions, in the exact mathematical sense, provide remarkable examples of completely integrable systems with an infinite number of degrees of freedom.
Solitons are of major interest to mathematician and physicists. These are essential to describe phenomena such as propagation of some hydrodynamic waves, localized waves in astrophysical plasmas, the propagation of signals in optical fibres, charge transport in conducting polymers, localized modes in magnetic crystals and the dynamics of biological molecules such as DNA and proteins. All these systems are approximately described by the nonlinear evolution equations which have soliton solutions. Therefore, solitons provide a fruitful approach to describe the physics of a nonlinear systems [5].

1.3.1 Different types of traveling wave solutions

The study of equations that model wave phenomena requires the study of traveling wave solutions. Traveling wave solution is a solution of permanent form moving with a constant velocity. The traveling wave solutions are usually obtained by reducing the nonlinear evolution equations to associated ODEs. This is mostly handled by using the ansatz \( u(x, t) = u(\xi) \), where \( \xi = x - ct \) and \( c \) is the wave speed. This will transform the PDE into an ODE which can be solved by several appropriate methods. There are many types of traveling wave solutions that are of particular interest in solitary wave theory which are rapidly developing in many scientific fields from water waves in shallow water to plasma physics. As stated before, traveling waves appear in many forms and here only some of the most common forms which prevail in literature are addressed in brevity.

1. Solitary waves and solitons

A solitary wave is a localized “wave of translation” that arises from a balance between nonlinear and dispersive effects, traveling with constant speeds and shapes and asymptotically goes zero at large distances. Solitons are special kinds of solitary waves. The soliton solution is spatially localized solution which keeps its identity upon interacting with other solitons. The KdV equation is the pioneer model for analytic bell-shaped \( \text{sech}^2 \) solitary wave solutions. Figure 1.1 shows the structure of a bell-shaped \( \text{sech}^2 \) soliton solution characterized by infinite wings or...
infinite tails.

2. Periodic solutions

Periodic solutions are traveling wave solutions that are periodic such as $\cos(x,t)$. The standard wave equation $u_{tt} = u_{xx}$ gives periodic solutions. Figure 1.2 shows a periodic solution $u(x, t) = \cos(x-t), -\pi \leq (x, t) \leq \pi$ for a standard wave equation.

3. Kink waves

Kink waves are traveling waves which rise or descend from one asymptotic state to another. The kink solution approaches a constant level at infinity. The standard dissipative Burgers equation $u_t + uu_x - \eta u_{xx} = 0$, where $\eta$ is the viscosity coefficient, is a well-known equation that gives kink solution of the form $u(x, t) = 1 - \tanh(x-t), -10 \leq x, t \leq 10$. Other equations provide kink solutions as well. Figure 1.3 represents a kink wave solution. Kinks (and antikinks) show all the collisional properties of solitons; that is, they emerge unscathed from collision, suffering only a phase shift.

4. Peakons

Peakons are peaked solitary wave solutions whose peaks have a discontinuous first derivative. In this case, the traveling wave solutions are smooth except for a peak at a corner of its crest. Peakons are the points at which spatial derivative changes sign so that peakons have a finite jump in first derivative of the solution $u(x, t)$. This means that peakons have discontinuities in the $x$-derivative but both one-sided derivatives exist and differ only by a sign $[9]$. The peakons are solitons retaining their shape and speed after interacting. In $[9]$, peakons were investigated and classified as periodic peakons and peakons with exponential decay. The integrable Camassa-Holm (CH) and the Degasperis-Procesi (DP) equations have peaked solitary wave solutions of the form $u(x, t) = ce^{-|x-ct|}$, where $c$ is the wave speed. Figure 1.4 shows a peakon solution for CH equation with $c = 1$.

5. Cuspons

Cuspons are other forms of solitons where solutions exhibit cusps at their crests. Unlike peakons
where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon
diverges. In Figure 1.5 we show a cuspon with a cusp on its crest. The derivatives at the cusp
diverges. It is emphasized that the soliton solution \( u(x, t) \), along with its derivatives, tends to
zero as \(|x| \to \infty\). The CH and the DP equations give cuspons for specific cases.

6. Compactons

A compacton is a new class of solitons which have finite (compact) support such that each com-
pacton is a soliton confined to a finite core. Compactons are defined by solitary waves with the
remarkable soliton property, i.e. after colliding with other compactons, they reemerge with the
same coherent shape [10]. These particle like waves exhibit elastic collision that are similar to
the soliton collision. It was found that a compacton is a solitary wave with a compact support
where the nonlinear dispersion confines it to a finite core, therefore the exponential wings van-
ish. The nonlinear dispersive \( K(n, n) \) equations, a family of nonlinear KdV like equations, of
the form

\[
    u_t + a(u^n)_x + (u^n)_{xx} = 0, \quad a > 0, \quad n > 1,
\]

supports compact solitary traveling structures for \( a > 0 \). The definitions given so far for com-
pactons are: (i)These are solitons with finite wavelength; (ii) These are solitary waves with
compact support; (iii) These are solitons free of exponential tails; (iv) These are solitons char-
acterized by the absence of infinite wings; (v) These are robust soliton-like solutions.

Two important features of compacton structures are observed, namely:

(i) unlike the standard KdV soliton where \( u(\xi) \to 0 \) as \( \xi \to \infty \), the compacton is characterized
by the absence of the exponential tails or wings, where \( u(\xi) \) does not tend to \( 0 \) as \( \xi \to \infty \); (ii)
unlike the standard KdV soliton where width narrows as the amplitude increases, the width of
the compacton is independent of the amplitude.

Figure 1.6 shows a graph of a compacton \( u(x, t) = \cos^{1/2}(x - ct), \quad c = 1, \quad 0 \leq u(x, t) \leq 1 \).

7. Oscillons

In physics, an oscillon is a soliton-like entity that occurs in granular and other dissipative me-
dia. Oscillons in granular media result from vertically vibrating a plate with a layer of uniform
particles placed freely on top. When the sinusoidal vibrations are of the correct amplitude and
frequency and the layer of sufficient thickness, a localized wave, referred to as an oscillon,
can be formed by locally disturbing the particles. This meta-stable state will remain for a long
time (many hundreds of thousands of oscillations) in the absence of further perturbation. An
oscillon changes form with each collision of the grain layer and the plate, switching between a
peak that projects above the grain layer to a crater like depression with a small rim. This self-sustaining state was named by analogy with the soliton, which is a localized wave that maintains its integrity as it moves. Whereas solitons occur as traveling waves in a fluid or as electromagnetic waves in a waveguide oscillons may be stationary. Astonishingly, oscillons of opposite phase will attract over short distances and form “bonded” pairs. Oscillons of like phase repel each other. Oscillons have been observed forming “molecule” like structures and long chains. In comparison, solitons do not form bound states. Oscillons have also been experimentally observed in thin perimetrically vibrated layers of viscous fluid and colloidal suspensions. Nonlinear electrostatic oscillations on a plasma boundary can also appear in the form of oscillons.

8. Breathers

A breather is a nonlinear wave in which energy concentrates in a localized and oscillatory fashion. The term breather originates from the characteristic that most breathers are localized in space and oscillate (breathe) in time. But also the opposite situation: oscillations in space and localized in time, is denoted as a breather. A breather is a localized periodic solution of either continuous media equations or discrete lattice equations. The exactly solvable sine-Gordon equation and the focusing nonlinear Schrödinger (NLS) equation are examples of one-dimensional PDEs that possess breather solutions. Discrete nonlinear Hamiltonian lattices in many cases support breather solutions. Breathers are solitonic structures. There are two types of breathers: standing or traveling ones. Standing breathers correspond to localized solutions whose amplitude vary in time (they are sometimes called oscillons). A necessary condition for the existence of breathers in discrete lattices is that the breather main frequency and all its multipliers are located outside of the phonon spectrum of the lattice. A breather solution of the sine-Gordon (sG) equation

$$u_{tt} - u_{xx} + \sin(u) = 0,$$  \hspace{1cm} (1.6)
has been found by using the inverse scattering transform as

\[ u = 4 \arctan \left( \frac{\sqrt{1 - \omega^2 \sin(\omega t)}}{\omega \cosh(\sqrt{1 - \omega^2} x)} \right), \]  

which, for \( \omega < 1 \), is periodic in time \( t \) and decays exponentially when moving away from \( x = 0 \). Although it is time dependent. Solution (1.7) is at rest in the sense that it corresponds to an oscillation localized around position \( x = 0 \), which does not propagate and shown in Fig. 1.7.

9. Envelope solitons

Solitary-wave descriptions of the envelopes of waves, such as those arise from the propagation of modulated plane waves in a dispersive nonlinear medium with an amplitude-dependent dispersion relation. One typically uses the descriptor **bright** to describe solitary waves whose peak intensity is larger than the background (reflecting applications in optics) and the descriptor **dark** to describe solitary waves with lower intensity than the background. Figs. 1.8 and 1.9 shows bright and dark solitons respectively.

10. Gap solitons

Solitary waves that occur in finite gaps in the spectrum of continuous systems. For example, gap solitons have been studied rather thoroughly in NLS equations with spatially periodic potentials and have been observed experimentally in the context of both nonlinear optics and Bose-Einstein condensation.

In the literature, the kink (bell)-shaped solitons are called topological (non-topological) solitons. In the former (latter) the two asymptotic states at \( \pm \infty \) are different from (the same with) each other. Also, for historical reasons, the same type of soliton may be called by different names in different physical systems. For example, the kink solitons assume the name of “walls” in magnetic systems and liquid crystals, “fluxons” in long Josephson junctions, and “discommensurations” in crystals with competing interactions. A topological soliton, also called a topological defect, is any solution of a set of PDEs that is stable against decay to the “trivial solution”. Examples of topological solitons include the screw dislocation in a crystalline lattice, the Dirac string and the magnetic monopole in electromagnetism, the Skyrmion and the Wess-Zumino-Witten model in quantum field theory, and cosmic strings and domain walls in cosmology.
1.3.2 Some observations on solitons

Since the discovery of the many unusual and fascinating properties of solitons, soliton excitations have been found in several condensed matter, molecular, biological and optical systems. Seeger et al. [11] had shown that the domain walls in ferromagnets are solitonic structures. The central peaks observed in scattering experiments on quasi-one-dimensional materials that exhibit structural transitions have been accredited [12] to solitons in the form of domain walls. The nonlinear current-voltage produced by pinned charge density wave materials has been attributed to solitons [13]. Soliton excitations have been found in quasi-one-dimensional isotropic Heisenberg magnets [14] and in Josephson junctions [15]. Solitons have also been originated in polycrystalline [16] and biological molecules [17] including DNA [18]. Solitons are related to the low-frequency collective motion in proteins and DNA. Solitons also have surprising manifestations in optics. McCall and Hahn [19] showed that if the frequency of an intense coherent pulse of radiation incident on an absorbing medium is close to the medium’s resonance frequency, the linear theory of absorption completely breaks down. In particular, the medium absorbs energy from the leading edge of the pulse, but re-radiates, leading to the formation and propagation of a soliton. This results in an anomalously low energy loss of the optical pulse, a phenomenon known as self-induced transparency. An significant technical application of solitons is found in optical communications, where the balance between the nonlinearity in the dielectric constant and dispersion can be used to transmit signals without degradation by using optical solitons [20, 21]. Solitons have also been found in highly anisotropic Bose-Einstein condensates of dilute atomic gasses [22]. Sievers and Takeno [23] have speculated that the breather excitations may also be quite widespread throughout nature, perhaps even occurring in the vibrational spectrum of three-dimensional ionic crystals and α-uranium [24]. Very recently, Chabchoub and Kimmoun et. al [25] have first ever observed dark solitons on the surface of water. It takes the form of an amplitude drop of the carrier wave which does not change shape in propagation. The shape and width of the soliton depend on the water depth, carrier frequency and the amplitude of the background wave. Some types of tidal bore, a wave phenomenon of a few rivers including the River Severn, are “undular”: a wavefront followed by a train of solitons. Other solitons occur as the undersea internal waves, initiated by seabed topography, that propagate on the oceanic pycnocline. Atmospheric solitons also exist, such as the “Morning Glory Cloud” of the Gulf of Carpentaria, where pressure solitons traveling in a temperature inversion layer produce vast linear roll clouds. The recent and not widely accepted
soliton model [26] in neuroscience proposes to explain the signal conduction within neurons as pressure solitons. The observations of Scott Russell and their full theoretical description by Korteweg and de-Vries signaled the birth of the field of nonlinear physics. Nonlinear physics is a subject that is thriving and will continue to thrive, but perhaps not as a distinct field. This subject may become integrated in the other branches of physics since interactions form essential parts of real physical systems and with increasing frequency, we comprehend that interactions cannot be treated as small perturbations.

1.4 Soliton equations

In past many exactly solvable models which have soliton solutions, including the KdV equation, the NLS equation, the coupled NLS equation, the sine-Gordon equation etc. were thoroughly investigated and have many interesting applications in a variety of fields. The mathematical theory of these equations is a broad and very active field of research. In this section some prototype soliton equations and their soliton solutions are given.

1.4.1 The Korteweg-de Vries equation

The KdV equation (with $\alpha = \text{constant}$) [1] is written as

$$u_t + \alpha u u_x + u_{xxx} = 0. \quad (1.8)$$

The single soliton solution is given by

$$u(x, t) = \frac{12}{\alpha} a^2 \text{sech}^2[a(x - 4a^2 t - x_0)], \quad (1.9)$$

which is a bell shaped and localized in space for fixed $t$, where $a$ and $x_0$ are arbitrary constants. Note that the amplitude $(12/\alpha)a^2$, the wave width $1/a$ and the velocity $4a^2$ are all related to each other—a characteristics property of solitons, which separate them from traveling wave solutions of linear equations where all three quantities are usually independent of each other. There is no “breather” soliton for this equation. The equation describes phenomena with weak nonlinearity and weak dispersion, including waves in shallow water, ion-acoustic and magneto hydrodynamic waves in plasmas, and phonon packets in nonlinear crystals.
1.4.2 The nonlinear Schrödinger equation

Another significant nonlinear PDE is the NLS equation, which takes the form [1]

\[ iu_t + u_{xx} + \alpha u|u|^2 = 0, \tag{1.10} \]

where \( u(x,t) \) is complex. The single soliton solution assumes the form

\[ u(x,t) = u_0 \text{sech}[(\alpha/2)^{1/2}u_0(x - at)] \exp[i(a/2)(x - bt)], \tag{1.11} \]

where \( a \) and \( b \) are the velocities of the envelope and the carrier respectively. Breather solitons do exist for it. The NLS equation describes phenomena with weak nonlinearity and strong dispersion, such as wave in deep water, self-focusing of laser in dielectrics, propagation of signals in optical fibers, one dimensional Heisenberg magnets and vortices in fluid flow.

1.4.3 The sine-Gordan (sG) equation

The sG equation is given by [1]

\[ u_{xx} - u_{tt} + \sin u = 0. \tag{1.12} \]

This equation appeared in various problems of differential geometry. A more contemporary use for it is in relativistic field theory. It is often convenient to study Eq.(1.12) in the variables \( \eta = \frac{1}{2}(x - t) \) and \( \xi = \frac{1}{2}(x + t) \) which transforms it to \( u_{\eta\xi} = \sin u \). There exist three basic types of solitons

(i) Kink soliton

\[ u = 4 \tan^{-1} \left( \exp[(x - ct - x_0)/(1 - c^2)^{1/2}] \right), \tag{1.13} \]

(ii) Antikink soliton

\[ u = 4 \tan^{-1} \left( \exp[-(x - ct - x_0)/(1 - c^2)^{1/2}] \right). \tag{1.14} \]

(iii) Breather solution

\[ u = 4 \tan^{-1} \left( (\tan a) \sin[(\cos a)(t - t_0)] \sech[(\sin a)(x - x_0)] \right), \tag{1.15} \]

where \( c, x_0, t_0 \) and \( a \) are constants. A breather is not a traveling wave, may be considered as the bound state of kink-antikink pair.

A two-soliton solution

\[ u = 4 \tan^{-1} \left( \frac{c \sinh[x/(1 - c^2)^{1/2}]}{\cosh[ct/(1 - c^2)^{1/2}]} \right), \tag{1.16} \]
also exists, which describes the collision of two single solitons. The sG equation has also been
used to describe crystal dislocation, walls in magnetic systems, disclinations in liquid crystals,
magnetic fluxes on a Josephson line [1] and so on. Furthermore, because the sG equation is
Lorentz invariant, it is much studied by particle physicists as a model field theory [27] with the
hope that elementary particles may eventually be interpreted as solitons. Interestingly, in this
regard, the sG solitons with \( c > 1 \) will then correspond to particles traveling faster than light—the
tachyons.

### 1.4.4 The Kadomtsev-Petviashvili (KP) equation

Integrable soliton equations in two dimensions (2D) are much rarer than those in 1D. An im-
portant example of the former was given by Kadomtsev and Petviashvili [28], which appeared
first in the stability study of the KdV solitons to transverse perturbations. The KP equation
\[
(u_t + 6 uu_x + u_{xxx})_x + 3u_{yy} = 0,
\]
(1.17)
has single soliton solution
\[
u(x, y, t) = 2a^2 \text{sech}^2 a[x + by - (3b^2 + 4a^2)t + x_0],
\]
(1.18)
which travels in an arbitrary direction in the \((x, y)\) plane, as well as multi soliton solutions. The
KP equation describes water surface waves and ion-acoustic waves in a plasma. As is the case
with the KdV equation, only overtaking (but not head-on) collisions are described by the KP
equation.

### 1.4.5 The Klein-Gordon (KG) equation

The nonlinear KG equation [29]
\[
u_{xx} - \nu_{tt} = \nu(\nu + 1)(\nu - 1),
\]
(1.19)
can be derived obviously from the continuum Hamiltonian
\[
H = \int dx \left[ \frac{1}{2} \nu_t^2 + \frac{1}{2} \nu_x^2 + \frac{1}{4} (\nu^2 - 1)^2 \right],
\]
(1.20)
where \( \nu = \nu(x, t) \). Equation (1.19) has the kink and antikink solutions
\[
u = \pm \tanh[a(x - ct)/\sqrt{2}],
\]
(1.21)
19
with \( a = (1 - c^2)^{-1/2} \). The kink (anti kink) solution assumes the “+” (“−”) sign in Eq.(1.21) and has asymptotic states \( \mp 1 \) \((\pm 1)\) as \( x \to \mp \infty \). Similar to those in the sG equation, both kinks and antikinks here can move in either a positive or negative direction (i.e., \( c \) can assume any value between \(-1\) and \(+1\)). However, when a kink-antikink pair collides head-on, they do not penetrate (as is the case in the sG equation) but rebound from each other. This equation comes from quantum field theory and describes nonlinear wave interactions. The KG equation also known as Klein-Gordon-Fock equation, arises in the study of theoretical physics. This equation is the relativistic version of the Schrödinger’s equation. It represents the equation of motion of a quantum scalar or a pseudo-scalar field, which is a field whose quanta are spinless particles. The KG equation is second order in time and therefore it cannot be directly interpreted as the Schrödinger equation for a quantum state. Moreover, it does not admit a positive definite conserved probability density. Nevertheless, with a proper interpretation, the KG equation describes the quantum amplitude for finding a point particle in various places and it also describes the relativistic wave function. It also appears in nonlinear optics where it plays an important role for Bose-Einstein condensates trapped in strong optical lattices formed by the interference patterns of laser beams [30]. This equation also arises in many other physical problems including nonlinear dispersion [31] and nonlinear meson theory [32].

1.4.6 The double sine-Gordon equation

The double sG equation is given by [33]

\[
\frac{u_{xx}}{u_{tt}} = \pm \left[ \sin u + \frac{1}{2} \sin \left( u/2 \right) \right].
\] (1.22)

The Hamiltonian corresponding to double sG equation is written as

\[
H = \int dx \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \left[ \sin u + \cos \left( u/2 \right) \right] \right\}.
\] (1.23)

The double sG equation with the positive sign has applications in nonlinear optics; with negative sign it has applications in nonlinear optics and liquid \(^3\)He. For the positive sign case, there exists a soliton solution of double-kink form,

\[
u = 4 \tan^{-1}[\exp(u + a)] + 4 \tan^{-1}[\exp(u - a)],
\] (1.24)

where \( u = (\sqrt{5}/2)(x - ct)(1 - c^2)^{-1/2} \) and \( a = \ln(\sqrt{5} + 2) \); the spatial derivative of \( u \) has a twin peak structure. Numerical studies [34] suggest that the collision of two kinks can penetrate each other without deformation, like solitons, whereas the collision of a kink and an antikink
results in loss of energy through radiations.

For negative sign case of Eq.(1.22), two kinds of kink solitons exist [35]; namely,

\[ u_1 = 2\pi + 4 \tan^{-1}\left[\sqrt{3/5} \tanh(v/2)\right], \quad \text{and} \]
\[ u_2 = 4 \tan^{-1}\left[\sqrt{5/3} \tanh(v/2)\right], \]

where \( v = (\sqrt{15/16})(x - ct)(1 - c^2)^{-1/2} \). The kinks in Eq.(1.25) have a jump of \( 4\pi - 2b \) and \( 2b \), respectively, with \( b = 2 \cos^{-1}(-1/4) \). Numerical studies show that both kink-kink and kink-antikink collisions lose energy due to radiations.

### 1.4.7 The Fisher equation

The Fisher equation

\[ u_t = u_{xx} - u(u - 1), \]

has an explicit soliton solution [36]

\[ u = 1 + \exp\left[\left(\eta/\sqrt{6}\right)(x - \eta ct - x_0)\right]^{-2}, \]

where \( c = 5/\sqrt{6}, \eta = \pm 1, \) and \( x_0 \) is an arbitrary constant. The Fisher equation occurs in flame propagation, biological growth problems [3], nuclear reactor theory and so on. It is probably the simplest nonlinear diffusion equation with an explicit soliton solution.

### 1.5 Various methods for PDEs analysis

Nonlinear evolution equations are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics and chemical physics. For a given a nonlinear PDE, there is no general way of knowing whether it has soliton solutions or not, or how the soliton solutions can be found. In order to get a better understanding of the underlying phenomena as well as their further applications in practical life, it is important to seek their exact solutions. Analytical solutions to nonlinear PDEs play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Moreover, new exact solutions may help researchers to find new phenomena. The exact solutions, if available, of nonlinear PDEs facilitate the verification of results for numerical solvers and aid in the stability analysis of solutions.
A literature survey reveals that researchers usually employ a variety of distinct methods to analyze nonlinear evolution equations. The methods range from reasonable to difficult that require a huge size of work. In fact there is no unified method that can be used for all types of nonlinear evolution equations. For single soliton solutions, several methods, such as the inverse scattering transformation method [37], Hirota’s bilinear method [38], the truncated Painlevé expansion [1], Bäcklund transformation method [1], Darboux transformation method [39] homogeneous balance method [40], projective Riccati equation method [41], Jacobi elliptic functions method [42] and many others have been used in past. Lot of informations and details about these methods are presented in several texts. Therefore, in order to solve some practical problems, one should select an appropriate method from the list of available methods. The salient features of some of the commonly used direct methods for solving nonlinear PDEs are presented in the next few subsections.

1.5.1 The Hirota’s bilinear method

In 1970, Hirota developed an ingenious method for obtaining the exact multi-soliton solutions of the KdV equation and derived an explicit expression for its N-soliton solution. An elegant formulation of this method requires the use of bilinear operators, therefore it is called Hirota’s bilinear method [38]. This method is usually applied to completely integrable equations, is well suited for partially integrable equations as well. The fundamental idea behind the method is to use some dependent variable transformation to put the nonlinear PDE in a form where the new dependent variable appears bilinearly. Once the bilinear form of the equation is found, one introduces a formal perturbation expansion to construct its solution step by step. If soliton solutions exist this expansion will always truncate and the then finite series will lead to an exact solution.

The drawback of this method is that the bilinear form of the PDE must be known. In other words, the technique applies to any equation that can be written in bilinear form, either as a single bilinear equation or as a system of coupled bilinear equations. Hirota introduced the customary definition of the Hirota’s bilinear operators by

\[
D^n_t D^m_t (a, b) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m a(x, t)b(x', t')|x' = x, t' = t,
\]

(1.29)
where \( n \) and \( m \) are positive integers. Some of the bilinear differentials operators are given as

\[
D_x(a.b) = a_x b - ab_x,
\]

\[
D_x^2(a.b) = a_{xx} b - 2a_x b_x + ab_{xx},
\]

\[
D_x D_t(a.b) = D_x(a_t b - ab_t) = a_{xt} b - a_t b_x - a_x b_t + ab_{xt},
\]

\[
D_x D_t(a.a) = 2(au_{xt} - a_x u_t),
\]

\[
D_x^4(a.b) = a_{xxxx} b - 4a_{xxx} b_x + 6a_{xx} b_{xx} - 4a_x b_{xxx} + ab_{xxxx},
\]

\[
D_x^n(a.a) = 0.
\]  \( (1.30) \)

for \( n \) is odd and so on. To comprehend the method, consider the solution of the canonical KdV equation

\[
u_t + 6uu_x + u_{xxx} = 0, \]  \( (1.31) \)

can be expressed by dependent variable transformation

\[
u(x,t) = 2 \frac{\partial^2}{\partial x^2} \ln F, \]  \( (1.32) \)

then Eq.(1.31) takes the form

\[
 F_{xxxx} F - 4F_{xxx} F_x + 3F^2_{xx} + F_{xt} F - F_x F_t = 0. \]  \( (1.33) \)

Now expanding function \( F(x,t) \) in a formal power series in terms of a small parameter \( \epsilon \) as

\[
F(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t), \]  \( (1.34) \)

where \( \epsilon^n \) is a formal expansion parameter. For the one-soliton solution, the solution of Eq.(1.33) becomes

\[
F(x,t) = 1 + \epsilon^n, \quad \eta_1 = k_1 x - k_1^3 t + \eta_1^{(0)}. \]  \( (1.35) \)

Then finally one-soliton solution becomes

\[
u(x,t) = \frac{k_1^2}{2} \text{sech}^2 \frac{1}{2}(k_1 x - k_1^3 t + \eta_1^{(0)}). \]  \( (1.36) \)

And for the two-soliton solution, \( F(x,t) \) is taken as

\[
F(x,t) = 1 + \epsilon^n + \epsilon^{n_2} + \epsilon^{n_1 + n_2 + A_{12}}, \]  \( (1.37) \)

where \( A_{12} = [(k_1 - k_2)/(k_1 + k_2)]^2 \), then we obtain two-soliton solution

\[
u(x,t) = \frac{1}{2}(k_2^2 - k_1^2) \left( \frac{k_2^2 \text{csch}^2(\eta_2/2) + k_1^2 \text{sech}^2(\eta_1/2)}{k_2 \text{coth}(\eta_2/2) - k_1 \tanh(\eta_1/2)} \right). \]  \( (1.38) \)
Similarly, the $N$-soliton solution is obtained from

$$F(x, t) = \sum_{i=1}^{N} \exp(\eta_i).$$

(1.39)

where $\eta_i = k_i x - c_i t$, where $k_i$ and $c_i$ are arbitrary constants, $k_i$ is called the wave number. The relation between $k_i$ and $c_i$ can be obtained by determining the dispersion relation.

### 1.5.2 The sine-cosine method

The main features of this method can be summarized as follows. The starting point is consider a nonlinear PDE of the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0,$$

(1.40)

which describes the dynamical wave solution $u(x, t)$. To find the traveling wave solution of Eq.(1.40), introduce the wave variable $\xi = (x - ct)$, which converts PDE (1.40) into an ODE

$$P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, ...) = 0.$$

(1.41)

Next integrate (1.41) as long as all terms contain derivatives where integration constants are considered zeros. The solutions of the reduced ODE equation can be expressed in the form

$$u(x, t) = \begin{cases} 
\lambda \sin(\beta (\mu \xi)), & |\xi| < \frac{\pi}{\mu} \\
0 & \text{otherwise}, 
\end{cases}$$

(1.42)

or in the form

$$u(x, t) = \begin{cases} 
\lambda \cos(\beta (\mu \xi)), & |\xi| < \frac{\pi}{\mu} \\
0 & \text{otherwise}, 
\end{cases}$$

(1.43)

where $\lambda, \mu$ and $\beta$ are parameters to be determined, $\mu$ and $c$ are the wave number and the wave speed respectively.

Substituting Eq.(1.42) or (1.43) into Eq.(1.41) gives a trigonometric equation of $\sin^K(\mu \xi)$ or $\cos^K(\mu \xi)$ terms. The parameters are then determined by first balancing the exponents of each pair of cosine or sine to determine $K$. Now collect all coefficients of the same powers in $\sin^K(\mu \xi)$ or $\cos^K(\mu \xi)$, where these coefficients have to vanish. This gives a system of algebraic equations among the unknowns $c, \lambda, \mu$ and $\xi$. The solutions offered in Eq.(1.42) or (1.43) follow immediately. In recent years, this method has been successfully applied to a large number of nonlinear PDEs [4].
1.5.3 The tanh-coth method

To demonstrate the basic concepts of this method, again consider nonlinear PDE (1.40) or its ODE-version (1.41). Now, if one introduces a new independent variable \( Y = \tanh(\mu \xi) \) where \( \mu \) is the wave number, leads to the change of derivatives:

\[
\frac{d}{d\xi} = \mu(1 - Y^2) \frac{d}{dY},
\]

\[
\frac{d^2}{d\xi^2} = -2\mu^2 Y(1 - Y^2) \frac{d}{dY} + \mu^2(1 - Y^2) \frac{d^2}{dY^2},
\]

\[
\frac{d^3}{d\xi^3} = 2\mu^3 Y(1 - Y^2)(3Y^2 - 1) \frac{d}{dY} - 6Y\mu^3(1 - Y^2)^2 \frac{d^2}{dY^2} + \mu^3(1 - Y^2)^3 \frac{d^3}{dY^3},
\]

and so on. The tanh-coth method admits the use of the finite expansion

\[
u(\xi) = S(Y) = \sum_{n=0}^{M} a_n Y^n + \sum_{n=1}^{M} b_n Y^{-n}.
\]

where \( M \) is a positive integer to be determined. To determine the parameter \( M \), balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. For noninteger \( M \) a transformation formula is used to overcome this difficulty. Then collect all coefficients of power of \( Y \) in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters \( a_n, b_n, \mu, \xi \) and \( c \). Having determined these parameters, an analytic solution \( u(x, t) \) in a closed form is obtained. Note that the expansion (1.45) reduces to the standard tanh method for \( b_n = 0, 1 \leq k \leq M \). The solutions obtain by this method may be solitons in terms of \( \text{sech}^2 \), or may be kinks in terms of \( \tanh \). This method may also give periodic solutions as well. This method is used in chapter 4 to solve nonlinear diffusion problems. In past this method has been successfully applied to a large number of nonlinear PDEs [4].

1.5.4 The F-expansion method

For a given nonlinear evolution equation, say, for two variables \( x \) and \( t \)

\[
P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0.
\]

We assume that Eq.(1.46) has the following formal solutions

\[
u(x, t) = a_0 + \sum_{i=1}^{l} [a_i F(\xi)^i + b_i F(\xi)^{-i}]
\]
where \( a_0 = a_0(x,t), a_i = a_i(x,t), b_i = b_i(x,t), (i = 1, ..., l), \xi = \xi(x,t) \) are all arbitrary functions of indicated variables and \( F(\xi) \) satisfies the following elliptic equation [42]

\[
\left( \frac{dF}{d\xi} \right)^2 = c_0 + c_2F^2 + c_4F^4,
\]

where \( c_0, c_2 \) and \( c_4 \) are real constants related to the square of elliptic modulus \( m \) of Jacobi elliptic functions. When modulus number \( m \to 0 \) or \( 1(0 < m < 1) \), we can get trigonometric function solutions or hyperbolic function solutions respectively.

Determine the parameter \( l \) by balancing the highest order derivative terms with the nonlinear terms in Eq.(1.46). It represents a subtle balance of the dissipation effect and the dispersion effect in physics where soliton origins from.

Substituting Eqs.(1.47) and (1.48) into (1.46) yields a set of algebraic polynomials for \( F(\xi) \). Eliminating all the coefficients of the powers of \( F(\xi) \) and \( \sqrt{c_0 + c_2F^2 + c_4F^4} \) yields a series of differential equations, from which the parameters \( a_0, a_i, b_i(i = 1, ..., l) \) and \( \xi \) are explicitly determined.

Substituting \( a_0, a_i, b_i \) and \( \xi \) into Eq.(1.47) and selecting the Jacobian elliptic functions, we can derive all kinds of Jacobian elliptic function solutions of Eq.(1.46). As a matter of fact, this method has been applied to a number of specific cases involving both coupled and uncoupled nonlinear PDEs as well as to variable coefficients nonlinear PDEs [42, 43].

**The extended F-expansion method:**

We now describe the extended F-expansion method for nonlinear evolution equations. We concisely show what is extended F-expansion method and how to use it to find various periodic wave solutions to nonlinear wave equations.

In this method a nonlinear PDE

\[
P(u, u_t, u_x, u_y, u_{xt}, u_{tt}, u_{gt}, u_{xx}, ...) = 0,
\]

(1.49)

can be converted to an nonlinear ODE

\[
O(u, u', u'', ...) = 0,
\]

(1.50)

upon using a wave variable \( \xi = \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 + ... + \lambda_lx_l - \omega t \), so that \( u(x_1, x_2, x_3, ..., t) = u(\xi) \) and the localized wave solution \( u(\xi) \) travels with speed of \( \omega \).

Now suppose that \( u(\xi) \) can be expanded as follows

\[
u(\xi) = \sum_{j=0}^{n} \sum_{i=0}^{j} c_{ji}F^i(\xi)G^{j-i}(\xi), \quad c_{nn} \neq 0,
\]

(1.51)
or
\[ u(\xi) = a_0 + \sum_{i=1}^{n} a_i F^i(\xi) + \sum_{j=1}^{n} b_j F^{-j}(\xi), \quad a_n \neq 0, \] (1.52)

where \( c_{ji}, a_0, a_i \) and \( b_j \) are constants to be determined, \( F(\xi) \) and \( G(\xi) \) satisfy the following relations
\[
\begin{align*}
F^2(\xi) &= P_1 F^4(\xi) + Q_1 F^2(\xi) + R_1, \quad G^2(\xi) = P_2 G^4(\xi) + Q_2 G^2(\xi) + R_2, \quad (1.53) \\
G^2(\xi) &= \mu F^2(\xi) + \nu, \quad R_1 = \frac{Q_1^2 - Q_2^2 + 3P_2R_2}{3P_1}, \quad \mu = \frac{P_1}{P_2}, \quad \nu = \frac{Q_1 - Q_2}{3P_2}, \quad \nu \neq 0. \quad (1.54)
\end{align*}
\]

The integer \( n \) is determined by considering the homogeneous balance between the governing nonlinear terms and highest order partial derivatives of \( u \) in Eq.(1.50).

Substituting Eq.(1.51) or Eq.(1.52) into Eq.(1.50) and using Eqs.(1.53) and (1.54), we obtain a series in \( F^p G^q \) or \( (F^p) \) to zero yields a system of algebraic equations for \( c_{ji} (j = 0, 1, 2, ... n; i = 0, ..., j) \) and \( \lambda_i, \omega, \) or \( (a_i, b_j, i = 1, 2, ... n; j = 1, 2, ... n; \lambda_i, \omega) \).

Now solving these equations by use of either Mathematica or Maple, \( c_{ij} \), \( \lambda_i \) and \( \omega \) can be expressed in terms of \( P_i, Q_i, R_i, \mu, \nu \) and the parameters of Eq.(1.50). Substituting these results into (1.51) or (1.52) gives the general form of traveling wave solutions.

By using the relations (1.53) and (1.54), the appropriate kinds of the Jacobi elliptic function solutions of Eq.(1.49) including the single functions and the combined function solutions are found. As we know, when \( m \to 1 \), Jacobi elliptic functions degenerate as hyperbolic functions and \( m \to 0 \), Jacobi elliptic functions degenerate as trigonometric functions. So we can get the corresponding hyperbolic function solutions and trigonometric function solutions. We employ this method to the nonlinear problems in third and fourth chapters.

### 1.5.5 The Fan’s sub-equation method

The extended tanh-function method, the modified extended tanh-function method and the F-expansion method belong to a class of method called sub-equation method for which it appears some basic relationships among the complicated nonlinear PDEs in study and some simple and solvable nonlinear ODEs. Thus, these sub-equation methods consist of looking for the solutions of the nonlinear PDEs in consideration as a polynomial in variable which satisfies an equation or equations (named sub-equation). Fan [44] developed a new algebraic method,
belonging to the sub-equation method, to seek more new solutions of nonlinear PDEs that can be expressed as polynomial in an elementary function which satisfies a more general sub-equation than other sub-equations like Riccati equation, auxiliary ordinary equation, elliptic equation and generalized Riccati equation. With regard to its applications to nonlinear PDE (1.40) or its ODE-version (1.41), one makes an ansatz for the solution \( u(\xi) \) as

\[
u(\xi) = \sum_{i=0}^{m} a_i \phi^i(\xi),
\]

where \( m \) is again a positive integer which is determined by the balancing procedure as given in the tanh-coth method and \( \phi(\xi) \) expresses the solutions of an sub-equation, the choice of which depends on the system under study. In the following, we consider the general elliptic equation as

\[
\left( \frac{d\phi(\xi)}{d\xi} \right)^2 = h_0 + h_1 \phi(\xi) + h_2 \phi^2(\xi) + h_3 \phi^3(\xi) + h_4 \phi^4(\xi).
\]

In some special cases, when \( h_0 \neq 0, h_1 \neq 0, h_2 \neq 0, h_3 \neq 0, \) and \( h_4 \neq 0, \) it may exist three parameters \( r, p \) and \( q \) such that

\[
\left( \frac{d\phi(\xi)}{d\xi} \right)^2 = h_0 + h_1 \phi(\xi) + h_2 \phi^2(\xi) + h_3 \phi^3(\xi) + h_4 \phi^4(\xi)
\]

\[
= [r + p\phi(\xi) + q\phi^2(\xi)]^2
\]

Eq.(1.57) is satisfied only if the following relations holds: \( h_0 = r^2, h_1 = 2rp, h_2 = 2rq + p^2, h_3 = 2pq \) and \( h_4 = q^2. \) Thus for Eq.(1.57), the general elliptic equation is reduced to the generalized Riccati equation. When \( h_0 = h_1 = 0, \) the general elliptic equation is reduced to the auxiliary ordinary equation

\[
\left( \frac{d\phi(\xi)}{d\xi} \right)^2 = h_2 \phi^2(\xi) + h_3 \phi^3(\xi) + h_4 \phi^4(\xi).
\]

When \( h_1 = h_3 = 0, \) the general elliptic equation is reduced to the elliptic equation

\[
\left( \frac{d\phi(\xi)}{d\xi} \right)^2 = h_0 + h_2 \phi^2(\xi) + h_4 \phi^4(\xi).
\]

Eq.(1.59) includes the Riccati equation

\[
\left( \frac{d\phi(\xi)}{d\xi} \right)^2 = [A + \phi^2(\xi)]^2,
\]

when \( h_0 = A^2, h_2 = 2A, h_4 = 1 \) and solutions of Eq.(1.60) can be deduced from those of Eq.(1.59) in a specific case where the modulus \( m \) of the Jacobi elliptic functions (solutions of
Eq. (1.59) is driven to 1 and 0. When \( h_2 = h_4 = 0 \), the general elliptic equation is reduced to the following equation
\[
\left( \frac{d\phi(\xi)}{d\xi} \right)^2 = h_0 + h_1\phi(\xi) + h_3\phi^3(\xi),
\]
where \( h_0, h_1, h_2, h_3 \) and \( h_4 \) are real parameters. The above set of ODEs admits various solutions depending on the choices of constants appearing in Eqs. (1.56)-(1.61). If we substitute ansatz Eq. (1.55) with along an auxiliary equation in Eq. (1.58) and equating to zero the coefficients of all powers of \( \phi(\xi) \) yields a set of algebraic equations. On solving these algebraic equations, we get the values of constants \( a_i \)'s in Eq. (1.55).

With regard to generalizations of this method, the ansatz becomes
\[
u(\xi) = \sum_{i=0}^{2m} a_i \phi^i(\xi).
\]
This method has been applied to a variety of systems and both the periodic and solitary wave solutions are investigated. The cases of coupled nonlinear PDEs with variable coefficients are also studied [45, 46, 47].

1.5.6 The \((G'/G)\)-expansion method

Recently Wang et al. [48] introduced a very elegant and powerful \((G'/G)\)-expansion method to find traveling wave solutions of a given nonlinear PDE. So to highlight the strength of this method, we again take the example of Eq. (1.40) (or Eq. (1.41)). The solution of Eq. (1.41) can be expressed by a polynomial in \((G'/G)\) as
\[
u(\xi) = \sum_{i=0}^{n} \alpha_i (\frac{G'}{G})^i,
\]
where \( G = G(\xi) \) satisfies the second order linear ODE of the form
\[
G'' + \lambda G' + \mu G = 0,
\]
where \( G' = \frac{dG(\xi)}{d\xi}, G'' = \frac{d^2G(\xi)}{d\xi^2} \) and \( \alpha, \lambda, \mu \) are constants to be determined later and \( \alpha_n \neq 0 \). The positive integer \( n \) can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in ODE (1.41). More precisely, we define the degree of \( \nu(\xi) \) as \( D[\nu(\xi)] = n \), which gives rise to the degree of other expressions as follows,
\[
D\left[ \frac{d^p \nu}{d\xi^q} \right] = n + q, \quad D\left[ \nu^p \left( \frac{d^q \nu}{d\xi^q} \right)^s \right] = np + s(n + q).
\]
Therefore, we can get the value of $n$ in Eq.(1.63). Substituting Eq.(1.63) into Eq.(1.41) and using Eq.(1.64), collecting all terms with the same order of $\left( \frac{G'}{G} \right)$ together, and then equating each coefficient of the resulting polynomial to zero yields a set of algebraic equations for $\alpha_i$, $c$, $\lambda$ and $\mu$. The solutions of Eq.(1.64) depending on whether $\lambda^2 - 4\mu > 0$, $\lambda^2 - 4\mu < 0$, $\lambda^2 - 4\mu = 0$ are given as

$$
\left( \frac{G'}{G} \right) = \begin{cases} 
\frac{\sqrt{\lambda^2-4\mu}}{2} \left( A_1 \sinh \left( \frac{1}{2} \sqrt{\lambda^2-4\mu} \xi + A_2 \cosh \left( \frac{1}{2} \sqrt{\lambda^2-4\mu} \xi \right) \right) - \frac{\lambda}{2}, \lambda^2 - 4\mu > 0, \\
\frac{\sqrt{4\mu-\lambda^2}}{2} \left( -A_1 \sin \left( \frac{1}{2} \sqrt{4\mu-\lambda^2} \xi + A_2 \cos \left( \frac{1}{2} \sqrt{4\mu-\lambda^2} \xi \right) \right) - \frac{\lambda}{2}, \lambda^2 - 4\mu < 0, \\
\frac{A_2}{A_1+A_2\xi} - \frac{\lambda}{2}, \lambda^2 - 4\mu = 0.
\end{cases}
$$

The above results can be written in simplified forms as

$$
\left( \frac{G'}{G} \right) = \begin{cases} 
\frac{\sqrt{\lambda^2-4\mu}}{2} \tanh \left( \frac{1}{2} \sqrt{\lambda^2-4\mu} \xi + \xi_0 \right) - \frac{\lambda}{2}, \lambda^2 - 4\mu > 0, \tanh \xi_0 = \frac{A_1}{A_2}, |\frac{A_1}{A_2}| > 1, \\
\frac{\sqrt{\lambda^2-4\mu}}{2} \coth \left( \frac{1}{2} \sqrt{\lambda^2-4\mu} \xi + \xi_0 \right) - \frac{\lambda}{2}, \lambda^2 - 4\mu > 0, \coth \xi_0 = \frac{A_1}{A_2}, |\frac{A_1}{A_2}| < 1, \\
\frac{\sqrt{4\mu-\lambda^2}}{2} \cot \left( \frac{1}{2} \sqrt{4\mu-\lambda^2} \xi + \xi_0 \right) - \frac{\lambda}{2}, \lambda^2 - 4\mu > 0, \cot \xi_0 = \frac{A_2}{A_1}, \\
\frac{A_2}{A_1+A_2\xi} - \frac{\lambda}{2}, \lambda^2 - 4\mu = 0.
\end{cases}
$$

The above results are the simplified forms of results obtained by $\left( \frac{G'}{G} \right)$-expansion method and hence called simplified $\left( \frac{G'}{G} \right)$-expansion method [49]. Now substituting $\alpha_i$, $c$ and the general solutions of Eq.(1.64) from Eqs.(1.66) or (1.67) into Eq.(1.63), we obtain traveling wave solutions of nonlinear differential equation (1.41).

1.5.7 The projective Riccati equation method

The well known projective Riccati equations read as

$$
f' (\xi) = pf (\xi) g(\xi), \quad (1.68)
$$

$$
g' (\xi) = R + pg^2 (\xi) - rf (\xi), \quad (1.69)
$$

where $p \neq 0$ is a real constant, $R$ and $r$ are two real constants. The relation between $f$ and $g$ is

$$
g^2 = -p \left[ R - 2rf + \frac{r^2 + \delta}{R} f^2 \right], \quad \delta = \pm 1, \ R \neq 0. \quad (1.70)
$$

In this method, a given nonlinear PDE in the unknown $a(x, y, z, ..., t)$, is reduced into a ODE
by the traveling wave reduction $u(x, y, z, ..., t) \rightarrow u(\xi = \lambda_1 x + \lambda_2 y + \lambda_3 z + ... + \lambda_n t)$. The solution $u(\xi)$ assumed of the form

$$u(\xi) = \sum_{i=1}^{n} f^{i-1}(\xi)[A_i f(\xi) + B_i g(\xi)] + A_0, \quad R \neq 0,$$

(1.71)

where $A_i$ and $B_i$ are constants to be fixed later and $f(\xi)$ and $g(\xi)$ are solutions of Eqs.(1.68) and (1.69). The parameter $n$ in Eq.(1.71) can be determined by balancing procedure. Now substituting Eq.(1.71) along with conditions (1.68), (1.69) and (1.70) into $O(u, u', u'', ...)$ = 0, and setting the coefficient of $f^i g^j (j = 0, 1, i = 0, 1, 2, 3, ...)$ to zero yields a set of over determined algebraic equations, from which the constants $A_i, B_i, R, r$ and $\lambda_i$ can be found explicitly.

The exact solutions of Eqs.(1.68) and (1.69) are of the form [50, 51]

Case 1: When $p = -1$, $\delta = -1$, $R \neq 0$

$$f_1(\xi) = \frac{R \text{sech}(\sqrt{R}\xi)}{r \text{sech}(\sqrt{R}\xi) + 1}, \quad g_1(\xi) = \frac{\sqrt{R} \tanh(\sqrt{R}\xi)}{r \text{sech}(\sqrt{R}\xi) + 1}.$$  (1.72)

Case 2: When $p = -1$, $\delta = 1$, $R \neq 0$

$$f_2(\xi) = \frac{R \text{csch}(\sqrt{R}\xi)}{r \text{csch}(\sqrt{R}\xi) + 1}, \quad g_2(\xi) = \frac{\sqrt{R} \coth(\sqrt{R}\xi)}{r \text{csch}(\sqrt{R}\xi) + 1}.$$  (1.73)

Case 3: When $p = 1$, $\delta = -1$, $R \neq 0$

$$f_3(\xi) = \frac{R \sec(\sqrt{R}\xi)}{r \sec(\sqrt{R}\xi) + 1}, \quad g_3(\xi) = \frac{\sqrt{R} \tan(\sqrt{R}\xi)}{r \sec(\sqrt{R}\xi) + 1},$$

$$f_4(\xi) = \frac{R \csc(\sqrt{R}\xi)}{r \csc(\sqrt{R}\xi) + 1}, \quad g_4(\xi) = \frac{-\sqrt{R} \cot(\sqrt{R}\xi)}{r \csc(\sqrt{R}\xi) + 1}.$$  (1.74)

Case 4: When $R = r = 0$

$$f_5(\xi) = \frac{C}{\xi} = C_{pg5}(\xi), \quad g_5(\xi) = \frac{1}{p\xi},$$  (1.76)

where $C$ is a constant.

Substitute the constants $A_i, B_i, R, r$ and $\lambda_i$ into Eq.(1.71) along with Eqs.(1.72)-(1.76) to obtain soliton and periodic (or rational) solutions of the nonlinear PDE of concern. This method is used in chapters 2, 3 and 4 of present thesis to solve nonlinear equations.

Apart from the above mentioned direct methods, several other direct and approximate methods have also been used in the literature to solve a given nonlinear PDE. For example, the variable separation method [1], the Adomain decomposition method [4], variational iteration method [4], homotopy analysis method [4], solitary wave ansatz method [52], He’s variational method
Motivation

Solitons, a fundamental nonlinear wave phenomenon, have been demonstrated to exist in many physical systems: surface waves in shallow water, plasma waves, high-energy physics, localized vibrational modes in biological systems, matter waves in Bose-Einstein condensates and nonlinear waves in optics. The availability of numerous material systems that are fully characterized by soliton equations and the ability to sample the waves directly as they propagate result in a field in which theory and experiments make rapid progress hand in hand. Equations having soliton solutions, in the exact mathematical sense, provide remarkable examples of completely integrable systems with an infinite number of degrees of freedom. Thus, the soliton theory has attracted the interest of numerous researchers. Therefore, keeping in view the promising technological applications of the solitons, the subject matter of this thesis is to find solitary wave solutions of some nonlinear physical systems and to explain their dynamical behavior.

1.6 The organization of the thesis

The organization of the thesis is as follows: In chapter 2, the optical solitary wave solutions for the nonlinear Schrödinger equation which describes the propagation of femtosecond light pulses in optical fibers in the presence of self-steepening and a self-frequency shift terms are derived. The solitary wave ansatz method is used to carry out the derivations of the solitons. Moreover by using He’s semi inverse method, variational formulation is established to obtain exact soliton solutions. The dark and bright 1-soliton solutions of time dependent form of this equation are also obtained. 1-soliton solutions of time-dependent inhomogeneous nonlinear Schrödinger equation are derived. Finally, we have applied F-expansion technique to the generalized nonlinear Schrödinger equation with varying parameters and successfully obtained Jacobi elliptic function solutions, Weierstrass elliptic function solutions, chirped and unchirped dark and bright solitary wave solutions and trigonometric function solutions.

In chapter 3, the exact spatiotemporal periodic traveling wave solutions to the generalized (3+1)-dimensional cubic-quintic nonlinear Schrödinger equation with spatial distributed coefficients are obtained. We then demonstrate the nonlinear tunneling effects and controllable compression technique of 3-dimensional bright and dark solitons when they pass unchanged through the potential barriers and wells affected by special choices of the diffraction and/or the nonlinearity parameters. The exact solution of (2+1)-dimensional nonlinear Schrödinger equation are
obtained by using the extended F-expansion and the projective Riccati equation methods.

In chapter 4, we present exact solutions of density independent and dependent nonlinear diffusion reaction equations. The bright and dark soliton solutions of variable coefficient modified KdV equation, complex modified KdV equation and generalized Gardner equation are obtained by using solitary wave ansatz method. We also investigated the Maccari equation by using extended F-expansion and projective Riccati equation methods.

Finally in concluding chapter 5, the major findings and future scope of the present thesis work are summarized.