Chapter 4

Length Biased Dynamic Inaccuracy Measures

4.1 Introduction

The concept of weighted distribution introduced by Rao [96] is widely used in statistics and other applications. Jain et al. [61], Gupta and Kirmani [48] and Nanda and Jain [83] have used the weighted distribution in many practical problems to model unequal sampling probabilities. Such distributions arise when the observations generated from a stochastic process are recorded with some weight function. Let $X$ be a non-negative continuous random variable with probability density function $f(x)$, and let $X^w$ be a weighted random variable corresponding to $X$ with weight function $w(x)$ which is positive for all value of $x \geq 0$. Then the corresponding p.d.f. $f^w(x)$ of the random variable $X^w$ is given by

$$f^w(x) = \frac{w(x)f(x)}{E[w(X)]}, \quad 0 \leq x < \infty$$

(4.1)

with $0 < E[w(X)] < \infty$. 

65
When \( w(x) = x \), \( X^w \) is said to be a length biased (or a size biased) random variable and the p.d.f. (4.1) in this case becomes

\[
f^L(x) = \frac{xf(x)}{E[X]}.
\] (4.2)

The length biased distribution function and the length biased survival function are defined respectively as

\[
F^L(t) = \frac{1}{E[X]} \int_0^t xf(x)dx, \quad \bar{F}^L(t) = \frac{1}{E[X]} \int_t^\infty xf(x)dx
\] (4.3)

respectively. These functions characterize weighted distributions that arise in sampling procedures where the sampling probabilities are proportional to the sample values, refer to Patil et al. [95], Furman and Zitikis [45].

In literature Belis and Guiasu [15] raised the important issue of integrating the quantitative concept of information with the qualitative concept, called utility and characterized weighted information measure, called the quantitative-qualitative measure of information, refer to (1.13). Information theoretic measures of weighted relative information and of weighted inaccuracy have been given by Taneja and Tuteja [118] and Taneja [120] respectively. However in these studies the weights attached to the outcomes of a random variables were independent of their probabilities of occurrence.

In the preceding chapter we have proposed the dynamic (both, residual and past) inaccuracy measures. In the present chapter we extend the concept of dynamic inaccuracy measure to the length biased dynamic inaccuracy measures and study the characterization results pertaining to the measures proposed. The chapter is organized as follows. In Section 4.2, we propose a measure of length biased residual inaccuracy and express it in terms of residual inaccuracy measure studied in Chapter 3. Section 4.3 considers a characterization result that under proportional hazard model the measure proposed characterizes the distribution function uniquely; and
also we have derived an upper bound to it. In Section 4.4 we consider a length biased measure of past inaccuracy and in Section 4.5 we prove a characterization result for this measure under proportional reversed hazard model. Some further results concerning the length biased past inaccuracy measure have been considered in Section 4.6. The chapter ends with the conclusion.

4.2 Length Biased Residual Inaccuracy Measure

We have observed in Chapter 3 that if a system has survived up to time $t$, the corresponding dynamic measures of uncertainty, refer to Ebrahimi [34], of discrimination, refer to Ebrahimi and Kirmani [38], and of inaccuracy, refer to Taneja et al. [117], are given as

$$H(f; t) = - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx,$$

(4.4)

$$H(f/g; t) = \int_t^\infty \frac{f(x)}{F(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/G(t)} dx,$$

(4.5)

and

$$H(f, g; t) = - \int_t^\infty \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx,$$

(4.6)

respectively.

When $t = 0$, then (4.4), (4.5) and (4.6) reduce to measures of Shannon entropy [109], Kullback discrimination [70] and Kerridge inaccuracy [67] respectively. These information measures do not take into account the weightage of the random variable but only its probability density function.

Di Crescenzo and Longobardi [31] considered a length-biased shift dependent information measure related to the differential entropy in which higher weights are
assigned to the larger values of the observed random variables. The residual measure of entropy (4.4) has been extended to the length biased weighted residual entropy given as

$$H^L(f, t) = -\int_t^\infty x \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \, dx.$$  \hfill (4.7)

The factor $x$ in the integral on right-hand-side yields a "length-biased" shift dependent information measure assigning greater importance to the larger values of the random variable $X$.

In agreement with Taneja and Tuteja [118], we refer to the measure

$$H^L(f, g) = -\int_0^\infty x f(x) \log g(x) \, dx ,$$  \hfill (4.8)

as the length biased weighted inaccuracy, and propose the measure

$$H^L(f, g; t) = -\int_t^\infty x \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx ,$$  \hfill (4.9)

as the length biased residual inaccuracy measure.

When $g(x) = f(x)$, the measure (4.9) reduces to (4.7), the length biased weighted residual entropy given by Di Crescenzo and Longobardi [31]. In case the weights are independent of $x$ then (4.9) reduces to (4.4), the dynamic measure of uncertainty proposed by Ebrahimi [34], and also when $t = 0$, the measure (4.9) reduces to the measure (4.8), the length biased weighted inaccuracy.

### 4.2.1 Weighted Residual Inaccuracy in Terms of Residual Inaccuracy

Rewriting $H^L(f, g; t)$ as

$$H^L(f, g; t) = -\int_t^\infty dx \int_0^x \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dy$$

68
Using (4.6) in (4.10), we obtain

$$ H^L(f, g; t) = t H(f, g; t) - \left( \int_t^\infty \, dy \left( \int_y^\infty \, \alpha(x; t) \, dx \right) \right), $$

where $\alpha(x; t) = \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)}$.

Writing $\int_y^\infty \alpha(x; t) \, dx$ as $\beta(y; t)$, this can be written as

$$ H^L(f, g; t) = t H(f, g; t) - \left( \int_t^\infty \, \beta(y; t) \, dy \right). $$

Differentiating (4.11) w.r.t. $t$ both sides using Leibnitz rule for differentiation under integration, we obtain

$$ \frac{d}{dt} H^L(f, g; t) = t \frac{d}{dt} H(f, g; t) + H(f, g; t) + \beta(t; t) $$

$$ = t \frac{d}{dt} H(f, g; t) + H(f, g; t) + \int_t^\infty \alpha(x; t) \, dx. $$

Thus

$$ \frac{d}{dt} H^L(f, g; t) = t \frac{d}{dt} H(f, g; t), $$

a relation giving the rate of change of weighted residual inaccuracy (4.9) in terms of rate of change of residual inaccuracy (4.6). This relation is used in the characterization problem considered in the next section.

### 4.3 Characterization Problem

The general characterization problem is to determine whether the residual measure characterizes the distribution function uniquely. In this section, we study characterization problem for the weighted residual inaccuracy measure (4.9) under the

$$ \int_0^t \int_t^\infty \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx \, dy - \int_t^\infty \int_y^\infty \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx \, dy. \quad (4.10) $$

Using (4.6) in (4.10), we obtain

$$ H^L(f, g; t) = t H(f, g; t) - \left( \int_t^\infty \, dy \left( \int_y^\infty \alpha(x; t) \, dx \right) \right), $$

where $\alpha(x; t) = \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)}$.

Writing $\int_y^\infty \alpha(x; t) \, dx$ as $\beta(y; t)$, this can be written as

$$ H^L(f, g; t) = t H(f, g; t) - \left( \int_t^\infty \, \beta(y; t) \, dy \right). $$

Differentiating (4.11) w.r.t. $t$ both sides using Leibnitz rule for differentiation under integration, we obtain

$$ \frac{d}{dt} H^L(f, g; t) = t \frac{d}{dt} H(f, g; t) + H(f, g; t) + \beta(t; t) $$

$$ = t \frac{d}{dt} H(f, g; t) + H(f, g; t) + \int_t^\infty \alpha(x; t) \, dx. $$

Thus

$$ \frac{d}{dt} H^L(f, g; t) = t \frac{d}{dt} H(f, g; t), $$

a relation giving the rate of change of weighted residual inaccuracy (4.9) in terms of rate of change of residual inaccuracy (4.6). This relation is used in the characterization problem considered in the next section.

### 4.3 Characterization Problem

The general characterization problem is to determine whether the residual measure characterizes the distribution function uniquely. In this section, we study characterization problem for the weighted residual inaccuracy measure (4.9) under the
proportional hazard model (PHM) as already introduced in Chapter 3. Under this model, refer to Cox [24] and Efron [42], the survival functions of two random lifetime variables are related by

\[ \bar{G}(x) = [\bar{F}(x)]^\beta, \quad \beta > 0, \]

(4.13)

where \( \beta \) is the proportionality constant. We note that based on the proportional hazard model (4.13), the hazard rate functions \( \lambda_F(.) \) and \( \lambda_G(.) \) satisfy the relation \( \lambda_G(x) = \beta \lambda_F(x) \). We consider the following characterization theorem.

**Theorem 4.1** If the two random variables \( X \) and \( Y \) satisfy the proportional hazard model (4.13) with proportionality constant \( \beta (> 0) \), and if \( H^L(f, g; t) \) is increasing in \( t \) with \( H^L(f, g; t) < \infty \), then \( H^L(f, g; t) \) uniquely determines \( \bar{F}(.) \), the survival function of \( X \).

**Proof** Consider the weighted residual inaccuracy

\[ H^L(f, g; t) = - \int_t^\infty x \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx. \]

Rewriting this as

\[ \int_t^\infty x f(x) \log g(x) dx = \log \bar{G}(t) \left[ \int_t^\infty x f(x) dx \right] - \bar{F}(t) H^L(f, g; t), \]

or,

\[ \int_t^\infty x f(x) \log g(x) dx = \log \bar{G}(t) \left[ t \bar{F}(t) + \int_t^\infty \bar{F}(y) dy \right] - \bar{F}(t) H^L(f, g; t). \]

Differentiating this both sides w.r.t. \( t \), and then using (4.12) and substituting \( \lambda_F(t) = \frac{f(t)}{F(t)} \), and \( \lambda_G(t) = \frac{g(t)}{G(t)} \), we obtain

\[
-t \lambda_F(t) \log \lambda_G(t) + t \lambda_G(t) = -\lambda_G(t) \left[ \int_t^\infty \frac{\bar{F}(y) dy}{\bar{F}(t)} \right] \\
+ \lambda_F(t) H^L(f, g; t) - t \frac{d}{dt} H(f, g; t). \quad (4.14)
\]
Now under proportional hazard model $\lambda_G(t) = \beta \lambda_F(t)$. Using this, Eq. (4.14) becomes

$$-t \lambda_F(t) \log \beta \lambda_F(t) + \beta t \lambda_F(t) = -\beta \lambda_F(t) \frac{\int_t^\infty \bar{F}(y)dy}{F(t)} + \lambda_F(t) H^L(f, g; t) - t \frac{d}{dt} H(f, g; t).$$

Thus for any fixed $t$, $\lambda_F(t)$ is a positive solution of the equation $h(x) = 0$, where

$$h(x) = x \left\{ \beta t - t \log \beta x + \frac{\beta \int_t^\infty \bar{F}(y)dy}{F(t)} - H^L(f, g; t) \right\} + t \frac{d}{dt} H(f, g; t). \tag{4.15}$$

Here $h(0) = t \frac{d}{dt} H(f, g; t) \geq 0$, since we have assumed that $H(f, g; t)$ is increasing in $t$, and also as $x \to \infty$, $h(x) \to -\infty$. Further differentiating (4.15) with respect to $x$, we get

$$\frac{d}{dx} h(x) = \beta t - t \log \beta x + \frac{\beta \int_t^\infty \bar{F}(y)dy}{F(t)} - H^L(f, g; t) - t.$$

Now, $\frac{d}{dx} h(x) = 0$ if, and only if,

$$x = \frac{1}{\beta} \exp \left[ -\frac{1}{t} \left\{ t - \beta t - \frac{\beta \int_t^\infty \bar{F}(y)dy}{F(t)} + H^L(f, g; t) \right\} \right] = x_0, \text{ say.}$$

In view of the above, $h(x) = 0$ has a unique positive solution. Thus $\lambda_F(t)$, and hence $\bar{F}(t)$ is uniquely determined by the weighted residual inaccuracy measure $H^L(f, g; t)$ under the assumption that $\frac{d}{dt} H(f, g; t) \geq 0$. This concludes the proof.

### 4.3.1 A Lower Bound to $H^L(f, g; t)$

To derive a lower bound for the weighted residual inaccuracy measure (4.9), we consider the following conditional mean value of a random variable $X$ as

$$\delta_t = E(X \mid X > t) = \frac{1}{F(t)} \int_t^\infty x f(x) dx, \tag{4.16}$$
a result which finds applications in insurance and economics, refer to Furman and Zitikis [45]. We have the following result:

**Theorem 4.2** If the hazard rate function $\lambda_G(t)$ is decreasing in $t$, then

$$H^L(f, g; t) \geq -\delta_t \log \lambda_G(t). \quad (4.17)$$

**Proof** From (4.9), we have

$$H^L(f, g; t) = -\frac{1}{F(t)} \int_t^\infty xf(x) \log \lambda_G(x) dx - \frac{1}{F(t)} \int_t^\infty xf(x) \log \frac{G(x)}{G(t)} dx.$$

Since $\log \frac{G(x)}{G(t)} \leq 0$, for $x \geq t$, and by assumption that hazard rate is decreasing in $t$, we have $\log \lambda_G(x) \leq \log \lambda_G(t)$, thus

$$H^L(f, g; t) \geq -\frac{1}{F(t)} \int_t^\infty xf(x) \log \lambda_G(x) dx$$

$$\geq -\frac{\log \lambda_G(t)}{F(t)} \int_t^\infty xf(x) dx,$$

which gives

$$H^L(f, g; t) \geq -\delta_t \log \lambda_G(t).$$

**Example 4.1** If the true distribution function $F(x)$ and the reference distribution function $G(x)$ are exponentially distributed with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively, then

$$f(x) = \lambda_1 e^{-\lambda_1 x}, \quad g(x) = \lambda_2 e^{-\lambda_2 x},$$

$$\overline{F}(x) = 1 - F(x) = e^{-\lambda_1 x},$$

and, $$\overline{G}(x) = 1 - G(x) = e^{-\lambda_2 x}.$$
residual inaccuracy measure as

\[ H^L(f, g; t) = - \int_t^\infty x \frac{\lambda_1 e^{-\lambda_1 x}}{e^{-\lambda_1 t}} \log \frac{\lambda_2 e^{-\lambda_2 x}}{e^{-\lambda_2 t}} \, dx \]

\[ = \frac{\lambda_2}{\lambda_1} \left( t + \frac{2}{\lambda_1} \right) - \left( t + \frac{1}{\lambda_1} \right) \log \lambda_2. \]

Further, we note that hazard rate is constant for an exponential distribution, that is, \( \lambda(t) = \lambda \), and the conditional mean value is \( \delta_t = t + \frac{1}{\lambda} \). Thus (4.17) holds.

### 4.4 Length Biased Past Inaccuracy Measure

In many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. For instance if at time \( t \), a system which is observed only at certain preassigned inspection times is found to be down, then the uncertainty of the system’s life relies on the past, that is, at which instant in the interval \( (0, t) \) the system has failed.

Based on this idea, measures of past entropy \([29]\), discrimination \([30]\) and of inaccuracy \([72]\) over \( (0, t) \) are given respectively as

\[ H^*(f; t) = - \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x) F(t)}{F(t)} \, dx, \quad (4.18) \]

\[ H^*(f/g; t) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x) F(t)}{g(x) G(t)} \, dx, \quad (4.19) \]

and

\[ H^*(f, g; t) = - \int_0^t \frac{f(x)}{F(t)} \log \frac{g(x) G(t)}{G(t)} \, dx. \quad (4.20) \]

Further, the concept of past entropy given by (4.18) has been extended to the length
biased past entropy given as

\[
H^{*L}(f, t) = - \int_0^t x \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \, dx,
\]

(4.21)

refer to Di Crescenzo and Longobardi [31].

In sequel to this, we extend the past inaccuracy measure (4.20) to the length biased past inaccuracy given by

\[
H^{*L}(f, g; t) = - \int_0^t x \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx.
\]

(4.22)

This may be considered as the differential weighted inaccuracy of the random variable \([X \mid X \leq t]\). When \(g(x) = f(x)\), then (4.22) is the weighted past entropy (4.21), and when the weights are independent of \(x\), then (4.22) reduces to the past inaccuracy measure studied in Chapter 3.

4.4.1 Weighted Past Inaccuracy Measure in Term of Past Inaccuracy

Rewriting \(H^{*L}(f, g; t)\) as

\[
H^{*L}(f, g; t) = - \int_0^t dx \int_0^x \frac{f(x)}{F(t)} \log \frac{g(y)}{G(t)} \, dy
\]

\[
= t \int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx - \int_0^t dy \int_0^y \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx.
\]

(4.23)

Using (4.20) in (4.23) , we obtain

\[
H^{*L}(f, g; t) = tH^*(f, g; t) - \int_0^t H^*(f, g; y) \, dy.
\]

(4.24)

Differentiating (4.24) w.r.t. \(t\) both sides, we obtain

\[
\frac{d}{dt} H^{*L}(f, g; t) = t \frac{d}{dt} H^*(f, g; t) ,
\]

(4.25)
a result analogous to the result (4.12) studied in context with weighted residual inaccuracy. We shall use this result in the characterization problem studied next in Section 4.5.

When \( g(x) = f(x) \), then (4.25) reduces to

\[
\frac{d}{dt} H^* \left( f(t) \right) = t \frac{d}{dt} H^* (f; t),
\]

(4.26)
a result given by Di Crescenzo and Longobardi [31].

### 4.5 Characterization Problem

The general characterization problem is to determine when the dynamic information-theoretic measure determines the distribution function uniquely. In this section, we study the characterization problem for the weighted past inaccuracy measure (4.22) under the proportional reversed hazard model as already stated in Chapter 3 and restated as follows.

Two random variables \( X \) and \( Y \) satisfy the proportional reversed hazard model (PRHM) with proportionality constant \( \beta (> 0) \), if

\[
\mu_G(x) = \beta \mu_F(x), \quad \beta > 0,
\]

(4.27)

where \( \mu_F(x) = \frac{f(x)}{F(x)} \).

We know that the PRHM is equivalent to the model

\[
G(x) = [F(x)]^\beta,
\]

(4.28)

where \( F(x) \) can be considered as the baseline distribution function and \( G(x) \) as some reference distribution function, refer to Gupta et al. [50].

Next, we consider the following characterization theorem.
Theorem 4.3 If two random variables $X$ and $Y$ satisfy the proportional reversed hazard model (4.27) with proportionality constant $\beta (> 0)$ and $H^*(f, g; t)$ is decreasing for all $t > 0$, then $H^*(f, g; t)$ uniquely determines $\bar{F}(.)$, the survival function of $X$.

Proof Consider the weighted past inaccuracy measure (4.22) given by

$$H^*(f, g; t) = -\int_0^t x \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx .$$

Rewriting this as

$$\int_0^t x f(x) \log g(x) \, dx = \log G(t) \left[ \int_0^t x f(x) \, dx \right] - F(t) H^*(f, g; t) \quad (4.29)$$

or,

$$\int_0^t x f(x) \log g(x) \, dx = \log G(t) \left[ t F(t) - \int_0^t F(y) \, dy \right] - F(t) H^*(f, g; t) .$$

Differentiating both sides w.r.t. $t$, and using $\mu_F(t) = \frac{f(t)}{F(t)}$ and $\mu_G(t) = \frac{g(t)}{G(t)}$, we obtain

$$t \mu_F(t) \log \mu_G(t) - t \mu_G(t) = -\mu_G(t) \left[ \int_0^t \frac{F(y) \, dy}{F(t)} \right] + \mu_F(t) H^*(f, g; t) - \frac{d}{dt} H^*(f, g; t) . \quad (4.30)$$

Under proportional reversed hazard model (4.27), this gives

$$t \mu_F(t) \log \beta \mu_F(t) - \beta t \mu_F(t) = -\beta \mu_F(t) \int_0^t \frac{F(y) \, dy}{F(t)} - \mu_F(t) H^*(f, g; t) - \frac{d}{dt} H^*(f, g; t) .$$

Thus for any fixed $t$, $\mu_F(t)$ is a positive solution of the equation $h_1(x) = 0$, where

$$h_1(x) = \frac{d}{dt} H^*(f, g; t) + x \left\{ -\beta t + \log \beta x + \frac{\beta \int_0^t F(y) \, dy}{F(t)} + H^*(f, g; t) \right\} \quad (4.31)$$

Here $h_1(0) = \frac{d}{dt} H^*(f, g; t) \leq 0$; since we have assumed that $H^*(f, g; t)$ is decreasing in $t$, and also, when $x \to \infty$, $h_1(x) \to \infty$. 

76
Differentiating (4.31) with respect to $x$, we get

$$\frac{d}{dx} h_1(x) = -\beta t + t \log \beta x + \frac{\beta \int_0^t F(y) dy}{F(t)} + \overline{H}^w (f, g; t) + t .$$

So that $\frac{d}{dx} h_1(x) = 0$ if, and only if

$$x = \frac{1}{\beta} \exp \left[ -\frac{1}{t} \left\{ t - \beta t + \frac{\beta \int_0^t F(y) dy}{F(t)} + H^{*L}(f, g; t) \right\} \right] = \bar{x}_0, \text{ say.} \quad (4.32)$$

Therefore $h_1(x) = 0$ has a unique positive solution. Thus $\mu_F(t)$, and hence $\bar{F}(t)$, is uniquely determined by the weighted past inaccuracy measure $H^{*L}(f, g; t)$. This concludes the proof.

**Example 4.2** If a random variable $X$ is uniformly distributed over $(a, b)$, $a < b$, then its density and distribution functions are given respectively by

$$f(x) = \frac{1}{b - a} \quad \text{and} \quad F(x) = \frac{x - a}{b - a} , \quad a < x < b.$$ 

Further if $X$ and $Y$ satisfy the PRHM with proportionality constant $\beta > 0$, then distribution function of the variable $Y$ is

$$G(x) = \left[ \frac{x - a}{b - a} \right]^\beta, \quad \text{which gives} \quad g(x) = \frac{\beta(x - a)^{\beta - 1}}{(b - a)\beta}, \quad a < x < b.$$ 

Substituting these in (4.22) and simplifying, we obtain the weighted past inaccuracy measure as

$$H^{*L}(f, g; t) = \left( \frac{t + a}{2} \right) \log \left( \frac{t - a}{\beta} \right) + (\beta - 1) \left( \frac{t + 3a}{4} \right) . \quad a < t < b. \quad (4.33)$$

For $\beta = 1$, this reduces to

$$H^{*L}(f; t) = \left( \frac{t + a}{2} \right) \log(t - a),$$
Example 4.3 If true distribution function $F(x)$ and reference distribution function $G(x)$ are exponentially distributed with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively, then

$$f(x) = \lambda_1 e^{-\lambda_1 x}, \quad \overline{F}(x) = 1 - F(x) = e^{-\lambda_1 x}, \quad x > 0$$

and,

$$g(x) = \lambda_2 e^{-\lambda_2 x}, \quad \overline{G}(x) = 1 - G(x) = e^{-\lambda_2 x}, \quad x > 0.$$  

(4.34)

Substituting for $\overline{G}$, $\overline{F}$, $f$ and $g$ in (4.22), we obtain the weighted past inaccuracy measure as

$$H^{*L}(f, g; t) = \frac{1}{[1 - e^{-\lambda_1 t}]} \left\{ \log \frac{1 - e^{-\lambda_2 t}}{\lambda_2} \left[ \frac{1}{\lambda_1} - \frac{e^{-\lambda_1 t}}{\lambda_1} - te^{-\lambda_1 t} \right] \right\}$$

$$+ \frac{1}{[1 - e^{-\lambda_1 t}]} \left\{ \frac{2\lambda_2}{\lambda_1^2} - \frac{2\lambda_2 e^{-\lambda_1 t}}{\lambda_1^2} - \frac{2\lambda_2 t e^{-\lambda_1 t}}{\lambda_1} - \lambda_2 t^2 e^{-\lambda_1 t} \right\}. \quad (4.35)$$

In addition to the general case as given by (4.35), the following two particular cases are of specific interest. The case I is of PRHM and case II is of PHM. But we must note that $H^{*L}(f, g; t)$ characterizes the distribution uniquely only under PRHM.

Case I: $\lambda_1 = \lambda_2$, that is, $G(x) = F(x)$. In this case the weighted past inaccuracy $H^{*L}(f, g; t)$ reduces to the weighted past entropy $H^{*L}(f; t)$, given by

$$H^{*L}(f; t) = \frac{1}{[1 - e^{-\lambda_1 t}]} \left\{ \log \frac{1 - e^{-\lambda_1 t}}{\lambda_1} \left[ \frac{1}{\lambda_1} - \frac{e^{-\lambda_1 t}}{\lambda_1} - te^{-\lambda_1 t} \right] \right\}$$

$$+ \frac{1}{[1 - e^{-\lambda_1 t}]} \left\{ \frac{2}{\lambda_1^2} - \frac{2 e^{-\lambda_1 t}}{\lambda_1^2} - \frac{2 t e^{-\lambda_1 t}}{\lambda_1} - \lambda_1 t^2 e^{-\lambda_1 t} \right\}. \quad (4.36)$$
Fig. 4.1: Plot of $H^{*L}(f, g; t)$ when $\lambda_1 \neq \lambda_2$ for $t \in [0 , 4]$.

Case II: $\lambda_2 = n\lambda_1$, that is, $G(x) = [F(x)]^n$. This corresponds to $n$-components series system where each component is having independent and identically distributed lifetime $X_i$, $i = 1, 2, ... n$, with distribution function $F(x)$, and here $G(x)$ is the distribution function of the lifetime $Y = \min\{X_1, X_2, .....X_n\}$ of the system. The weighted past inaccuracy (4.35) in this case becomes

$$H^{*L}(f, g; t) = \frac{1}{[1 - e^{-\lambda_1 t}]} \left\{ \log \frac{1 - e^{-n\lambda_1 t}}{n\lambda_1} \left[ \frac{1}{\lambda_1} - \frac{e^{-\lambda_1 t}}{\lambda_1} - te^{-\lambda_1 t} \right] \right\}$$

$$+ \frac{1}{[1 - e^{-\lambda_1 t}]} \left\{ \frac{2n}{\lambda_1} - \frac{2ne^{-\lambda_1 t}}{\lambda_1} - 2nte^{-\lambda_1 t} - n\lambda_1 t^2 e^{-\lambda_1 t} \right\}. \quad (4.37)$$

The plots in case of the measures (4.35), (4.36) and (4.37) for various values of $\lambda_1$ and $\lambda_2$ are given respectively in Figs. 4.1, 4.2 and 4.3.
\begin{align*}
\lambda_1 = \lambda_2 &= 2 \\
\lambda_1 = \lambda_2 &= 3 \\
\lambda_1 = \lambda_2 &= 4 \\
\lambda_1 = \lambda_2 &= 5 
\end{align*}

\begin{align*}
\lambda_1 &= 3, \lambda_2 = 6 \\
\lambda_1 &= 3, \lambda_2 = 9 \\
\lambda_1 &= 3, \lambda_2 = 12 \\
\lambda_1 &= 3, \lambda_2 = 15 
\end{align*}

Fig. 4.2: Plot of $H^L(f, g; t)$ when $\lambda_1 = \lambda_2$ for $t \in [0, 4]$.

Fig. 4.3: Plot of $H^L(f, g; t)$ when $\lambda_2 = n\lambda_1$ for $t \in [0, 4]$, where $\lambda_1 = 3$ fixed and $n = 2, 3, 4, 5$. 

80
4.6 Some Further Results on Past Inaccuracy

4.6.1 An Upper Bound to $H^*L(f, g; t)$

To obtain an upper bound to the weighted past inaccuracy measure (4.22), we define the mean past lifetime of the system as

$$\tau(t) = E(X \mid X \leq t) = \int_0^t x f(x) \frac{F(x)}{F(t)} dx = t - \frac{1}{F(t)} \int_0^t F(y) dy.$$  (4.38)

Next, we consider the following result.

**Theorem 4.4** If $\mu_G(t) = \frac{g(t)}{G(t)}$, the reversed hazard rate is decreasing in $t$, then

$$H^*L(f, g; t) \leq -\tau F(t) \left[\log \mu_G(t) + 1\right] + \frac{G(t)}{F(t)} \int_0^t x f(x) \frac{G(x)}{G(t)} dx.$$  (4.39)

**Proof** From (4.22), we have

$$H^*L(f, g; t) = -\frac{1}{F(t)} \int_0^t x f(x) \log \mu_G(x) dx + \frac{1}{F(t)} \int_0^t x f(x) \log \frac{G(t)}{G(x)} dx.$$  

Since $\mu_G(t) = \frac{g(t)}{G(t)}$ is decreasing in $t$, we have $\log \mu_G(x) \geq \log \mu_G(t)$ for $0 < x < t$. Moreover, $\log x \leq x - 1$ for $x > 0$. We obtain

$$H^*L(f, g; t) \leq -\frac{\log \mu_G(t)}{F(t)} \int_0^t x f(x) dx + \frac{1}{F(t)} \int_0^t x f(x) \frac{G(t)}{G(x)} - 1] dx$$

and, after simplification, we get (4.39).

**Example 4.4** Let $X$ be a non-negative random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x & ; \text{ if } 0 \leq x < 1 \\ 0 & ; \text{ otherwise} \end{cases}$$
and let random variable $Y$ be uniformly distributed over $(0, a)$ with density and distribution functions given respectively by

$$g_Y(x) = \frac{1}{a} \quad \text{and} \quad G_Y(x) = \frac{x}{a}, \quad 0 < x < a.$$ 

Substituting these values in (4.22), we obtain the length biased past inaccuracy measure as

$$H^*_{L}(f, g; t) = \frac{2t \log t}{3}, \quad 0 < t < 1.$$ (4.40)

Also the right hand side of (4.39) gives

$$- \tau_F(t) \left[ \log \mu_G(t) + 1 \right] + \frac{G(t)}{F(t)} \int_0^t \frac{xf(x)}{G(x)} dx = \frac{2t \log t}{3} + \frac{t}{3}.$$

Comparing this with (4.40), it is easily seen that (4.39) is fulfilled.

### 4.6.2 Weighted Inaccuracy in Terms of Residual and Past Inaccuracy

We express weighted inaccuracy measure (4.8) in terms of length biased past inaccuracy measure (4.22) and length biased residual inaccuracy measure (4.9). We prove the following result.

**Theorem 4.5** For a random variable $X$ having finite mean $E(X)$ for all $t > 0$, the weighted inaccuracy measure $H^L(f, g)$ as given by (4.8) can be expressed as

$$H^L(f, g) = F(t)H^*_{L}(f, g; t) + \bar{F}(t)H^L(f, g; t) - E(X) \left\{ F^L(t) \log G(t) + \bar{F}^L(t) \log \bar{G}(t) \right\}.$$ (4.41)

**Proof** We have

$$H^L(f, g) = -\int_0^\infty xf(x) \log g(x) dx.$$
This can be rewritten as

\[ H^L(f, g) = -F(t) \int_0^t x \frac{f(x)}{F(t)} \log g(x) dx - \bar{F}(t) \int_t^\infty x \frac{f(x)}{F(t)} \log g(x) dx. \]

Using (4.22) and (4.9), we obtain

\[ H^L(f, g) = F(t) H^*_L(f, g; t) + \bar{F}(t) H^L(f, g; t) \]

\[ - \log G(t) \left\{ \int_0^t x f(x) dx \right\} - \log \bar{G}(t) \left\{ \int_t^\infty x f(x) dx \right\}. \]

Using (4.3), we obtain (4.41), the desired result.

### 4.7 Conclusion

The concept of weighted distributions and hence that of weighted information measures is of wide interest when a stochastic process is recorded with some weight function. We have seen here that the dynamic inaccuracy measures (both residual and past) studied in Chapter 3 find a natural extension to the corresponding length biased (weighted) residual and past inaccuracy measures. These measures also characterize the underlying distribution uniquely. So far we have concentrated only on p.d.f. based information-theoretic measures which have their own inherent limitations. In the subsequent chapters we consider distribution function based information theoretic measures which overcome the limitations of density based measures.