Chapter 3

Dynamic Inaccuracy Measures

3.1 Introduction

Most of the work on characterization of lifetime distribution function of a system in the reliability context centers around the hazard rate or the mean residual life function. In a variant approach, Ebrahimi [34] proposed the residual entropy function as a useful tool to analyze the stability of a component or a system. In the preceding chapter we have considered a one parameter non-additive residual information measure and based on that we have characterized a few specific lifetime distributions.

Several researchers, refer to [8, 9, 12], have employed information measures like time dependent Kullback-Leibler directed divergence [71] and its generalizations in characterizing lifetime distributions. The Kerridge inaccuracy measure [67] can be viewed as a generalization of Shannon’s entropy [109] in the sense that when the predicted probability distribution of a random variable X coincides with the actual probability distribution, then the Kerridge inaccuracy measure reduces to the
Shannon entropy measure. Therefore, there is a scope for extending the results based on Shannon’s entropy and its generalizations to the inaccuracy measures. Motivated by this, in the present chapter we extend the definition of the inaccuracy to the truncated situation and propose dynamic measures of inaccuracy, both residual and past. The chapter is organized as follows. In Section 3.2, we introduce a residual inaccuracy measure by considering the measures of residual entropy and residual discrimination. In Section 3.3, we prove a characterization result that if the proposed and actual probability distributions satisfy the proportional hazard model then the residual inaccuracy measure determines the underlying probability distribution uniquely. Section 3.4 introduces the concept of past inaccuracy and in the subsequent Section 3.5 we prove a characterization result. Also we have derived some specific properties of the measures introduced. The chapter ends with conclusion.

### 3.2 Residual Inaccuracy Measure

Let $X$ and $Y$ be two non-negative random variables representing time to failure of two systems with p.d.f. respectively $f(x)$ and $g(x)$. Let $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ be failure distributions, $\lambda_F(x) = \frac{f(x)}{F(x)}$ and $\lambda_G(x) = \frac{g(x)}{G(x)}$ be hazard rates, and $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ be survival functions of $X$ and $Y$ respectively. Shannon’s measure of uncertainty [109] associated with the random variable $X$ and Kullback’s measure of discrimination [71] of $X$ about $Y$ are given respectively by

$$H(f) = -\int_0^\infty f(x) \log f(x) dx ,$$

(3.1)

and

$$H(f/g) = \int_0^\infty f(x) \log \frac{f(x)}{g(x)} dx .$$

(3.2)

In survival analysis and in life testing, since the current age of the system under
consideration is also taken into account, thus for calculating the uncertainty of a system or the discrimination between two systems, the measures (3.1) and (3.2) are not suitable. Given that the system has survived up to time $t$, the corresponding dynamic measure of uncertainty [34], and of discrimination [37, 38] are given by

$$H(f; t) = -\int_t^\infty \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \, dx,$$

and

$$H(f/g; t) = \int_t^\infty \frac{f(x)}{F(t)} \log \frac{f(x)}{g(x)/G(t)} \, dx$$

respectively.

When $t = 0$, then (3.3) reduces to (3.1), and (3.4) reduces to (3.2).

Adding (3.3) and (3.4), we obtain

$$H(f; t) + H(f/g; t) = -\int_t^\infty \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \, dx + \int_t^\infty \frac{f(x)}{F(t)} \log \frac{f(x)}{g(x)/G(t)} \, dx$$

$$= -\int_t^\infty \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx,$$

$$= H(f, g; t), \text{ say.}$$

In case we have a system with true survival function $\bar{F}(.)$ and the reference survival function $G(.)$, then the measure $H(f, g; t)$ can be interpreted as a measure of inaccuracy associated with the density functions $f_t$ and $g_t$, where $f_t = \frac{f(x)}{F(t)}$ and $g_t = \frac{g(x)}{G(t)}$.

We define the measure

$$H(f, g; t) = -\int_t^\infty \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx,$$

as a dynamic measure of inaccuracy associated with two residual lifetime distributions $F(.)$ and $G(.)$ analogous to the Kerridge inaccuracy [67] given by

$$H(f, g) = -\int_0^\infty f(x) \log g(x) \, dx.$$
Obviously, at $t = 0$, (3.6) reduces to (3.7).

When $g(x) = f(x)$, then (3.6) becomes (3.3), the dynamic measure of uncertainty given by Ebrahimi [34].

### 3.3 Characterization Problem For Residual Inaccuracy Measure

The general characterization problem is to determine when the dynamic inaccuracy measure determines the distribution functions uniquely. We study characterization problem for the dynamic inaccuracy measure under the assumption that the distribution functions of the random variables $X$ and $Y$ satisfy the proportional hazard model. Under this model, refer to [24] and [42], their survival functions $\bar{F}(.)$ and $\bar{G}(.)$ are related by

$$\bar{G}(x) = [\bar{F}(x)]^\beta, \; \beta > 0.$$  \hspace{1cm} (3.8)

We note that based on the proportional hazard model (3.8), the hazard rate functions $\lambda_F(.)$ and $\lambda_G(.)$ satisfy the relation

$$\lambda_G(x) = \beta \lambda_F(x).$$  \hspace{1cm} (3.9)

Next, we prove the following characterization result.

**Theorem 3.1** Let $X$ and $Y$ be two non-negative random variables satisfying the proportional hazard model (3.8), and let $H(f,g;t) < \infty, \forall \; t \geq 0$, then $H(f,g;t)$ determines the survival function $\bar{F}(.)$ uniquely.

**Proof** Let $f_1, g_1$ and $f_2, g_2$ be two sets of the probability density functions satisfying the proportional hazard model, that is, $\lambda_{G_1}(x) = \beta \lambda_{F_1}(x)$, and $\lambda_{G_2}(x) = \beta \lambda_{F_2}(x)$, and let

$$H(f_1, g_1; t) = H(f_2, g_2; t), \; \forall \; t \geq 0.$$  \hspace{1cm} (3.10)
Consider
\[ H(f, g; t) = - \int_t^\infty \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx , \quad (3.11) \]
\[ = \log G(t) - \int_t^\infty \frac{f(x)}{F(t)} \log g(x) \, dx . \quad (3.12) \]

Differentiating (3.12) w.r.t. \( t \) and using (3.9), we obtain
\[ H'(f, g; t) = -\lambda G(t) + \lambda_F(t) \log g(t) - \lambda_F(t) \int_t^\infty \frac{f(x)}{F(t)} \log g(x) \, dx . \quad (3.13) \]
\[ = \lambda_F(t) \left[ -\beta + \log g(t) - \int_t^\infty \frac{f(x)}{F(t)} \log g(x) \, dx \right] \]
\[ = \lambda_F(t) \left[ -\beta + \log g(t) - \log G(t) - \int_t^\infty \frac{f(x)}{F(t)} \log \frac{G(x)}{G(t)} \, dx \right] \]
\[ = \lambda_F(t) \left[ -\beta + \log g(t) - \log G(t) + H(f, g; t) \right] . \]

This gives
\[ H'(f, g; t) = \lambda_F(t) \left[ -\beta + \log g(t) + H(f, g; t) \right] . \quad (3.14) \]

Using (3.14), from (3.10) we obtain
\[ \lambda_{F_1}(t) \left[ -\beta + \log g(t) + \log \lambda_{F_1}(t) + H(f_1, g_1; t) \right] \]
\[ = \lambda_{F_2}(t) \left[ -\beta + \log g(t) + \log \lambda_{F_2}(t) + H(f_2, g_2; t) \right] \quad (3.15) \]

To prove that (3.10), under the assumption of proportional hazard model (3.8), implies \( F_1(t) = F_2(t) \), it is sufficient to prove that
\[ \lambda_{F_1}(t) = \lambda_{F_2}(t), \forall \ t \geq 0. \quad (3.16) \]

Define a set
\[ A = \{ t : t \geq 0, \text{ and } \lambda_{F_1}(t) \neq \lambda_{F_2}(t) \} \quad (3.17) \]
and assume the set $A$ to be non empty. Thus for some $t_0 \in A$, $\lambda_{F_1}(t_0) \neq \lambda_{F_2}(t_0)$. Without loss of generality suppose that $\lambda_{F_1}(t_0) > \lambda_{F_2}(t_0)$ and since (3.15) holds, then either

$$-\beta + \log \beta + \log \lambda_{F_1}(t_0) + H(f_1, g_1; t_0) < -\beta + \log \beta + \log \lambda_{F_2}(t_0) + H(f_2, g_2; t_0) \quad (3.18)$$

or

$$-\beta + \log \beta + \log \lambda_{F_1}(t_0) + H(f_1, g_1; t_0) = -\beta + \log \beta + \log \lambda_{F_2}(t_0) + H(f_2, g_2; t_0) = 0. \quad (3.19)$$

Suppose (3.18) holds, then using (3.10) the inequality (3.18) reduces to $\lambda_{F_1}(t_0) < \lambda_{F_2}(t_0)$. If (3.19) holds, then using (3.10), it reduces to $\lambda_{F_1}(t_0) = \lambda_{F_2}(t_0)$. Combining these two we get $\lambda_{F_1}(t_0) \leq \lambda_{F_2}(t_0)$. This contradicts the assumption $\lambda_{F_1}(t_0) > \lambda_{F_2}(t_0)$ and, therefore, the set $A$ is empty and this concludes the proof.

### 3.3.1 Properties of the Residual Inaccuracy Measure

Before working for the properties of the residual measure of inaccuracy we give following definitions.

**Definition 3.1** A distribution function $F(.)$ is said to be decreasing (increasing) mean residual life DMRL (IMRL), if its mean residual life function $\delta_F(t)$ is decreasing (increasing) in $t \geq 0$.

**Definition 3.2** A survival function $F(.)$ has decreasing (increasing) inaccuracy in residual life DIRL (IIIRL), if $H'(f, g; t)$ is decreasing (increasing) in $t$, $t \geq 0$.

**Definition 3.3** Let $\phi(x)$ be a monotone function. If $Y(\phi(X)) = Y(X)$ for all continuous random variable $X$, then $\phi(x)$ is affine transformation.

We know that Shannon entropy is not invariant under affine transformation because it is shift invariant but not scale invariant, that is $H(aX + b) = H(X) + \log a$. 

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The dynamic measure of inaccuracy $H(f, g; t)$ satisfies the following important properties:

I  For a common increasing transformation $\phi$ of $X$ and $Y$

$$H(X, Y; \phi^{-1}(t)) = H(\phi(X), \phi(Y); t).$$

Proof Consider

$$H(\phi(X), \phi(Y), t) = -\int_{t}^{\infty} \frac{f(\phi^{-1}(x))}{\phi'(\phi^{-1}(x))F(\phi^{-1}(t))} \log \frac{g(\phi^{-1}(x))}{G(\phi^{-1}(t))} dx \quad (3.20)$$

$$= -\int_{\phi^{-1}(t)}^{\infty} \frac{f(y)}{F(\phi^{-1}(t))} \log \frac{g(y)}{G(\phi^{-1}(t))} dy$$

$$= H(X, Y; \phi^{-1}(t)).$$

II  If $\bar{F}(.)$ and $\bar{G}(.)$ satisfy the proportional hazard model (3.8), and $\delta_F(t)$ is finite, then

$$H(f, g, t) \leq \beta - \log \beta + \log \delta_F(t), \quad (3.21)$$

where $\delta_F(t)$ is the mean residual lifetime function.

Proof The dynamic measure of inaccuracy is

$$H(f, g; t) = -\int_{t}^{\infty} \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx.$$

Under proportional hazard model (3.9), we can express it as

$$H(f, g; t) = \beta - \log \beta - \int_{t}^{\infty} \frac{f(x)}{F(t)} \log \lambda_F(x) dx$$

$$= (\beta - \log \beta - 1) + 1 - \int_{t}^{\infty} \frac{f(x)}{F(t)} \log \lambda_F(x) dx. \quad (3.22)$$

Also, the dynamic measure of entropy is

$$H(f; t) = 1 - \int_{t}^{\infty} \frac{f(x)}{F(t)} \log \lambda_F(x) dx \leq 1 + \log \delta_F(t),$$
refer to [34]. Using this in (3.22), we get (3.21).

**III The maxima of dynamic inaccuracy measure under proportional hazard model exists when $F$ is exponential.**

**Proof** From (3.22), under proportional hazard model, we have

$$H(f, g; t) = (\beta - \log \beta - 1) + H(f; t).$$

(3.23)

Since the maxima of $H(f; t)$ exists, when $f(x) = \theta \exp(-\theta x)$, $\theta > 0$ and $\max H(f; t) = 1 - \log \theta$, refer to [36], thus from (3.23) the maxima of $H(f, g; t)$ under proportional hazard model also exists only when $f(x) = \theta \exp(-\theta x)$, and it is given by

$$\max H(f, g; t) = (\beta - \log \beta - 1) + (1 - \log \theta)$$

$$= \beta - \log \beta - \log \theta.$$  

**IV If $\bar{F}(.)$ and $\bar{G}(.)$ satisfy the proportional hazard model with proportionality constant $\beta$ and $\bar{F}(.)$ is decreasing mean residual life (DMRL), then it is decreasing inaccuracy in residual life (DIRL).**

**Proof** From (3.14), we have

$$H'(f, g; t) = \lambda_F(t)[-\beta + \log \beta + \log \lambda_F(t) + H(f, g; t)].$$

Using (3.21), this gives

$$H'(f, g; t) \leq \lambda_F(t)[\log \lambda_F(t) + \log \delta_F(t)]$$

$$\leq \lambda_F(t) \log[\lambda_F(t)\delta_F(t)]$$

$$\leq \lambda_F(t) \log[1 + \delta'_F(t)]$$

$$\leq 0,$$

for all $t \geq 0$. The last inequality comes from the assumption that $\delta_F(t)$ is decreasing. This proves the result.
3.4 Past Inaccuracy Measure

In many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. For instance if at time $t$, a system which is observed only at certain preassigned inspection times, is found to be down, then the uncertainty of the system’s life relies on the past, that is, at which instant in $(0, t)$ the system has failed. Based on this idea, Di Crescenzo and Longobardi [29, 30] have studied measures of entropy and discrimination based on the past entropy over $(0, t)$ given respectively as

$$H^*(f; t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \, dx ,$$  \hspace{1cm} (3.24)

and

$$H^*(f/g; t) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} / \frac{g(x)}{G(t)} \, dx .$$  \hspace{1cm} (3.25)

In sequel to these measures of entropy and discrimination based on the past entropy over $(0, t)$, we propose

$$H^*(f, g; t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx ,$$  \hspace{1cm} (3.26)

as a dynamic measure of past inaccuracy over the interval $(0, t)$.

Here we observe that

$$H^*(f; t) + H^*(f/g; t) = H^*(f, g; t)$$  \hspace{1cm} (3.27)

in confirmation with the result

$$H(f) + H(f/g) = H(f, g) ,$$

in the literature, refer to Kerridge [67]. Here $H(f)$, $H(f/g)$ and $H(f, g)$ are given respectively by (3.1), (3.2) and (3.7).

In case we have a system with the baseline distribution function $F(.)$ and the reference distribution function $G(.)$, then the measure $H^*(f, g; t)$ can be interpreted
as a measure of inaccuracy associated with the probability density functions $f_t$ and $g_t$, where $f_t = \frac{f(x)}{F(t)}$ and $g_t = \frac{g(x)}{G(t)}$.

We observe that the measure of past inaccuracy defined by (3.26) can be considered analogous to the measure of residual inaccuracy defined by (3.6). When $t \to \infty$, then (3.26) reduces to (3.7), the Kerridge measure of inaccuracy [67], and further, when $g(x) = f(x)$, then (3.26) becomes (3.24), the dynamic measure of past entropy given by Di Crescenzo and Longobardi [29].

Next, consider the past dynamic inaccuracy measure (3.26) when the random variables satisfy the assumption of proportional reversed hazard model (PRHM). We recall that if $X$ is a non-negative random variable with distribution function $F(\cdot)$, denoting the lifetime of a component, then the reversed hazard rate of $X$, denoted by $\mu_X(x)$, is given by

$$\mu_X(x) = \frac{d}{dx} \log F(x) = \frac{f(x)}{F(x)},$$

where $f$ is the probability density function (p.d.f.) of $X$.

Here $\mu_X(x)dx$ provides the probability of failing a component in the interval $(x - dx, x)$, when it has been found in failed state at time $x$. For example, if lifetime $X$ of a component is uniformly distributed in the interval $[a, b]$, then the reversed hazard rate is, $\mu_X(x) = \frac{f(x)}{F(x)} = \frac{1}{x-a}$.

Next, the two random variables $X$ and $Y$ satisfy the proportional reversed hazard model (PRHM) with proportionality constant $\beta \ (> 0)$, if

$$\mu_Y(x) = \beta \mu_X(x), \ \beta > 0, \quad (3.28)$$

which is equivalent to

$$G(x) = [F(x)]^\beta, \ \beta > 0, \quad (3.29)$$

where $F(x)$ is the baseline distribution function and $G(x)$ can be considered as some reference distribution function. This model was proposed by Gupta et al. [50] in contrast to the proportional hazard model (PHM) given by Cox [24] and Efron [42].
As an example, for some positive integral value of $\beta$, if $X_1, X_2, \ldots, X_\beta$ are independent and identically distributed (i.i.d.) random variables each with distribution function $F(x)$ representing the lifetimes of components in a $\beta$-components parallel system, then the lifetime of the system is given by $Y = \max(X_1, X_2, \ldots, X_\beta)$ with distribution function $G(x)$ given by (3.29).

Consider

$$H^*(f, g; t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx$$

$$= \log G(t) - \int_0^t \frac{f(x)}{F(t)} \log g(x) dx$$

(3.30)

$$= \log G(t) - \int_0^t \frac{f(x)}{F(t)} \log \mu_x dx - \frac{1}{F(t)} \int_0^t f(x) \log G(x) dx.$$  (3.31)

Using (3.28) and (3.29) in (3.31) we obtain

$$H^*(f, g; t) = \beta - \log \beta - \int_0^t \frac{f(x)}{F(t)} \log \mu_x dx .$$  (3.32)

When $\beta = 1$, that is, $G(x) = F(x)$, then (3.32) becomes the past entropy given by Di Crescenzo and Longobardi [29].

**Remark 3.1** We observe that the three inaccuracy measures viz. $H(f; g)$, $H(f, g; t)$ and $H^*(f, g; t)$ considered above, satisfy the relation

$$H(f; g) = F(t) H(f, g; t) + F(t) H^*(f, g; t) + H[F(t), G(t)],$$  (3.33)

where

$$H[F(t), G(t)] = -F(t) \log G(t) - [1 - F(t)] \log[1 - G(t)],$$

corresponds to the Kerridge inaccuracy [67].

When $g = f$, then (3.33) reduces to

$$H(f) = H[F(t), F(t)] + F(t) H^*(t) + F(t) H(t),$$

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a result obtained by Di Crescenzo and Longobardi [29], where $H(p, 1 - p) = -p \log p - (1 - p) \log(1 - p)$ is the entropy of a Bernoulli random variable.

3.5 Characterization Based on Past Inaccuracy Measure

The characterization of specific distributions using relations between reliability measures has become of increasing interest. Several characterizations of probability models have been obtained based on the failure rate or mean residual life (MRL) functions. Asadi and Ebrahimi [8] have studied the characterization based on Shannon residual entropy. Characterizations based on aging measures and dynamic information measures have also been given by Belzunce et al. [16], Ruiz and Navarro [103] and Nanda et al. [84, 85]. In the preceding chapter in Section 2.3 we have characterized some specific lifetime distributions based on the non-additive dynamic entropy measure (2.10). In this section we characterize uniform distribution in term of the past inaccuracy measure (3.26) under the assumption that the two random variables $X$ and $Y$ satisfy the proportional reversed hazard model (3.28). We give the following theorem.

**Theorem 3.2** If two random variables $X$ and $Y$ satisfy the proportional reversed hazard model (PRHM) with proportionality constant $\beta > 0$, then random variable $X$ over $(a, b)$, $a < b$, has uniform distribution if, and only if

$$H^*(f, g; t) = \beta - \log \beta - 1 + \log(t - a), \quad a < t < b. \quad (3.34)$$

**Proof** The 'only if' part of the theorem is straightforward since in case of uniform distribution of $X$ over $(a, b)$

$$F(x) = \frac{x - a}{b - a} \quad \text{and} \quad f(x) = \frac{1}{b - a}. \quad 58$$
Hence, under PRHM, \( G(x) = \left[ \frac{x-a}{b-a} \right]^\beta \). This gives \( g(x) = \frac{\beta(x-a)^{\beta-1}}{(b-a)^{\beta}} \). Substituting these in (3.30) and simplifying, we obtain

\[
H^*(f, g; t) = \beta - \log \beta - 1 + \log(t - a).
\]

To prove the 'if part' let (3.34) be valid. Differentiating (3.30) w.r.t. \( t \) and using \( \mu_G(x) = \beta \mu_F(x) \), we obtain

\[
\frac{d}{dt}H^*(f, g; t) = \mu_G(t) - \mu_F(t) \log g(t) + \mu_F(t) \int_0^t \frac{f(x)}{F(t)} \log g(x)dx
\]

\[
= \mu_F(t)[\beta - \log g(t) - \int_0^t \frac{f(x)}{F(t)} \log g(x)dx]
\]

\[
= \mu_F(t)[\beta - \log g(t) + \log G(t) + \int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)}dx]
\]

\[
= \mu_F(t)[\beta - \log \mu_G(t) - H^*(f, g; t)]
\]

\[
= \mu_F(t)[\beta - \log \beta - \log \mu_F(t) - H^*(f, g; t)]
\]  

(3.36)

This gives

\[
\frac{d}{dt}H^*(f, g; t) - \mu_F(t)[\beta - \log \beta - \log \mu_F(t) - H^*(f, g; t)] = 0.
\]

Hence for a fixed \( t > 0 \), \( \mu_F(t) \) is a solution of \( g_1(x) = 0 \), where

\[
g_1(x) = \frac{d}{dt}H^*(f, g; t) - x[\beta - \log \beta - \log x - H^*(f, g; t)].
\]  

(3.37)

Differentiating (3.37) with respect to \( x \), we obtain

\[
g_1'(x) = [1 - \beta + \log \beta + \log x + H^*(f, g; t)],
\]

and \( g_1'(x) = 0 \) gives

\[
x = \exp[\beta - 1 - \log \beta - H^*(f, g; t)] = x_0, \text{ (say)}.
\]
Then from (3.37), we have
\[ g_1(0) = \frac{d}{dt} H^*(f, g; t) > 0. \]

Also we can show that \( g_1(x) \) is a convex function with minima at \( x = x_0 \). So \( g_1(x) = 0 \) has a unique solution and if \( g_1(x_0) = 0 \), then we have
\[ x_0 = \exp[\beta - 1 - \log \beta - H^*(f, g; t)]. \]

Using (3.34), we get
\[ x_0 = \frac{1}{t-a}, \quad t > a \]
and
\[ g_1(x_0) = \frac{d}{dt} H^*(f, g; t) - x_0[\beta - \log \beta - \log x_0 - H^*(f, g; t)] = 0. \]

Thus \( g_1(x) = 0 \) has a unique solution given by \( x = x_0 \). But \( \mu_F(t) \) is a solution to (3.37). Hence \( \mu_F(t) = x_0 = (t - a)^{-1}, \quad t > a \) is the unique solution to \( g_1(x) = 0 \). Thus the distribution is uniform, and this proves the result.

**Example 3.1** Consider an \( n \)-components parallel system with components having independent and identically distributed (i.i.d) lifetimes \( X'_i \), \( i = 1, 2, \ldots, n \), where \( X'_i \) are exponentially distributed random variables with the same parameter \( \theta \), and let \( Y = \max\{X_1, X_2, \ldots, X_n\} \) be the lifetime of the system. Further, let \( f(x) \) and \( F(x) \) be respectively the p.d.f. and c.d.f. of \( X_i \). If \( G \) is the distribution function for \( Y \), then under PRHM, the c.d.f. of \( Y \) is \( G(x) = [F(x)]^n \) and its p.d.f. is \( g(x) = n[F(x)]^{n-1}f(x) \).

Here
\[ f(x) = \theta e^{-\theta x}, \]
\[ F(x) = 1 - e^{(-\theta x)}, \]
\[ G(x) = (1 - e^{(-\theta x)})^n, \]
and, \[ g(x) = n\theta e^{-\theta x}[1 - e^{(-\theta x)}]^{n-1}. \]

Also,
\[ H^*(f, g; t) = \log G(t) - \int_0^t \frac{f(x)}{F(t)} \log g(x) dx. \]
Substituting for $G, F, f$ and $g$, this gives

$$H^*(f, g; t) = n - \log n \theta + \log(1 - e^{-\theta t}) - \frac{\theta te^{-\theta t}}{1 - e^{-\theta t}}. \quad (3.38)$$

Taking limit as $t \to \infty$, we obtain

$$\lim_{t \to \infty} H^*(f, g; t) = n - \log n \theta , \quad (3.39)$$
a result in confirmation with the inaccuracy measure $H(f, g)$ under PRHM for

$$f(x) = \theta e^{-\theta x}.$$ 

The graph of $H^*(f, g; t)$ versus $t$ for $t \in [0, 2]$ is shown below in Fig. 3.1. It suggests that when $n$, the number of components increases in a parallel system then the past inaccuracy measure $H^*(f, g; t)$ also increases. Otherwise, for fix $n$, $H^*(f, g; t)$ is an increasing function of $t$.

![Graph of $H^*(f, g; t)$ versus $t$ for different values of $n$.](image)

**Fig. 3.1:** Plot of $H^*(f, g; t)$ versus $t$ for different values of $n.$
Next, we consider another example where \( F(x) \) and \( G(x) \) does not satisfy proportional reversed hazard model.

**Example 3.2** Let \( X \) and \( Y \) be two nonnegative random variables having distribution functions respectively

\[
F(x) = \begin{cases} 
\frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\
\frac{x^2+2}{6}, & \text{for } 1 \leq x < 2 \\
1 & \text{for } x \geq 2
\end{cases}
\]

and

\[
G(x) = \begin{cases} 
\frac{x^2+x}{4}, & \text{for } 0 \leq x < 1 \\
\frac{x}{2}, & \text{for } 1 \leq x < 2 \\
1 & \text{for } x \geq 2.
\end{cases}
\]

The past inaccuracy measure (3.26) is given by

\[
H_\ast(f, g; t) = \begin{cases} 
\frac{1}{2} - \frac{1}{2t} + \frac{1}{4t^2} \log(2t+1) + \log \frac{t^2+t}{2t+1}, & \text{for } 0 < t < 1 \\
\log \frac{t}{2} + \left(\frac{t^2-1}{t^2+2}\right) \log 2 + \frac{6}{t^2+2} \log 2 - \frac{9}{4(t^2+2)} \log 3, & \text{for } 1 \leq t < 2 \\
\frac{3}{2} \log 2 - \frac{3}{8} \log 3, & \text{for } t \geq 2.
\end{cases}
\]

The graph of the past inaccuracy measure for \( t \in [0, 1) \), is shown in Fig. 3.2 on the next page.
3.5.1 An Upper Bound to $H^*(f, g; t)$

To find an upper bound to $H^*(f, g; t)$, we prove the following result.

**Theorem 3.3** If $\tilde{F}(.)$ and $\tilde{G}(.)$ satisfying the proportional reversed hazard model (3.28) and $\mu_F(t)$ is decreasing in $t$, then

$$H^*(f, g, t) \leq \beta - \log \beta - \log \mu_F(t),$$

(3.40)

where $\mu_F(t)$ is the reversed failure rate function.

**Proof** The measure of past inaccuracy is

$$H^*(f, g; t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx.$$ 

Using the proportional reversed hazard model (3.28), this gives

$$H^*(f, g; t) = \beta - \log \beta - \int_0^t \frac{f(x)}{F(t)} \log \mu_F(x) \, dx$$

$$= (\beta - \log \beta - 1) + 1 - \int_0^t \frac{f(x)}{F(t)} \log \mu_F(x) \, dx.$$ 

(3.41)
Also, in case of measure of past entropy

\[ H^*(f; t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \, dx \leq 1 - \log \mu_F(t), \quad (3.42) \]

refer to Di Crescenzo and Longobardi [29]. Using this in (3.41), we get (3.40).

### 3.6 Conclusion

The concept of inaccuracy given by Kerridge [67], measures the inaccuracy in the statement when the true distribution is not the same as the actual one. For a system which has survived up to time \( t \), for the residual time \([X \mid X \geq t]\), the residual inaccuracy measure is \( H(f, g; t) \). It characterizes the base line distribution \( F(.) \) uniquely when \( F(.) \) and \( G(.) \) satisfy the proportional hazard model. For the past time \([X \mid X \leq t]\) distribution, the past inaccuracy measure is given by \( H^*(f, g; t) \). It characterizes a specific distribution (uniform) under proportional reversed hazard model. So far we have carried over the study when the process is observed by assigning equal weights to all the observations. In the next chapter we will study this concept for the weighted distributions.