Chapter 2

Generalized Dynamic Entropy Measure

2.1 Introduction

The description of the behavior of biological and engineering systems normally requires the use of concepts of information theory, and in particular of entropy. Shannon’s entropy [109] is probably the most widely used index of alpha diversity in ecology, also the Kullback’s relative information measure [71] has received scant attention from ecologists as dissimilarity measure between two communities. Shannon’s theory has been used to study genomic sequences by calculating the amount of information contributed by individual nucleotides during these encoding and decoding processes, refer to [107]. Novel applications of Shannon [109] and Kullback-Leibler [71] information measures are promoting increased understanding of the mechanisms by which genetic information is converted to work and order. More recently it has been used in the context of theoretical neurobiology, refer to Johnson and Glantz [62].
Let $X$ be a non-negative continuous random variable which denote the lifetime of a device or a system with probability density function $f(x)$ and survival function $\bar{F}(x) = 1 - F(x)$, where $F(.)$ is the failure distribution function of $X$. Then the average amount of uncertainty associated with the random variable $X$ is given by the differential entropy \[ H(f) = - \int_{0}^{\infty} f(x) \log f(x) \, dx, \] (2.1)

which is the continuous analogous of the Shannon entropy measure for the discrete probability distribution $P = (p_1, p_2, \cdots, p_n)$ given by

\[ H(P) = - \sum_{i=1}^{n} p_i \log p_i, \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{n} p_i = 1. \] (2.2)

In life testing experiments, normally the experimenter has information about the current age of the system under consideration. Obviously the measure like (2.1) is not suitable in such situations and needs to be modified to take into account the current age also. Accordingly Ebrahimi [34] proposed a dynamic measure of uncertainty known as residual entropy for the residual lifetime distribution and defined the residual entropy $H(f; t)$ based on the measure (2.1) as

\[ H(f; t) = - \int_{t}^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} \, dx, \] (2.3)

where $\bar{F}(x)$ is the survival function of $X$. We note that the measure (2.3) is the Shannon entropy of the random variable $X_t = (X - t | X > t)$, and also, when $t = 0$, (2.3) becomes (2.1).

Ebrahimi [34] has showed that the dynamic (residual) measure (2.3) uniquely determines the survival function $\bar{F}(.)$. Sankaran and Gupta [105] have characterized some specific residual lifetime distributions using (2.3) in terms of hazard rate function and mean residual life function.
The measure (2.2) is additive in nature in the sense that if $X$ and $Y$ are two independent random variables, then

$$H(X \bullet Y) = H(X) + H(Y).$$ \hfill (2.4)

With ever increasing applications of information theoretic measures, sub-additivity rather than additivity has become an acceptable basis. In many social and physical systems the additivity does not quite prevail. For instance, in biological systems the interactions between the various drugs call for non-additivity of the individual effects rather than additivity. Thus non-additive entropy measures are of vital importance from applications point of view. An important non-additive entropy measure given by Havrda and Charvat [57] is

$$H^\alpha(P) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \sum_{i=1}^{n} p_i^\alpha - 1 \right], \quad \alpha \neq 1, \quad \alpha > 0. \hfill (2.5)$$

It satisfies the non-additivity

$$H(X \bullet Y) = H(X) + H(Y) + (2^{1-\alpha} - 1)H(X)H(Y).$$ \hfill (2.6)

The continuous analogous to the measure (2.5) is

$$H^\alpha(f) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_0^\infty f^\alpha(x) \, dx - 1 \right], \quad \alpha \neq 1, \quad \alpha > 0. \hfill (2.7)$$

When $\alpha \to 1$, the measure (2.7) tends to the differential entropy (2.1).

Among the existing Shannon-Like entropies, the Havrda and Charvat entropy is perhaps the best known and most widely used entropy. This is mainly because Havrda and Charvat entropy has a number of desirable properties which are crucial in many applications. It is more general than the Shannon entropy and simpler than the Renyi entropy [101]. Further the importance of this measure arises from the fact that it is frequently employed in other fields with slight variations. One such variations is $q$-entropy

$$H_q(f) = \frac{1}{(1-q)} \left[ \int_0^\infty f^q(x) \, dx - 1 \right], \hfill (2.8)$$
where $q$ can be seen as measuring the degree of nonextensivity. This is also the well-known Tsallis entropy [121, 122]. In recent years, authors have shown more interest in studying the properties and applications of Tsallis entropy, refer to Boghosian [19], Compte and Jou [22], Hamity and Barraco [55] and, Ion and Ion [60].

In this chapter we propose a dynamic (residual) measure of entropy, based on the non-additive entropy (2.7) and study it. The chapter is organized as follows. In Section 2.2, the generalized dynamic measure of entropy is proposed and a characterization result that $H^\alpha(f; t)$ uniquely determines the survival function $\overline{F}(\cdot)$ has been studied. By considering a relation between dynamic entropy measure and hazard rate function, some specific residual lifetime distributions have been characterized in Section 2.3. Section 2.4 deals with some properties, like upper bound, monotonicity etc. of the measure prescribed. The chapter ends with the concluding remarks.

2.2 Generalized Dynamic Entropy Measure

Let $X$ be a non-negative random variable representing the lifetime of a system with the average uncertainty given by the non-additive entropy (2.7). Suppose that the system has survived up to time $t$, then the measure of uncertainty of the remaining lifetime denoted by the random variable $X_t = [X - t \mid X > t]$, based on the generalized entropy (2.7) is proposed as

$$H^\alpha(f; t) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_0^\infty f_t^\alpha(x) dx - 1 \right], \quad \alpha \neq 1, \quad \alpha > 0, \quad (2.9)$$

where $f_t(x)$ is the p.d.f. of the random variable $X_t = (X - t \mid X > t)$ given by
\[ f_t(x) = \begin{cases} \frac{f(x)}{F(t)}, & \text{if } x > t \\ 0, & \text{otherwise.} \end{cases} \]

The measure (2.9) may be considered as the residual measure of entropy. This can be rewritten as
\[ H^\alpha(f; t) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \frac{\int_t^\infty f^\alpha(x)dx}{F^\alpha(t)} - 1 \right], \alpha > 0, \alpha \neq 1. \tag{2.10} \]

Obviously \( H^\alpha(f; 0) = H^\alpha(f) \) is the Havrda and Charvat information measure (2.7), and when \( \alpha \to 1 \), then (2.10) reduces to (2.3), the residual entropy \( H(f; t) \).

### 2.2.1 Characterization Result

A natural question arises that whether the proposed generalized residual measure of entropy \( H^\alpha(f; t) \) determines the lifetime distribution \( F(.) \) uniquely. In this context we prove the following Theorem.

**Theorem 2.1** Let \( X \) be a non-negative continuous random variable with probability density function \( f(x) \). If \( H^\alpha(f; t) < \infty, \forall \alpha > 0, \alpha \neq 1 \) and is increasing in \( t \), then \( H^\alpha(f; t) \) determines the distribution function \( F(.) \) uniquely.

**Proof** Rewriting the residual entropy (2.10) as
\[ (2^{1-\alpha} - 1)H^\alpha(f; t) + 1 = \frac{\int_t^\infty f^\alpha(x)dx}{F^\alpha(t)}. \tag{2.11} \]

Differentiating (2.11) with respect to \( t \), we obtain
\[ (2^{1-\alpha} - 1)H'(f; t) = -[\lambda_F(t)]^\alpha + \alpha \lambda_F(t) \frac{\int_t^\infty f^\alpha(x)dx}{F^\alpha(t)}, \tag{2.12} \]
where $\lambda_F(t) = \frac{f(t)}{F(t)}$ is the hazard rate of the random variable $X$.

Using (2.11), it can be rewritten as

$$(2^{1-\alpha} - 1)H'\alpha(f; t) = -[\lambda_F(t)]^\alpha + \alpha\lambda_F(t) + \alpha\lambda_F(t)(2^{1-\alpha} - 1)H^\alpha(f; t).$$ (2.13)

This gives

$$[\lambda_F(t)]^\alpha = \alpha\lambda_F(t) + \alpha\lambda_F(t)(2^{1-\alpha} - 1)H^\alpha(f; t) - (2^{1-\alpha} - 1)H'\alpha(f; t).$$ (2.14)

Hence for fixed $t > 0$, $\lambda_F(t)$ is a solution of the equation

$$g(x) = x^\alpha - \alpha x - \alpha x(2^{1-\alpha} - 1)H^\alpha(f; t) + (2^{1-\alpha} - 1)H'\alpha(f; t).$$ (2.15)

Differentiating it both sides with respect to $x$, we have

$$g'(x) = \alpha x^{\alpha-1} - \alpha - \alpha(2^{1-\alpha} - 1)H^\alpha(f; t).$$ (2.16)

For extreme value of $g(x)$, we must have $g'(x) = 0$, which gives

$$x_t = [1 + (2^{1-\alpha} - 1)H^\alpha(f; t)]^{\frac{1}{\alpha-1}}. $$

Further

$$g''(x) = \alpha(\alpha-1)x^{\alpha-2}. $$

**Case I:** Let $\alpha > 1$, then $g''(x_t) > 0$. Thus $g(x)$ attains minimum at $x_t$. Also, $g(0) < 0$ and $g(\infty) = \infty$. Further $g(x)$ decreases for $0 < x < x_t$ and increases for $x > x_t$, so $x = \lambda_F(t)$ is the unique solution to $g(x) = 0$.

**Case II:** Let $\alpha < 1$, then $g''(x_t) < 0$. Thus $g(x)$ attains maximum value at $x_t$. Also, $g(0) > 0$ and $g(\infty) = -\infty$. Further it can be easily seen that $g(x)$ decreases for $x > x_t$, and increases for $0 < x < x_t$, so $x = \lambda_F(t)$ is the unique solution to $g(x) = 0$.

Thus the generalized dynamic entropy measure (2.10) determines the hazard rate function, and hence, the distribution function uniquely. This completes the proof.
2.3 Characterizing Some Specific Lifetime Distribution Functions

In this section, by considering a relationship between the non-additive residual entropy $H^\alpha(f; t)$ and the hazard rate function $\lambda_F(t)$, we characterize some specific lifetime distributions based on the generalized dynamic entropy measure (2.10). We prove the following theorem:

**Theorem 2.2** Let $X$ be a non-negative continuous random variable with survival function $\bar{F}(.)$, hazard rate $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ and non-additive residual entropy $H^\alpha(f; t)$, then

$$H^\alpha(f; t) = \frac{c}{\alpha} + \frac{\lambda_F^{-1}(t) - \alpha}{\alpha(2^{1-\alpha} - 1)}, \quad (2.17)$$

if, and only if for

(i) $c = 0$, $X$ has exponential distribution for $\alpha \neq 1$, $\alpha > 0$,

(ii) $c > 0$, $X$ has distribution with p.d.f.

$$f(t) = Apq\exp[-A(1+pt)^{q}] = Apq\exp[-A(1+pt)^{q}], \quad t \geq 0, \quad 0 < \alpha < 1, \quad (2.18)$$

(iii) $c < 0$, $X$ has distribution with p.d.f.

$$f(t) = Apq\exp[-A(1-pt)^{q}] = Apq\exp[-A(1-pt)^{q}], \quad t \geq 0, \quad 0 < \alpha < 1, \quad (2.19)$$

where

$$p = \frac{k\alpha}{d}, \quad q = \frac{\alpha - 1}{\alpha - 2}, \quad A = \left[\frac{d}{q}\right]^q \frac{1}{k\alpha}, \quad k = (2^{1-\alpha} - 1) \text{ and } d > 0$$

are constants.

**Proof** (i) Let $X$ be an exponential random variable with parameter $\theta > 0$, then its p.d.f. is given by

$$f(x) = \theta e^{-\theta x} \quad (2.20)$$
and the failure rate function is \( \lambda_F(t) = \theta \). The residual entropy \( H^\alpha(f; t) \) in this case becomes

\[
H^\alpha(f; t) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_t^\infty \frac{f^\alpha(x)dx}{F^\alpha(t)} - 1 \right]
\]

\[
= \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_t^\infty (\theta e^{-\theta x})^\alpha dx \right]
\]

\[
= \frac{1}{(2^{1-\alpha} - 1)} \left[ \frac{\theta^{\alpha-1} - \alpha}{\alpha} \right]
\]

\[
= \left[ \frac{\lambda_F^{-1}(t) - \alpha}{\alpha(2^{1-\alpha} - 1)} \right],
\]

which is (2.17) for \( c = 0 \).

Conversely, consider

\[
\frac{1}{(2^{1-\alpha} - 1)} \left[ \int_t^\infty \frac{f^\alpha(x)dx}{F^\alpha(t)} - 1 \right] = \frac{1}{(2^{1-\alpha} - 1)} \left[ \frac{\lambda_F^{-1}(t) - \alpha}{\alpha} \right],
\]

Substituting for \( \lambda_F(t) = \frac{f(t)}{F(t)} \) and simplifying, we obtain

\[
\alpha \int_t^\infty f^\alpha(x)dx = F(t)f^{\alpha-1}(t).
\]

Differentiating (2.22) w.r.t. \( t \) both sides, we obtain

\[
f'(t)F(t) + f^2(t) = 0,
\]

which further gives

\[
\lambda_F'(t) = 0 \Rightarrow \lambda_F(t) = a, \text{ a constant.}
\]

Since exponential distribution is the only distribution with failure rate as a constant, thus \( X \) follows the exponential distribution.
(ii) Let $X$ be a random variable with p.d.f. as given in (2.18), then

$$(2^{1-\alpha} - 1)H^\alpha(f; t) = [\int_t^\infty \frac{f^\alpha(x)dx}{F^\alpha(t)} - 1], \; \alpha > 0, \; \alpha \neq 1,$$

becomes

$$(2^{1-\alpha} - 1)H^\alpha(f; t) = \frac{(Apq)^{\alpha-1}}{\alpha}(1 + pt)^q + \frac{(Apq)^{\alpha-1}}{A\alpha^2} - 1,$$

or,

$$H^\alpha(f; t) = \frac{[(Apq)(1 + pt)^q]^{\alpha-1} - \alpha}{\alpha(2^{1-\alpha} - 1)} + \frac{(Apq)^{\alpha-1}}{A\alpha^2(2^{1-\alpha} - 1)}. \tag{2.23}$$

Further since the hazard rate function of the p.d.f. (2.18) is

$$\lambda_F(t) = (Apq)(1 + pt)^q,$$

thus (2.23) can be rewritten as

$$H^\alpha(f; t) = \frac{[\lambda_F^{\alpha-1}(t) - \alpha]}{\alpha(2^{1-\alpha} - 1)} + \frac{c}{\alpha}, \; 0 < \alpha < 1, \tag{2.24}$$

where $c = \left[\frac{(Apq)^{\alpha-1}}{A\alpha(2^{1-\alpha} - 1)}\right] > 0$, and this proves the if part.

To prove the 'only if' part consider (2.17) to be valid. This is equivalent to

$$\int_t^\infty \frac{f^\alpha(x)dx}{F^\alpha(t)} = \frac{kc}{\alpha} + \frac{\lambda_F^{\alpha-1}(t)}{\alpha},$$

or,

$$\alpha \int_t^\infty f^\alpha(x)dx = kcF^\alpha(t) + f^{\alpha-1}(t)F(t). \tag{2.25}$$

Differentiating both sides of this equation with respect to $t$, we get

$$\frac{\alpha - 1}{\alpha} \lambda_F^{\alpha-3}(t) \left[\lambda_F^2(t) + \frac{f'(x)}{F(t)}\right] = kc. \tag{2.26}$$

Using the fact that

$$\lambda_F'(t) = \frac{f'(t)}{F(t)} + \lambda_F^2(t),$$
Eq. (2.26) becomes
\[ \frac{\lambda_F'(t)}{\lambda_F^{3-\alpha}(t)} = \frac{k\alpha}{\alpha - 1}. \] (2.27)
Solving this for \( \lambda_F(t) \), we obtain
\[ \lambda_F(t) = \left[ \left( \frac{2-\alpha}{1-\alpha} \right) (k\alpha t + d) \right]^{\frac{1}{\alpha-1}} \] (2.28)
\[ = \left( \frac{d}{q} \right) (1 + pt)^{(q-1)} ; \ p, q, t > 0, \]
which is the hazard rate function of the probability density function (2.18), and this concludes the proof for part (ii).

(iii) The proof for the case \( c < 0 \) is similar to that of (ii) except that the signs of \( p \) and \( A \) become negative.

2.3.1 Behavior of Hazard Rate Function Versus Time

We know that a lifetime distribution \( F(.) \) is classified according to the shape of its hazard rate function \( \lambda_F(t) \) as follows. Distribution \( F(.) \) is increasing failure rate (IFR) (or, decreasing failure rate (DFR)), if its hazard rate function \( \lambda_F(t) \) is non-decreasing (or, non-increasing) in \( t \); bathtub (BT) (or, upside bath tub (UBT)) curve, if \( \lambda_F(t) \) has a bath tub (or, upside-down bath tub) shape.

The patterns of failures over time are normally classified as infant mortality, useful life, and wear-out recognized respectively by decreasing, constant, and increasing hazard rate functions. The three patterns combine to produce the well known bath tub curve. The bath tub shaped failure rate functions play an important role in reliability applications, such as human life and electronic devices.

The graph of the hazard rate function (2.28) for some specific values of the parameters have been shown in Figs. 2.1-2.4.
Fig. 2.1: Plot of $\lambda_F(t)$ versus $t$

Fig. 2.2: Plot of $\lambda_F(t)$ versus $t$
### Useful Life Period

![Useful Life Period Diagram]

- $c = 3$, $d = 2$, $\alpha = 1$

**Fig. 2.3:** Plot of $\lambda_F(t)$ versus $t$

### Bath Tub Curve

![Bath Tub Curve Diagram]

- $c = .3$, $d = 2$, $\alpha = 2.25$

**Fig. 2.4:** Plot of $\lambda_F(t)$ versus $t
Figs. 2.1 and 2.2 has increasing hazard rate function a case of wear out. In Fig. 2.3, the period is characterized by a relatively constant failure rate. The length of this period is referred to as useful life of a unit. In Fig. 2.4, the life of a unit can be divided into three distinct periods. The first period is of infant mortality period, the next period is of useful life and the third period, which begins at the point where the slope begins to increase and extends to the end of the graph, is wear out period.

### 2.4 Properties of Generalized Dynamic Entropy Measure

In this section we study some properties of the non-additive residual entropy measure (2.10). We recall, refer to Section 1.5.4, that if $X$ is a random variable with distribution function $F(\cdot)$, then the mean residual life of $X$ is given by

$$
\delta_F(t) = E[X - t|X > t] = \int_t^\infty \frac{x f(x)}{F(t)} \, dx,
$$

$$
= t + \frac{1}{F(t)} \int_t^\infty F(x) \, dx.
$$

This represents the expected time a system will work further provided that it has survived to a certain point of time $t$, refer to [75].

1. **Upper Bound to $H^\alpha(f; t)$**: We have the following result:

**Theorem 2.3** If $X$ is the lifetime of a system with probability density function $f(x)$, survival function $\overline{F}(x)$, then

$$
H^\alpha(f; t) \leq \frac{\delta_{\alpha}(t) - \alpha}{\alpha(2^{1/\alpha} - 1)} , \quad \forall \ t \geq 0, \ \alpha > 0, \ \alpha \neq 1 ,
$$

(2.29)

where $\delta_F(t)$ is the mean residual life function of the exponential distribution.
Proof For a given $t$, let the random variable $Y_t$ be defined as $[Y_t = Y | Y > t]$ and $g_t(y)$ be its probability density function. Then

$$g_t(y) = \frac{d}{dy} P(Y_t \leq y) = \frac{d}{dy} [P(Y \leq y | Y > t)]$$

$$= \begin{cases} \frac{f(y)}{F(t)} & \text{if } y > t \\ 0 & \text{if } y \leq t \end{cases}$$

It is easy to see that $\int^\infty_t yg_t(y)dy = \delta F(t) + t$. If we define $Z_t = Y_t - t$, then the probability density function of $Z_t$ is $h_t(\eta)$, where $h_t(\eta) = g_t(\eta + t)$ and $E[Z_t] = \delta F(t)$. Thus the Havrda and Charvat entropy of $Z_t$ is

$$H^\alpha(Z_t) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_0^\infty h_t^\alpha(\eta)d\eta - 1 \right]$$

$$= \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_0^\infty g_t^\alpha(\eta + t)d\eta - 1 \right]$$

$$= \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_t^\infty g_t^\alpha(\eta)d\eta - 1 \right]$$

$$= H^\alpha(g; t).$$

Under the assumption that $\delta F(t) < \infty$, and if the support of a random variable is $[0, \infty)$, then the exponential distribution with mean residual life $\delta F(t)$ has the maximum entropy, refer to [10]. Now the dynamic Havrda and Charvat entropy is

$$H^\alpha(f; t) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_0^\infty f_t^\alpha(x)dx - 1 \right],$$

and for exponential distribution $\delta F(t) = \frac{1}{\lambda_F(t)} = \frac{1}{\theta}$, thus from (2.21) we have

$$H^\alpha(f; t) \leq \frac{(\delta_F^1)^{1-\alpha}(t) - \alpha}{\alpha(2^{1-\alpha} - 1)}.$$

(2.30)
This completes the proof.

**Remark 2.1** Further if $\delta_F(t)$ is a decreasing function of $t$, then

$$H^\alpha(f; t) \leq \frac{(\mu^{1-\alpha} - \alpha)}{\alpha (2^{1-\alpha} - 1)},$$

where $\delta_F(0) = E[X] = \mu$ is the mean lifetime of the unit.

2. **Monotonicity of $H^\alpha(f; t)$**: In reliability and life testing situations, a number of non-parametric classes of lifetime distributions are considered to model the life times of individuals as well as of mechanical systems or components. Most of these classes characterize the aging properties of the underlying phenomenon. Some of the most commonly used classes are the ones defined in terms of failure rate and mean residual life functions. Here we identify the conditions under which the residual entropy measure $H^\alpha(f; t)$ given by (2.10) is monotone. First we give the following definitions.

**Definition 2.1** A distribution function $F(.)$ has increasing (or, decreasing) residual entropy of order $\alpha$ (IREO(\(\alpha\))) (or, DREO(\(\alpha\))), if $H'^\alpha(f; t)$ is increasing (or, decreasing) in $t$, $t > 0$, where $H'^\alpha(f; t)$ is the derivative of $H^\alpha(f; t)$ w.r.t. $t$.

This implies that $F(.)$ has IREO(\(\alpha\)) (DREO(\(\alpha\))) if $H'^\alpha(f; t) \geq (\leq) 0$.

When $F(.)$ is both IREO(\(\alpha\)) and DREO(\(\alpha\)), then $H'^\alpha(f; t) = 0$ and consequently the distribution is exponential. This means that the exponential distribution is the only distribution which is both IREO(\(\alpha\)) and DREO(\(\alpha\)).

**Definition 2.2** A distribution function $F(.)$ is said to be decreasing (or, increasing) mean residual life DMRL (or, IMRL) if its mean residual life function is decreasing (or, increasing) in $t \geq 0$.

Next we prove the following results in context with the monotonicity of $H^\alpha(f; t)$. 
**Theorem 2.4** (a) If $F(.)$ is DMRL, then it is DREO($\alpha$).

(b) If $F(.)$ is IREO($\alpha$), then it is IMRL.

**Proof** From Eq. (2.13), we have

$$H'\alpha(f; t) = -\frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha} - 1)} + \frac{\alpha\lambda_F(t)}{(2^{1-\alpha} - 1)} + \alpha\lambda_F(t)H\alpha(f; t).$$

Using Theorem 2.3, we get

$$H'\alpha(f; t) \leq -\frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha} - 1)} + \frac{\alpha\lambda_F(t)}{(2^{1-\alpha} - 1)} + \lambda_F(t)[\delta_F^{1-\alpha}(t) - \alpha]$$

$$= \frac{\lambda_F(t)}{(2^{1-\alpha} - 1)} \{\delta_F^{1-\alpha}(t) - \lambda_F^{-1}(t)\}$$

$$= \frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha} - 1)} \{[\lambda_F(t)\delta_F(t)]^{1-\alpha} - 1\}. \quad (2.31)$$

Using the relationship $\lambda_F(t)\delta_F(t) = 1 + \delta'_F(t)$, we obtain

$$H'\alpha(f; t) \leq \frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha} - 1)} \{1 + [1 + \delta'_F(t)]^{1-\alpha}\}. \quad (2.32)$$

We consider the following two cases.

**Case I:** Let $0 < \alpha < 1$, then $(2^{1-\alpha} - 1) > 0$ and thus $H'\alpha(f; t) \leq 0$.

**Case II:** Let $\alpha > 1$, then $(2^{1-\alpha} - 1) < 0$ and thus $H'\alpha(f; t) \leq 0$.

This completes the proof.

(b) The proof is similar to that of part (a), and hence omitted.

**Remark 2.2** When $\alpha \to 1$, then (2.32) reduces to

$$H'(f; t) \leq \lambda_F(t) [\log(1 + \delta'_F(t))],$$

a result given by Ebrahimi and Kirmani [39].
**Theorem 2.5** (a) If $X$ is IREO($\alpha$) and if $\phi$ is non-negative, increasing and convex, then $\phi(X)$ is DREO($\alpha$).

(b) If $X$ is DREO($\alpha$) and if $\phi$ is non-negative, increasing and convex, then $\phi(X)$ is IREO($\alpha$).

**Proof** (a) The probability density function of $Y = \phi(X)$ is $g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$. Thus

$$H^\alpha(g; t) = \frac{1}{(2^{1-\alpha} - 1)} \left[ \int_1^\infty g^\alpha(y) dy \frac{1}{G^\alpha(t)} - 1 \right].$$

This gives

$$H^\alpha(g; t) = \frac{1}{(2^{1-\alpha} - 1)} \left( \frac{1}{F^\alpha(\phi^{-1}(t))} \int_{\phi^{-1}(t)}^\infty f^\alpha(\phi^{-1}(y)) dy \frac{1}{\phi'^\alpha(\phi^{-1}(y))} - 1 \right). \quad (2.33)$$

By taking $x = \phi^{-1}(t)$, we have

$$H^\alpha(g; t) = \frac{1}{(2^{1-\alpha} - 1)} \left( \frac{1}{F^\alpha(\phi^{-1}(t))} \int_{\phi^{-1}(t)}^\infty f^\alpha(x) \phi'^{-\alpha}(x) dx - 1 \right). \quad (2.34)$$

Differentiating w.r.t. $t$ under the integral sign, we obtain

$$(2^{1-\alpha} - 1) \frac{d}{dt} H^\alpha(g; t) = -\frac{f^\alpha(\phi^{-1}(t))\phi'^{-\alpha}(\phi^{-1}(t))}{\phi'(t)F^\alpha(\phi^{-1}(t))}$$

$$+ \int_{\phi^{-1}(t)}^\infty f^\alpha(x) \phi'^{-\alpha}(x) dx \left[ \frac{\alpha f(\phi^{-1}(t))}{\phi'(t)F^{\alpha+1}(\phi^{-1}(t))} \right]. \quad (2.35)$$

This gives

$$(2^{1-\alpha} - 1) \frac{d}{dt} H^\alpha(g; t) = -\frac{\lambda_f^\alpha(\phi^{-1}(t))\phi'^{-\alpha}(\phi^{-1}(t))}{\phi'(t)}$$

$$+ \frac{\alpha \lambda_f(\phi^{-1}(t))}{\phi'(t)} \left( \frac{1}{F^\alpha(\phi^{-1}(t))} \int_{\phi^{-1}(t)}^\infty f^\alpha(x) \phi'^{-\alpha}(x) dx - 1 \right). \quad (2.36)$$
Let $\alpha > 1$. $\phi'(x)$ is increasing function because $\phi(x)$ is a convex function and so, $\phi^{1-\alpha}(x)$ is a decreasing function, that is,

$$\phi^{1-\alpha}(x) \leq \phi^{1-\alpha}(\phi^{-1}(t)), \quad \forall \ x > \phi^{-1}(t).$$

Hence, (2.36) becomes

$$(2^{1-\alpha} - 1) \frac{d}{dt} H^\alpha(g; t) \leq -\frac{\lambda_F^\alpha(\phi^{-1}(t)) \phi^{n-\alpha}(\phi^{-1}(t))}{\phi'(t)}$$

$$+ \alpha \frac{\lambda_F(\phi^{-1}(t)) \phi^{n-\alpha}(\phi^{-1}(t))}{\phi'(t)} \left( \frac{\int_{\phi^{-1}(t)}^\infty f^\alpha(x) dx}{\bar{F}(\phi^{-1}(t))} - 1 \right).$$

(2.37)

Using (2.11), we obtain

$$(2^{1-\alpha} - 1) \frac{d}{dt} H^\alpha(g; t) = -\frac{\lambda_F^\alpha(\phi^{-1}(t)) \phi^{n-\alpha}(\phi^{-1}(t))}{\phi'(t)}$$

$$+ \alpha \frac{\lambda_F(\phi^{-1}(t)) \phi^{n-\alpha}(\phi^{-1}(t))}{\phi'(t)} \left\{ (2^{1-\alpha} - 1) H^\alpha(f; \phi^{-1}(t)) + 1 \right\}$$

$$= \frac{\phi^{n-\alpha}(\phi^{-1}(t))}{\phi'(t)} \left[ -\lambda_F^\alpha(\phi^{-1}(t)) + \alpha \lambda_F(\phi^{-1}(t)) \left( (2^{1-\alpha} - 1) H^\alpha(f; \phi^{-1}(t)) + 1 \right) \right].$$

Using (2.13), we get

$$\frac{d}{dt} H^\alpha(g; t) = \frac{\phi^{n-\alpha}(\phi^{-1}(t))}{\phi'(t)} \left[ H^\alpha(f; \phi^{-1}(t)) \right] \leq 0.$$
This bound can be obtained using (2.13). The proof is simple and hence omitted.

**Remark 2.3** Since the distribution function and the hazard rate function are equivalent in the sense that one can be obtained from the other uniquely, thus using the relationship

\[
\tilde{F}(t) = \exp \left[ - \int_0^t \lambda_F(x) dx \right],
\]

Theorem 2.6 can give a bound to the distribution function also. The result is stated as follows.

**Corollary 2.1** Let \( F(.) \) be an IREO(\( \alpha \)), (DREO(\( \alpha \))), then

\[
\tilde{F}(t) \geq (\leq) \exp \left[ - \int_0^t \left\{ \alpha + (2^{(1-\alpha)} - 1)\alpha H^\alpha(f; u) \right\}^{\frac{1}{\alpha-1}} du \right] \forall t \geq 0.
\]

### 2.5 Conclusion

The concept of entropy \( H(f) \) introduced by Shannon [109] in the literature measures the average uncertainty associated with a random variable \( X \) with probability density function \( f(.) \). For a component, which has survived up to time \( t \), \( H(f; t) \) measures the uncertainty about the remaining lifetime \([X|X \geq t]\). Considering the importance of non-additive entropy measure we have proposed one parameter generalized residual entropy measure \( H^\alpha(f; t) \) and have observed that the proposed measure determines the distribution uniquely. Further we have seen that it characterizes three specific lifetime distributions. Some properties like upper bound to the measure proposed, and monotonicity etc. have been studied. In the subsequent chapters we extend the scope of dynamic entropy measures to the concept of inaccuracy given by Kerridge [67].