Chapter 6

Dynamic Cumulative Inaccuracy Measures

6.1 Introduction

The average amount of uncertainty associated with the random variable $X$ with p.d.f. $f(x)$, as given by Shannon differential entropy [109], is

$$H(f) = - \int_{0}^{\infty} f(x) \log f(x) \, dx .$$

(6.1)

The concept of entropy has been generalized in a number of different ways. An extension of Shannon’s idea has been given by Kerridge [67], as Kerridge’s inaccuracy. If $f(x)$ is the actual probability density function (p.d.f.) and $g(x)$ is the reference p.d.f. of a random variable $X$ associated with a system, then Kerridge’s measure of inaccuracy [67] is

$$H(f; g) = - \int_{0}^{\infty} f(x) \log g(x) \, dx .$$

(6.2)
The measure of inaccuracy suggested by Kerridge has many useful applications in statistics and has been studied by many researchers from various aspects. We have also studied the dynamic and length biased dynamic measures of inaccuracy in Chapter 3 and Chapter 4 respectively. In the preceding chapter we have considered the concept of cumulative residual entropy (CRE)

\[ \xi(X) = \xi(F) = - \int_{0}^{\infty} \bar{F}(x) \log \bar{F}(x) dx, \quad (6.3) \]

as given by Rao et al. [98] and have studied its one parameter and two parameters generalizations and also their dynamic versions.

Taking into considerations the advantage of distribution function based information theoretic measures over probability density function based measure as discussed in Chapter 5, in this chapter we study the distribution functions based inaccuracy measures analogous to the Kerridge inaccuracy measure (6.2). The distribution function based inaccuracy measure can also be viewed as a natural extension of the cumulative residual entropy measure suggested by [98]. The chapter has been organized as follows. In Section 6.2 we propose a measure of cumulative residual inaccuracy (CRI) and derive an upper bound to it. Section 6.3 considers dynamic cumulative residual inaccuracy (DCRI) measure in context with residual lifetime distribution \([X|X \geq t]\). A characterization theorem for the dynamic cumulative residual inaccuracy under proportional hazard rate model has been proved in Section 6.4 and some specific lifetime distributions have been characterized. Section 6.5 introduces the dynamic cumulative past inaccuracy (DCPI) in context with past lifetime distribution \([X|X \leq t]\), and characterization result for this has been studied in Section 6.6 which also includes a few other results on this measure. Section 6.7 concludes the chapter.
6.2 Cumulative Residual Inaccuracy

Let $X$ and $Y$ be two random variables with the same support, and let $\bar{F}(x)$ and $\bar{G}(x)$ be their survival functions then the cumulative residual inaccuracy (CRI) analogous to the inaccuracy measure (6.2) is defined as

$$\xi(F; G) = -\int_0^\infty \bar{F}(x) \log \bar{G}(x) dx.$$  \hspace{1cm} (6.4)

When these two distributions coincide the measure (6.4) reduces to the cumulative residual entropy (6.3).

If the two random variables $X$ and $Y$ satisfy the proportional hazard model (PHM), refer to Cox [24] and Efron [42], that is, if $\lambda_G(x) = \beta \lambda_F(x)$, or equivalently

$$\bar{G}(x) = [\bar{F}(x)]^\beta,$$  \hspace{1cm} (6.5)

for some constant $\beta > 0$, then obviously the cumulative residual inaccuracy (6.4) reduces to a constant multiple of the cumulative residual entropy (6.3).

**Example 6.1** Let a non-negative random variable $X$ be uniformly distributed over $(a, b)$, $a < b$, with density and distribution functions respectively given by

$$f(x) = \frac{1}{b - a} \quad \text{and} \quad F(x) = \frac{x - a}{b - a}, \quad a < x < b.$$  

If the random variables $X$ and $Y$ satisfy the proportional hazard model (PHM), then the distribution function of the random variable $Y$ is

$$\bar{G}(x) = [\bar{F}(x)]^\beta = \left[\frac{b - x}{b - a}\right]^\beta, \quad a < x < b, \; \beta > 0.$$  

Substituting these in (6.4) and simplifying we obtain the cumulative inaccuracy measure as

$$\xi(F; G) = \frac{\beta(b - a)}{4}.$$  

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6.2.1 A Lower Bound to $\xi(F; G)$

Before deriving the lower bound to $\xi(F; G)$, we state the log-sum inequality given as follows;

Let $m$ be a sigma finite measure. If $f$ and $g$ are positive and $m$-integrable then

$$\int \log \left( \frac{f}{g} \right) dm \geq \left[ \int fdm \right] \log \frac{\int fdm}{\int gdm} . \quad (6.6)$$

Also another result of interest which we will use is the inequality given by

$$x \log \frac{x}{y} \geq x - y, \quad (6.7)$$

for all non-negative $x$ and $y$. We prove the following result.

**Theorem 6.1** If $X$ and $Y$ are two non-negative random variables with finite means $E(X)$ and $E(Y)$ respectively and if CRE measure $\xi(X)$ given by (6.3) is finite, then

$$\xi(F; G) \geq \int_0^\infty F(x)F(x)dx + E(X) - E(Y). \quad (6.8)$$

**Proof** We have

$$\xi(F; G) = - \int_0^\infty \tilde{F}(x) \log \tilde{G}(x)dx$$

$$= - \int_0^\infty \tilde{F}(x) \log \tilde{F}(x)dx + \int_0^\infty \tilde{F}(x) \log \frac{\tilde{F}(x)}{\tilde{G}(x)}dx .$$

Using the log-sum inequality (6.6), we have

$$\xi(F; G) \geq \xi(X) + \int_0^\infty \tilde{F}(x)dx \log \frac{\int_0^\infty \tilde{F}(x)dx}{\int_0^\infty \tilde{G}(x)dx}$$

$$\geq \xi(X) + \frac{E(X)}{E(Y)} E(X) - E(Y) .$$

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The last inequality has been obtained using (6.7). This proves the result.

In the next section we extend the concept of cumulative residual inaccuracy to dynamic cumulative residual inaccuracy.

### 6.3 Dynamic Cumulative Residual Inaccuracy

In life-testing experiments normally the experimenter has information about the current age of the system under consideration. Obviously the CRI (6.4) is not suitable in such a situation and needs to be modified to take into account the current age also. Further, if $X$ is the lifetime of a component which has already survived up to time $t$, then the random variable $X_t = [X - t|X > t]$, called the residual lifetime random variable, has the survival function

$$
\bar{F}_t(x) = \begin{cases} 
\frac{F(x)}{F(t)} & \text{if } x > t \\
1 & \text{otherwise}
\end{cases}
$$

and similarly for $\bar{G}_t(x)$.

The cumulative inaccuracy measure (6.4), for the residual lifetime random variable $X_t$, is

$$\xi(F, G; t) = -\int_t^\infty \bar{F}_t(x) \log \bar{G}_t(x) \, dx$$  \hspace{1cm} (6.9)

$$= -\int_t^\infty \frac{F(x)}{F(t)} \log \frac{G(x)}{G(t)} \, dx .$$  \hspace{1cm} (6.10)

The measure (6.10) is defined as the dynamic cumulative residual inaccuracy measure (DCRI). Obviously when $t = 0$, then (6.10) becomes (6.4).

We observe that (6.10) is analogous to the residual inaccuracy measure

$$H(f, g; t) = -\int_t^\infty \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} \, dx$$  \hspace{1cm} (6.11)
as discussed in Chapter 3.

**Example 6.2** Let $X$ be a non-negative random variable with p.d.f.

\[
f_X(x) = \begin{cases} 
2x & \text{if } 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

and survival function $\bar{F}(x) = 1 - F(x) = (1 - x^2)$, and let the random variable $Y$ be uniformly distributed over $(0, 1)$ with density and survival functions given respectively by $g_Y(x) = 1$ and $\bar{G}_Y(x) = 1 - x$, $0 < x < 1$.

Substituting these values in (6.10), the dynamic cumulative residual inaccuracy measure is

\[
\xi(F, G; t) = \begin{cases} 
\frac{9(1-t) - 2(1-t)^2}{18(1+t)} & \text{if } 0 \leq t < 1 \\
0 & \text{otherwise}
\end{cases}
\]

The plot of the dynamic cumulative residual inaccuracy measure $\xi(F, G; t)$ for $t \in [0, 1]$ is shown in Fig. 6.1

![Plot of $\xi(F, G; t)$ against $t \in [0, 1]$](image.png)

**Fig. 6.1:** Plot of $\xi(F, G; t)$ against $t \in [0, 1]$.


6.4 Characterization Result of Dynamic Cumulative Inaccuracy

The general characterization problem is to determine when the proposed dynamic inaccuracy measure (6.10) characterizes the distribution function uniquely. We study the characterization problem under the proportional hazard model (6.5). Also we know that the hazard rate function \( \lambda_F(t) = \frac{f(t)}{1 - F(t)} \) and the mean residual life function \( \delta_F(t) = \frac{\int_t^\infty \frac{F(x)}{F(t)} dx}{1 - F(t)} \), characterize the distribution function of a random variable \( X \) and the relation between the two is given by

\[
\lambda_F(t) = 1 + \frac{\delta_F'(t)}{\delta_F(t)}. \tag{6.12}
\]

We shall use (6.12) in establishing the characterization result stated next.

**Theorem 6.2** Let \( X \) and \( Y \) be two non-negative random variables with survival functions \( \bar{F}(.) \) and \( \bar{G}(.) \) satisfying the proportional hazard model (6.5), and let \( \xi(F,G; t) < \infty, \forall t \geq 0 \) be an increasing function of \( t \), then \( \xi(F,G; t) \) determines the survival function \( \bar{F}(.) \) of the variable \( X \) uniquely.

**Proof** The dynamic cumulative residual inaccuracy measure (6.10) can be expressed as

\[
\xi(F,G; t) = -\frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{G}(x) dx + \delta_F(t) \log \bar{G}(t), \tag{6.13}
\]

where \( \delta_F(t) \) is the mean residual life function. Substituting (6.5) into (6.13) gives

\[
\xi(F,G; t) = -\frac{\beta}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx + \beta \delta_F(t) \log \bar{F}(t).
\]

Differentiating this w.r.t. \( t \) both sides, we obtain
\[ \xi'(F,G;t) = \beta \log \bar{F}(t)[1 + \delta'_F(t)] \]

\[- \beta \lambda_F(t) \int_t^\infty \frac{F(x)}{F(t)} \log F(x) dx - \beta \lambda_F(t)\delta_F(t), \quad (6.14)\]

where \( \lambda_F(t) \) is hazard rate function. Substituting (6.12) and (6.13) in (6.14), we obtain

\[ \xi'(F,G;t) = \lambda_F(t)\{\xi(F,G;t) - \beta \delta_F(t)\}. \quad (6.15) \]

Let \( F_1, G_1 \) and \( F_2, G_2 \) be two sets of the probability distribution functions satisfying the proportional hazard model, that is, \( \lambda_{G_1}(x) = \beta \lambda_{F_1}(x) \), and \( \lambda_{G_2}(x) = \beta \lambda_{F_2}(x) \), and let

\[ \xi(F_1,G_1;t) = \xi(F_2,G_2;t), \forall \ t \geq 0. \quad (6.16) \]

Differentiating it both sides w.r.t. \( t \) and using (6.15), we obtain

\[ \lambda_{F_1}(t)\{\xi(F_1,G_1;t) - \beta \delta_{F_1}(t)\} = \lambda_{F_2}(t)\{\xi(F_2,G_2;t) - \beta \delta_{F_2}(t)\}. \quad (6.17) \]

If for all \( t \geq 0 \), \( \lambda_{F_1}(t) = \lambda_{F_2}(t) \), then \( \bar{F}_1(t) = \bar{F}_2(t) \) and the proof is over, otherwise, let

\[ A = \{t : t \geq 0, \text{ and } \lambda_{F_1}(t) \neq \lambda_{F_2}(t)\} \quad (6.18) \]

and assume the set \( A \) to be non empty. Thus for at least one \( t_0 \in A \), \( \lambda_{F_1}(t_0) \neq \lambda_{F_2}(t_0) \).

Without loss of generality suppose that \( \lambda_{F_2}(t_0) > \lambda_{F_1}(t_0) \). Using this, (6.17) for \( t = t_0 \) gives

\[ \xi(F_1,G_1;t_0) - \beta \delta_{F_1}(t_0) > \xi(F_2,G_2;t_0) - \beta \delta_{F_2}(t_0), \]

which implies that

\[ \delta_{F_1}(t_0) < \delta_{F_2}(t_0), \]

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a contradiction. Thus the set A is empty set and this concludes the proof.

Next, we characterize some specific lifetime distributions using the dynamic cumulative inaccuracy measure (6.10). We give the following theorem.

**Theorem 6.3** Let $X$ and $Y$ be two non-negative continuous random variables satisfying the proportional hazard model (6.5). If $X$ is with mean residual life $\delta_F(t)$, then the dynamic cumulative residual inaccuracy measure

$$\xi(F,G; t) = c \delta_F(t), \ c > 0$$  \hspace{1cm} (6.19)

if, and only if $X$ follows the

(i) exponential distribution for $c = \beta$ ,

(ii) Pareto distribution for $c > \beta$ ,

(iii) finite range distribution for $0 < c < \beta$.

**Proof** First we prove the 'if' part.

(i) If $X$ has an exponential distribution with survival function $\bar{F}(x) = \exp(-\theta x)$, $\theta > 0$, then the mean residual life function $\delta_F(t) = \frac{1}{\theta}$. The dynamic cumulative residual inaccuracy measure (6.10) under PHM is given as

$$\xi(F,G; t) = \frac{\beta}{\theta} = c\delta_F(t),$$

for $c = \beta$.

(ii) If $X$ follows Pareto distribution with p.d.f.

$$f(x) = \frac{ab^a}{(x+b)^{a+1}}, \ a > 1, \ b > 0,$$
then the survival function is
\[ \bar{F}(x) = 1 - F(x) = \left(1 + \frac{x}{b}\right)^{-a} = \frac{b^a}{(x+b)^a}, \]
and the mean residual life is
\[ \delta_F(t) = \frac{\int_t^\infty \bar{F}(x) dx}{\bar{F}(t)} = \frac{t + b}{a - 1}. \] (6.20)

The dynamic cumulative inaccuracy measure (6.10), under PHM is given by
\[ \xi(F, G; t) = \beta a (t + b) \frac{1}{(a - 1)^2} = c \delta_F(t), \]
for \( c = \frac{\beta a}{a - 1} > \beta. \)

(iii) In case \( X \) follows finite range distribution with p.d.f.
\[ f(x) = a(1 - x)^{a-1}, \ a > 1, \ 0 \leq x \leq 1, \]
then the survival function is
\[ \bar{F}(x) = 1 - F(x) = (1 - x)^a, \]
and the mean residual life is
\[ \delta_F(t) = \frac{1 - t}{a + 1}. \]

The inaccuracy measure (6.10) under PHM is given by
\[ \xi(F, G; t) = \beta a (1 - t) \frac{1}{(a + 1)^2} = c \delta_F(t), \]
for \( c = \frac{\beta a}{a + 1} < \beta. \)
This proves the 'if' part.

To prove the 'only if' part, consider (6.19) to be valid.

Eq. (6.13) under PHM, using (6.19) gives
\[ -\frac{\beta}{F(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx + \beta \delta_F(t) \log \bar{F}(t) = c \delta_F(t). \]
Differentiating both sides w.r.t. $t$, we obtain

\[
\frac{c}{\beta} \delta_F'(t) = \delta_F(t) \log \bar{F}(t) - \lambda_F(t) \delta_F(t) + \log \bar{F}(t)
\]

\[
- \lambda_F(t) \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx,
\]

\[
= \delta_F'(t) \log \bar{F}(t) - \lambda_F(t) \delta_F(t) + \log \bar{F}(t) + \lambda_F(t) \left[ \frac{c}{\beta} \delta_F(t) - \delta_F(t) \log \bar{F}(t) \right].
\]

From (6.12), put $\delta_F'(t) = \lambda_F(t) \delta_F(t) - 1$ and simplify, we obtain

\[
\lambda_F(t) \delta_F(t) = \frac{c}{\beta},
\]

which implies

\[
\delta_F'(t) = \frac{c}{\beta} - 1.
\]

Integrating both sides of this w.r.t. $t$ over $(0, x)$ yields

\[
\delta_F(x) = \left( \frac{c}{\beta} - 1 \right)x + \delta_F(0).
\]

(6.21)

The mean residual life function $\delta_F(x)$ of a continuous non-negative random variable $X$ is linear of the form (6.21) if, and only if the underlying distribution is exponential for $c = \beta$, Pareto for $c > \beta$, or finite range for $0 < c < \beta$, refer to Hall and Wellner [54]. This completes the theorem.

Next, we extend the result (6.19) to a more general case taking $c$ as a function of $t$. We state the following result:
**Theorem 6.4** Let $X$ and $Y$ be two non-negative continuous random variables and satisfying the proportional hazard model (PHM) (6.5) and if

\[
\xi(F, G; t) = c(t)\delta_F(t), \text{ for } t \geq 0,
\]

then

\[
\delta_F(t) = \left[ k + \left( \int_0^t \left\{ \frac{c(x) - \beta}{\beta} \right\} e^{\frac{c(x)}{\beta}} \, dx \right) \right] e^{-\frac{c(t)}{\beta}},
\]

where $k = \delta_F(0) e^{\frac{c(0)}{\beta}}$.

**Proof** Substituting (6.22) in (6.15), we obtain

\[
\xi'(F, G; t) = \lambda_F(t)\delta_F(t)\{c(t) - \beta\}.
\]

Differentiating (6.22) w.r.t. $t$ and substituting for $\xi'(F, G; t)$, from (6.24) we obtain

\[
c'(t)\delta_F(t) + c(t)\delta_F'(t) = \lambda_F(t)\delta_F(t)\{c(t) - \beta\}.
\]

Substituting $\lambda_F(t)\delta_F(t) = 1 + \delta_F'(t)$ in above expression and simplifying, we obtain

\[
\delta_F'(t) + \frac{c'(t)}{\beta}\delta_F(t) = \frac{c(t) - \beta}{\beta},
\]

a linear differential equation in $\delta_F(t)$. Solving this we obtain (6.23).

**Example 6.3** Let $c(t) = at + b, \ t > 0$ and $a > 0$. From (6.23), we obtain the general model with mean residual life function

\[
\delta_F(t) = ke^{-\frac{(at+b)}{\beta}} + \frac{at - 2\beta + b}{a} - \frac{(b - 2\beta)e^{-\frac{at}{\beta}}}{a}.
\]

If $a = 0$, we obtain the characterization results given by Theorem 6.3.

**Remark 6.1** For $\beta = 1$, (6.26) reduces to

\[
\delta_F(t) = ke^{-at-b} + \frac{b - 2 + at}{a} - \frac{(b - 2)e^{-at}}{a},
\]

a result given by Navarro et al. [86] in context with the cumulative residual entropy (6.3).
6.5 Dynamic Cumulative Past Inaccuracy Measure

Measures of uncertainty in context with past lifetime distributions have been studied extensively in the literature, refer to, Di Crescenzo and Longobardi [29, 30] Nanda and Paul [85]. We have also studied such measures in the proceeding chapters. For instance if at time $t$ a system, which is observed only at certain preassigned inspection times, is found to be down, then the uncertainty of the system’s life relies on the past, that is, at which instant in $(0, t)$ the system has failed. In this situation, the random variable $\tau X = [X | X \leq t]$ is suitable to describe the time elapsed between the failure of a system and the time when it is found to be ‘down’.

The past lifetime random variable $\tau X$ is related with two relevant aging functions, the reversed hazard rate defined by $\mu_F(x) = \frac{f(x)}{F(x)}$, and the mean past lifetime (MPT) defined by $\delta^*_{\tau}(t) = E(t - X | X < t) = \frac{1}{F(t)} \int_0^t F(x) dx$, which are further related as follows

$$\mu_F(t) = \frac{1 - \delta^*_{\tau}(t)}{\delta^*_{\tau}(t)}, \quad (6.27)$$

where $\delta^*_{\tau}(t) = \frac{d}{dt} \delta^*_{\tau}(t)$. For further results on reversed hazard rate function refer to Gupta and Nanda [51].

In analogy with the cumulative residual entropy (CRE) measure (6.3), based on the survival function $\bar{F}(x)$, Di Crescenzo and Longobardi [32] introduced and studied the cumulative entropy, defined as

$$\xi^*(F) = - \int_0^\infty F(x) \log F(x) dx, \quad (6.28)$$

based on the failure function $F(x)$.

A dynamic version of the cumulative entropy (6.28) given as

$$\xi^*(F; t) = - \int_0^t \frac{F(x)}{F(t)} \log \frac{F(x)}{F(t)} dx, \quad (6.29)$$

was also studied by Di Crescenzo and Longobardi [32].
Analogous to the Kerridge measure of inaccuracy (6.2), we propose a cumulative inaccuracy measure as

$$
\xi^*(F; G) = - \int_0^\infty F(x) \log G(x) dx ,
$$

(6.30)

where $F(x)$ is the baseline distribution function and $G(x)$ can be considered as some reference distribution function. When these two distributions coincide, the measure (6.30) reduces to the measure (6.28) the cumulative entropy.

In case the two random variables $X$ and $Y$ satisfy the *proportional reversed hazard model* (PRHM), refer to Gupta et al. [50], that is, if $\mu_G(x) = \beta \mu_F(x)$, or equivalently

$$
G(x) = [F(x)]^\beta, \beta > 0 ,
$$

(6.31)

then obviously the cumulative inaccuracy measure (6.30) reduces to a constant multiple of the cumulative information measure (6.28).

The distribution function of the past lifetime random variable $[X|X \leq t]$ is given by

$$
F_{t,X}(x) = \begin{cases} 
\frac{F(x)}{F(t)} & ; \text{ if } x < t \\
1 & ; \text{ otherwise}
\end{cases}
$$

and similarly for $\bar{G}_{t}(x)$. Thus the cumulative inaccuracy measure analogous to the inaccuracy measure (6.30), for the past lifetime distribution is given by

$$
\xi^*(F, G; t) = - \int_0^{t} F_{t,X}(x) \log G_{t,X}(x) dx,
$$

$$
= - \int_0^{t} \frac{F(x)}{F(t)} \log \frac{G(x)}{G(t)} dx .
$$

(6.32)

We define the measure (6.32) as the *dynamic cumulative past inaccuracy* measure. When $t \to \infty$, the measure (6.32) reduces to (6.30).
Example 6.4 Let $X$ be a non-negative random variable with distribution function $F_X(x) = x^2$, $0 < x < 1$, and let the random variable $Y$ be uniformly distributed over $(0, 1)$ with distribution function given by $G_Y(x) = x$. Substituting these values in (6.32), we obtain the cumulative past inaccuracy measure as

$$\xi^*(F, G; t) = \frac{t}{9}.$$ 

Example 6.5 Let $X$ and $Y$ be two non-negative random variables having distribution functions respectively

$$F(x) = \begin{cases} 
\frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\
\frac{x^2 + 2}{6}, & \text{for } 1 \leq x < 2 \\
1 & \text{for } x \geq 2
\end{cases}$$

and

$$G(x) = \begin{cases} 
\frac{x^2 + x}{4}, & \text{for } 0 \leq x < 1 \\
\frac{x}{2}, & \text{for } 1 \leq x < 2 \\
1 & \text{for } x \geq 2.
\end{cases}$$

The cumulative past inaccuracy measure is given by

$$\xi^*(F, G; t) = \begin{cases} 
\frac{2t}{9} - \frac{(t-2)}{6t} - \frac{1}{3t^2} \log(t + 1), & \text{for } 0 < t < 1 \\
\frac{t}{9} + \frac{16t}{9(t^2+2)} - \frac{17}{18(t^2+2)} - \frac{18 \log 2 + 24 \log t}{18(t^2+2)}, & \text{for } 1 \leq t < 2 \\
\log 2 + \frac{1}{6} \log 5 - \frac{41}{54} - \frac{8}{3} \tan^{-1}(\frac{1}{2}) & \text{for } t \geq 2.
\end{cases}$$

Next, we study the characterization problem in case of the dynamic measure (6.32) under the proportional reversed hazard rate model (6.31). This is analogous to Theorem 6.2 proved in case of dynamic cumulative residual inaccuracy.
6.6 Characterization Results of Dynamic Cumulative Past Inaccuracy

We consider the characterization problem for the dynamic cumulative past inaccuracy measure under the proportional reversed hazard model (6.31). We state the following theorem.

**Theorem 6.5** Let $X$ and $Y$ be two non-negative random variables with distribution functions $F(.)$ and $G(.)$ satisfying the proportional reversed hazard rate model (6.31), and let, $\xi^*(F, G; t) < \infty, \forall t \geq 0$ be an decreasing function of $t$, then $\xi^*(F, G; t)$ determines the distribution function $F(.)$ uniquely.

The proof is similar to that of Theorem 6.2, hence omitted.

Next, we characterize a specific distribution by using the dynamic cumulative past inaccuracy measure (6.32). The result is stated as follows.

**Theorem 6.6** Let $F(.)$ and $G(.)$ be two distribution functions satisfying the proportional reversed hazard model (6.31). The dynamic cumulative past inaccuracy measure

$$\xi^*(F, G; t) = c\delta_F^*(t), \quad 0 < c < \beta,$$

(6.33)

if, and only if $F(x) = \left(\frac{x}{b}\right)^{\frac{c}{c-\beta}}, \quad b > 0$.

**Proof** Rewriting (6.32) as

$$\xi^*(F, G; t) = -\frac{1}{F(t)} \int_0^t F(x) \log G(x)dx + \delta_F^*(t) \log G(t).$$

(6.34)

Substituting (6.31), this gives
\[ \xi^*(F, G; t) = -\frac{\beta}{F(t)} \int_0^t F(x) \log F(x) dx + \beta \delta^*_F(t) \log F(t). \quad (6.35) \]

Differentiating this w.r.t. \( t \) both sides, we obtain

\[
\xi'^*(F, G; t) = \beta \log F(t) \{ \delta'^*_F(t) - 1 \} \\
+ \beta \mu_F(t) \int_0^t \frac{F(x)}{F(t)} \log F(x) dx + \beta \mu_F(t) \delta^*_F(t). \quad (6.36)
\]

Substituting (6.27) and (6.35) in Eq. (6.36), we obtain

\[
\xi'^*(F, G; t) = \mu_F(t) \{ \beta \delta^*_F(t) - \xi^*(F, G; t) \}. \quad (6.37)
\]

Let (6.33) be valid. Differentiating both sides w.r.t. \( t \), we get

\[
\xi'^*(F, G; t) = c \delta'^*_F(t). \quad (6.38)
\]

Substituting this in (6.37), we get

\[
c \delta'^*_F(t) = (\beta - c) \mu_F(t) \delta^*_F(t). \quad (6.39)
\]

Using (6.27) and simplifying, we obtain

\[
\delta'^*_F(t) = \left( \frac{\beta - c}{\beta} \right) = 1 - \frac{c}{\beta}. \quad (6.40)
\]

This gives

\[
\delta^*_F(t) = \left( \frac{\beta - c}{\beta} \right) t. \quad (6.41)
\]

Dividing (6.40) by (6.41), we obtain

\[
\frac{1 - \delta'^*_F(t)}{\delta^*_F(t)} = \mu_F(t) = \left( \frac{c}{\beta - c} \right) \frac{1}{t}. \quad (6.42)
\]

Using the relationship between reversed hazard rate and distribution function is given by

\[
F(x) = \exp \left[ \int_0^x \mu_F(t) dt \right],
\]

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we obtain
\[ F(x) = \left( \frac{x}{b} \right)^{\frac{c}{b-c}}, \quad b > 0. \] (6.43)

Conversely, when the distribution of \( X \) is specified by (6.43), using (6.35), we get

\[
\xi^*(F, G; t) = -\frac{\beta}{F(t)} \int_0^t F(x) \log F(x) dx + \beta \frac{F_t(t)}{k+1} \log \left( \frac{x}{b} \right).
\]

After simplification, we obtain

\[
\xi^*(F, G; t) = \frac{\beta kt}{(k+1)^2} = \frac{\beta k}{k+1} \delta^*_F(t),
\]

where \( \delta^*_F(t) = \int_0^t F(x) dx = \frac{t}{k+1} \). This prove the result.

**Example 6.6** Let \( X \) and \( Y \) be two non-negative random variables satisfying the proportional reversed hazard model (PRHM) and let

\[
f_X(x) = \begin{cases} \alpha x^{a-1} & \text{if } 0 \leq x < 1, \quad a > 0 \\ 0 & \text{otherwise} \end{cases}
\]

The distribution function \( F(x) = x^a \), and \( G(x) = [F(x)]^\beta, \quad \beta > 0 \).

Substituting these values in (6.32), after simplification we get

\[
\xi^*(F, G; t) = \frac{t}{(a+1)^2} = c \delta^*_F(t),
\]

where \( c = \frac{1}{a+1} \) and mean past lifetime is \( \delta^*_F(t) = \frac{t}{a+1} \).

Next, we extend the result (6.33) to a more general case taking \( c \) as a function of \( t \).

We state the following result:
**Theorem 6.7** If $X$ and $Y$ satisfy the PRHM (6.31), and
\[ \xi^*(F, G; t) = c(t)\delta^*_F(t), \text{ for } t \geq 0, \tag{6.44} \]
then
\[ \delta^*_F(t) = \left( \int_0^t \left\{ \beta - c(x) \right\} e^{\frac{c(x)}{\beta}} dx \right) e^{-\frac{c(t)}{\beta}}. \tag{6.45} \]

The proof is similar to that of Theorem 6.4, hence omitted.

**Example 6.7** Let $c(t) = at + b$, $t > 0$ and $a > 0$. From (6.45), we obtain the general model with mean inactivity time function
\[ \delta^*_F(t) = \frac{2\beta - at - b}{a} + \frac{(b - 2\beta)e^{-at}}{a}. \tag{6.46} \]

For $\beta = 1$, (6.46) reduces to
\[ \delta^*_F(t) = \frac{2 - at - b}{a} - \frac{(b - 2)e^{-at}}{a}, \]
a result in context with the cumulative entropy, refer to Di Crescenzo and Longobardi [32].

### 6.7 Conclusion

The distribution functions based measure of cumulative residual inaccuracy and cumulative past inaccuracy have been considered as natural extension of the distribution functions based dynamic entropy measures. The proposed cumulative inaccuracy measures determine the underlying distribution uniquely under PHM (for residual) and PRHM (for past) models; and also characterize certain specific probability distributions using relation between different reliability measure. It is expected that dynamic cumulative inaccuracy measures introduced in this chapter will further, extend the scope of study of information theoretic measures.