Chapter 1

CONCEPT OF FUZZY SET THEORY FOCUSED ON DECISION MAKING PROBLEMS (A BRIEF REVIEW)

1.1. Introduction

In this chapter, we briefly introduce the concepts of fuzzy sets, fuzzy numbers, interval arithmetic based on fuzzy $\alpha$-cuts, fuzzy function, extension principle, linguistic variable, fuzzy relations and fuzzy if–then rules, which are directly or indirectly involved in dealing with the decision-making problems in the thesis. Most of these concepts come from Bezdek [5], Kandel [58], Kaufmann and Gupta [59], Klir and Yuan [63], Novak [79], Ross [93], Yen and Langari [99] and Zimmermann [110]. The notations used throughout the thesis for a fuzzy set is a set symbol with a tilde over score say, $\tilde{A}$, where the functional mapping given by $\mu_{\tilde{A}}(x)\in[0,1]$ denotes the degree of membership of an element in the fuzzy set $\tilde{A}$. Therefore, $\mu_{\tilde{A}}(x)=$Degree$(x \in \tilde{A})$. Fuzziness of an element describes the ambiguity of the element. The element will belong to or not, but is the description of the element unambiguous enough to quantify its belongingness or non-belongingness. The domain of the membership function which is the domain of concern from which elements of the set are drawn is called the universe of discourse.

In Sec.1.2, we introduce the basic concepts of fuzzy sets. Some important fuzzy set operations are highlighted in Sec. 1.3 and in Sec. 1.4, the concepts of fuzzy numbers are introduced. In Sec.1.5, we discuss fuzzy arithmetic including interval arithmetic and fuzzy arithmetic based on $\alpha$-cuts. The ordering of fuzzy numbers is discussed in 1.6. In Sec. 1.7, fuzzy function, extension principle and relation between $\alpha$-cuts and interval arithmetic are investigated. A brief review of linguistic variables and linguistic hedges, fuzzy relations and fuzzy rules are discussed in Sec. 1.8, Sec. 1.9 and Sec. 1.10 respectively.
1.2. Fuzzy sets

If \( X \) is some universe of discourse, then a fuzzy subset \( A \) of \( X \) is defined by its membership function written as \( \mu_A(x) \) which produce values in \([0,1]\) for all \( x \in X \). So, \( \mu_A(x) \) is a function mapping from \( X \) into \([0,1]\). When \( \mu_A(x) \) is always equal to 1 or 0, we obtain a crisp (non-fuzzy) subset.

A notational convention for fuzzy sets when the universe of discourse \( X \) is discrete and finite, is as follows: For a fuzzy set \( A \),

\[
\tilde{A} = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots \right\} = \left\{ \sum_i \frac{\mu_A(x_i)}{x_i} \right\}
\]

When the universe of discourse \( X \) is continuous and infinite, the fuzzy set is denoted by

\[
\tilde{A} = \left\{ \int \frac{\mu_A(x)}{x} \right\}
\]

In the above notations, the horizontal bar is not a quotient but rather delimiter. In (1.2.1), the summation symbol is not for algebraic summation but rather denotes the aggregation or collection of each element and hence the “+” signs in the first notation are a function theoretic union. In the second notation, the integral sign is not an algebraic integral but a continuous function theoretic union notation for continuous variables.

The important concepts of fuzzy sets are the concepts of \( \alpha \)-cut, support, Height, normal and subnormal.

**Definition 1.2.1.** Given a fuzzy set \( \tilde{A} \) defined on \( X \) (the universe of discourse) and any number \( \alpha \in [0,1] \), the \( \alpha \)-cut \( \tilde{A}_\alpha \) and strong \( \alpha \)-cut \( \tilde{A}_\alpha^+ \) are the crisp sets: \( \tilde{A}_\alpha = \{ x : \mu_A(x) \geq \alpha \} \) and \( \tilde{A}_\alpha^+ = \{ x : \mu_A(x) > \alpha \} \) respectively.

The 1-cut \( \tilde{A}_1 \) is often called the core of \( \tilde{A} \).
**Definition 1.2.2.** The support of a fuzzy set $\tilde{A}$ i.e. $\text{supp} (\tilde{A})$ within the universal set $X$ is the crisp set that contains all the elements of $X$ that have non-zero membership grades in $\tilde{A}$.

**Definition 1.2.3.** The height of a fuzzy set $\tilde{A}$ denoted by $h(\tilde{A})$ is the largest membership grade obtained by any $x \in X$. Formally,

$$h(\tilde{A}) = \text{Sup } \mu_A(x).$$

A fuzzy set $\tilde{A}$ is said to be normal when $h(\tilde{A})=1$; it is called subnormal if $h(\tilde{A})<1$.

**Definition 1.2.4.** A fuzzy set $\tilde{A}$ on $\mathbb{R}$ (set of real numbers) is convex iff

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq (\mu_A(x_1) \land \mu_A(x_2)), x_1, x_2 \in \mathbb{R}, \forall \lambda \in [0,1]$$

where $\land$ denotes the minimum operator.

1.3. **Fuzzy set Operations**

Define three fuzzy sets $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ on the universe of discourse $X$. For a given $x \in X$, the following function theoretic operations for the set theoretic operations of union, intersection and complement can be generalized to fuzzy sets more than one way. However, one particular generalization that are referred to as standard fuzzy set operations in the literature are defined for $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ on $X$:

For all $x \in X$,

- **Union**
  $$\mu_{\tilde{A} \cup \tilde{B}}(x) = \mu_{\tilde{A}}(x) \lor \mu_{\tilde{B}}(x) \tag{1.3.1}$$

- **Intersection**
  $$\mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \land \mu_{\tilde{B}}(x) \tag{1.3.2}$$

- **Complement**
  $$\mu_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) \tag{1.3.3}$$

where $\lor$, $\land$ denote the maximum, minimum operators respectively and $\overline{\tilde{A}}$ denotes the complement of $\tilde{A}$. It is noted that the null set and the whole universe of discourse $X$ are not fuzzy sets:
\( \tilde{A} \subseteq X \Rightarrow \mu_\tilde{A}(x) \leq \mu_x(x) \) \hspace{1cm} (1.3.4)

For all \( x \in X, \mu_\emptyset(x) = 0 \) \hspace{1cm} (1.3.5)

For all \( x \in X, \mu_x(x) = 1 \) \hspace{1cm} (1.3.6)

De Morgan’s laws for classical sets also hold for fuzzy sets:

\[
\overline{\tilde{A} \cap \tilde{B}} = \overline{\tilde{A}} \cup \overline{\tilde{B}} \tag{1.3.7}
\]

\[
\overline{\tilde{A} \cup \tilde{B}} = \overline{\tilde{A}} \cap \overline{\tilde{B}} \tag{1.3.8}
\]

The excluded middle laws extended for fuzzy sets are defined by

\[
\tilde{A} \cup \overline{\tilde{A}} \neq X \tag{1.3.9}
\]

\[
\overline{\tilde{A}} \cap \tilde{A} \neq \Phi \tag{1.3.10}
\]

Other properties of fuzzy sets are defined as follows:

Commutative \( \tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A} \) and \( \tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A} \) \hspace{1cm} (1.3.11)

Associativity \( \tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C} \)
\( \tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C} \) \hspace{1cm} (1.3.12)

Distributive \( \tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C}) \)
\( \tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C}) \) \hspace{1cm} (1.3.13)

Idempotency \( \tilde{A} \cup \tilde{A} = \tilde{A} \) and \( \tilde{A} \cap \tilde{A} = \tilde{A} \) \hspace{1cm} (1.3.14)

Identity \( \tilde{A} \cup \Phi = \tilde{A} \) and \( \tilde{A} \cap X = \tilde{A} \)
\( \tilde{A} \cap \Phi = \Phi \) and \( \tilde{A} \cup X = X \) \hspace{1cm} (1.3.15)

Transitivity \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \) then \( \tilde{A} \subseteq \tilde{C} \) \hspace{1cm} (1.3.16)

Involution \( \overline{\overline{\tilde{A}}} = \tilde{A} \) \hspace{1cm} (1.3.17)

For any fuzzy set \( \tilde{A} \) defined on a finite universal set \( X \), the sigma count or scalar cardinality \( |\tilde{A}| \) is defined by the formula

\[
|\tilde{A}| = \sum_{x \in X} \mu_\tilde{A}(x) \tag{1.3.18}
\]
For any fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) defined on a finite universal set \( X \), the concept of Hamming distance \( d(\tilde{A}, \tilde{B}) \) between \( \tilde{A} \) and \( \tilde{B} \) is defined as

\[
d(\tilde{A}, \tilde{B}) = \sum_{x \in X} |\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)|
\]

(1.3.19)

Thus, from equations (1.3.18) and (1.3.19), we have

\[
|\tilde{A}| = d(\tilde{A}, \Phi)
\]

(1.3.20)

where \( \Phi \) denotes the empty set.

We now introduce fuzzy intersection, fuzzy union and fuzzy complement as binary operations from \([0,1] \times [0,1]\) to \([0,1]\). For more detail, one can refer to (Fang et al. [31] and Klir and Yuan [63]). In the literature, fuzzy intersections, unions are also known as \( t \)-norms and \( t \)-conorms respectively.

**Definition 1.3.1.** A fuzzy intersection (\( t \)-norm), \( i \) is a binary operation on the unit interval \([0,1]\) that satisfies at least the following axioms for all \( a, b, d \in [0,1] \):

- **Axiom i1** \( i(a,1) = a \) boundary condition
- **Axiom i2** \( b \leq d \Rightarrow i(a,b) \leq i(a,d) \) monotonicity
- **Axiom i3** \( i(a,b) = i(b,a) \) commutativity
- **Axiom i4** \( i(a,i(b,d)) = i(i(a,b),d) \) associativity

The important requirements of fuzzy intersection (\( t \)-norm) are expressed by the following axioms (for all \( a, a_1, a_2, b_1, b_2 \in [0,1] \)):

- **Axiom i5** \( i \) is a continuous function. Continuity
- **Axiom i6** \( i(a,a) \leq a \) Sub- idempotency
- **Axiom i7** \( a_1 < a_2 \) and \( b_1 < b_2 \Rightarrow i(a_1,b_1) < i(a_2,b_2) \) Strict monotonicity

Some of \( t \)-norms that are frequently used as fuzzy intersections are given below (for all \( a, b \in [0,1] \)):

- **Standard intersection** \( i(a,b) = \min(a,b) \) (1.3.21)
Algebraic product \( i(a,b) = ab \) \hfill (1.3.22)

Bounded difference \( (a,b) = \max(0,\ ab) \) \hfill (1.3.23)

Drastic intersection \( i(a,b) = \begin{cases} a & \text{when } b = 1 \\ b & \text{when } a = 1 \\ 0 & \text{otherwise} \end{cases} \) \hfill (1.3.24)

**Definition 1.3.2.** The Yager class of \( t \)-nормs \( i_w \) is defined by

\[
i_w(a,b) = 1 - \min(1, [(1-a)^w + (1-b)^w]^{1/w}) \quad (w > 0)
\]

**Theorem 1.3.1. (Klir and Yuan [63]).** Let \( i_w \) denote the Yager class of \( t \)-nормs. Then, \( i_{\text{min}}(a,b) \leq i_w(a,b) \leq \min(a,b) \) for all \( a,b \in [0,1] \).

Proof: Lower bound:

From definition 1.3.2, it is trivial that \( i_w(1,b) = b \) and \( i_w(a,1) = a \) which are independent of \( w \).

Now \( \lim_{w \to 0} [1 - a]^{1/w} = \infty \)

Hence, \( \lim_{w \to 0} i_w(a,b) = 0 \) for all \( a,b \in [0,1] \).

Upper bound:

Now we shall prove that \( \lim_{w \to \infty} \min(1, [(1-a)^w + (1-b)^w]^{1/w}) = \max[1-a,1-b] \). The result is obvious whenever \( 1-a = 0 \), \( 1-b = 0 \) or \( 1-a = 1-b \) because

\[
\lim_{w \to \infty} [(1-a)^w + (1-b)^w]^{1/w} = 1-a \quad (\because \lim_{w \to \infty} 2^{1/w} = 1)
\]

If \( 1-a \neq 1-b \), let us assume, without loss of generality, that \( 1-a < 1-b \).

Let \( Q = [(1-a)^w + (1-b)^w]^{1/w} \)

\[
\therefore \lim_{w \to \infty} Q = \lim_{w \to \infty} \frac{\ln[(1-a)^w + (1-b)^w]}{w}
\]

Using L’Hôpital’s Rule, we obtain
\[
\lim_{w \to \infty} \ln Q = \lim_{w \to \infty} \frac{(1-a)^w \ln(1-a) + (1-b)^w \ln(1-b)}{(1-a)^w + (1-b)^w} \\
= \lim_{w \to \infty} \frac{\left(\frac{1-a}{1-b}\right)^w \ln(1-a) + \ln(1-b)}{\left(\frac{1-a}{1-b}\right)^w + 1} \\
= \ln(1-b) \\
\therefore \lim_{w \to \infty} \ln Q = \lim_{w \to \infty} [(1-a)^w + (1-b)^w]^{\frac{1}{w}} \\
= (1-b) \\
= \max[1-a, 1-b]
\]

Therefore, \(i_e(a, b) = 1 - \max[1-a, 1-b] = \min(a, b)\). This concludes the proof.

Like fuzzy intersection (\(t\)-norm), the fuzzy union (\(t\)-conorm), \(u\) can be defined as a binary operation on the unit interval \([0, 1]\):

\[
u : [0, 1] \times [0, 1] \to [0, 1]
\]

**Definition 1.3.3.** A fuzzy union (\(t\)-conorm), \(u\) is a binary operation on the unit interval that satisfies the following axioms for all \(a, b, d, g \in [0, 1]\):

- **Axiom \(u_1\)** \(u(a, 1) = 1\) boundary condition
- **Axiom \(u_2\)** \(b \leq d \Rightarrow u(a, b) \leq u(a, d)\) monotonicity
- **Axiom \(u_3\)** \(u(a, b) = u(b, a)\) commutativity
- **Axiom \(u_4\)** \(u(a, u(b, d)) = u(u(a, b), d)\) associativity
- **Axiom \(u_5\)** \(u\) is a continuous function.
- **Axiom \(u_6\)** \(u(a, a) > a\) supper idempotency
- **Axiom \(u_7\)** \(a < b, d < g \Rightarrow u(a, d) < u(b, g)\) strict monotonicity
The examples of some \( t \)-conorms that are frequently used as fuzzy union are given as follows (for all \( a, b \in [0,1] \)):

- **Standard union** \( u(a, b) = \max(a, b) \) \hspace{1cm} (1.3.26)
- **Algebraic sum** \( u(a, b) = a + b - ab \) \hspace{1cm} (1.3.27)
- **Bounded sum** \( u(a, b) = \min(1, a + b) \) \hspace{1cm} (1.3.28)
- **Drastic union** \( u(a, b) = \begin{cases} 
  a & \text{for } b = 0 \\
  b & \text{for } a = 0 \\
  1 & \text{otherwise}
\end{cases} \) \hspace{1cm} (1.3.29)

The Yager class of \( t \)-co-norms is defined as

\[
u_w(a, b) = \min(1, [a^w + b^w]^{1/w}) \quad (w > 0)
\] \hspace{1cm} (1.3.30)

This class covers the whole class of \( t \)-conorms.

**Definition 1.3.4.** Let \( \tilde{A} \) be a fuzzy set on \( \mathcal{X} \). Then, a fuzzy complement \( \tilde{cA} \) is defined by a function

\[
c : [0,1] \rightarrow [0,1]
\] \hspace{1cm} (1.3.31)

which assigns a value \( c(\mu_{\tilde{A}}(x)) \) to each membership grade \( \mu_{\tilde{A}}(x) \) for any given \( x \in \mathcal{X} \).

The value \( c(\mu_{\tilde{A}}(x)) \) is interpreted as the value of \( c(\mu_{\hat{A}}(x)) \), i.e.,

\[
c(\mu_{\tilde{A}}(x)) = c(\mu_{\hat{A}}(x)) = \mu_{\hat{A}}(x)
\] \hspace{1cm} (1.3.32)

The function \( c \) defined by (1.3.31) must satisfy the following axiomatic requirement (for all \( a, b \in [0,1] \)):

- **Axiom \( c_1 \)** \( c(0) = 1 \text{ and } c(1) = 0 \) boundary condition \hspace{1cm} (1.3.33)
- **Axiom \( c_2 \)** \( a \leq b \Rightarrow c(a) \geq c(b) \) monotonicity \hspace{1cm} (1.3.34)
- **Axiom \( c_3 \)** \( c \) is a continuous function
- **Axiom \( c_4 \)** for each \( a \in [0,1] \), \( c(c(a)) = a \) involutive \hspace{1cm} (1.3.35)
Definition 1.3.5. Let $\tilde{A}$ be a fuzzy set on $X$. Then, the equilibrium of a complement $c$ is that degree of membership in the fuzzy set $\tilde{A}$ which equals the degree of membership in the complement $c\tilde{A}$.

Theorem 1.3.2. Every fuzzy complement has at most one equilibrium.

Proof: Let $\tilde{A}$ be a fuzzy set on universal set $X$. For any $x \in X$, suppose $\mu_{\tilde{A}}(x) = a \in [0,1]$. Let $c$ be an arbitrary fuzzy complement. Now, an equilibrium of $c$ is a solution of the equation $c(a) - a = 0$ where $a \in [0,1]$. We are to demonstrate that any equation $c(a) - a = b$ where $b$ is a real constant, must have at most one solution.

Assume that $a_1$ and $a_2$ be two different solutions of the equation $c(a) - a = b$ such that $a_1 < a_2$. Since $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$, we get

$$c(a_1) - a_1 = c(a_2) - a_2$$

(1.3.36)

By axiom $c_2$, $c$ is monotonic non-increasing and since $a_1 < a_2$, we have $c(a_1) \geq c(a_2)$. Therefore, $c(a_1) - a_1 \geq c(a_2) - a_2$. This inequality contradicts (1.3.36) thus showing that the equation must have at most one solution.

1.4. Fuzzy numbers

A general definition of fuzzy number may be found in Ganguly et al. [41], Zadeh [100], Zimmermann [110] etc. However, the fuzzy numbers in this thesis will be almost always triangular (shaped) or trapezoidal (shaped) fuzzy numbers.

Definition 1.4.1. A triangular fuzzy number $\tilde{N}$ on the real line $\mathbb{R}$ is defined by three numbers $a < b < c$ where the base of the triangle is the interval $[a,c]$ and its vertex is at $x = b$. Triangular fuzzy numbers will be denoted as $\tilde{N} = (a,b,c)$. It is shown in Fig. 1.4.1
Definition 1.4.2. A trapezoidal fuzzy number $\tilde{M}$ on the real line is defined by four numbers $a < b < c < d$ where the base of the trapezoid is the interval $[a, d]$ and its top (where the membership equals one) is over $[b, c]$ (Fig. 1.4.2). We write $\tilde{M} = (a, b, c, d)$.

To be a triangular shaped fuzzy number, the graph on $[a, b]$ or on $[b, c]$ are not straight line segments. Thus, for a triangular shaped fuzzy number, we require the graph to be continuous such that

1. monotonically increasing on $[a, b]$
2. monotonically decreasing on $[b, c]$
For triangular shaped fuzzy numbers \( \tilde{P} \) with base on \([a,c]\), we use the notation \( \tilde{P} \approx (a,b,c) \) (as shown in Fig 1.4.3). Similarly, we define trapezoidal shaped fuzzy number \( \tilde{Q} \approx (a,b,c,d) \) whose base is \([a,d]\) and top is over the interval \([b,c]\) (as shown in Fig.1.4.4).

![Fig. 1.4.3 Triangular shaped fuzzy number](image)

For any fuzzy number \( \tilde{N} \) and for \( 0 \leq \alpha \leq 1 \), the \( \alpha \)-cut \( \tilde{N}_\alpha \) is a closed and bounded interval. We will write this as

\[
\tilde{N}_\alpha = [n_1(\alpha), n_2(\alpha)]
\]  

(1.4.1.1)
where \( n_1(\alpha)(n_2(\alpha)) \) is an increasing (decreasing) function of \( \alpha \) with \( n_1(\alpha) \leq n_2(\alpha) \).

If \( \tilde{N} \) is a triangular shaped or a trapezoidal shaped fuzzy number, then, we have

1. \( n_1(\alpha) \) is a continuous, monotonically increasing function of \( \alpha \) in \([0,1]\)
2. \( n_2(\alpha) \) is a continuous, monotonically decreasing function of \( \alpha \), \( 0 \leq \alpha \leq 1 \)
3. \( n_1(l) = n_2(l) \) \((n_1(l) < n_2(l)\) for trapezoidals).

We can check the monotone increasing (decreasing) by showing that
\[
\frac{d}{d\alpha} n_1(\alpha) > 0 \left( \frac{d}{d\alpha} n_2(\alpha) < 0 \right)
\]
holds.

**Definition 1.4.3.** Let \( \tilde{N} = (a, b, c) \) be a triangular fuzzy number. Then, for some real number \( \delta \), \( \tilde{N} \geq \delta \) if \( a \geq \delta \); \( \tilde{N} > \delta \) when \( a > \delta \); \( \tilde{N} \leq \delta \) when \( c \leq \delta \) and \( \tilde{N} < \delta \) when \( c < \delta \). The same results hold for triangular shaped fuzzy number whose support is the interval \([a,c]\), The result can be extended for trapezoidal (shaped) fuzzy numbers.

If \( \tilde{A} \) and \( \tilde{B} \) be two fuzzy numbers of the universal set \( X \), then \( \tilde{A} \leq \tilde{B} \) means \( \mu_\tilde{A}(x) \leq \mu_\tilde{B}(x) \) for all \( x \in X \) or \( \tilde{A} \) is fuzzy subset of \( \tilde{B} \). Also, \( \tilde{A} < \tilde{B} \) holds \( \mu_\tilde{A}(x) < \mu_\tilde{B}(x) \) for all \( x \in X \).

### 1.5. Fuzzy Arithmetic

In order to add, subtract, multiply and divide two fuzzy numbers \( \tilde{A} \) and \( \tilde{B} \), there are different methods. But we will consider the method based on \( \alpha \)-cuts and interval arithmetics. So, we now present the basics of interval arithmetic.

**1.5.1. Interval Arithmetic**

Here, we only give a brief introduction of interval arithmetic. For more detail reference can be made to Moore [75], Neumaier [78]. Let \([a_1,b_1]\) and \([a_2,b_2]\)
be two closed, bounded intervals of real numbers. If \( \Theta \) denotes addition, subtraction, multiplication or division, then

\[
[a_1, b_1] \Theta [a_2, b_2] = [\alpha, \beta]
\]

where \( [\alpha, \beta] = \{ a \Theta b | a_1 \leq a \leq b_1, a_2 \leq b \leq b_2 \} \)

(1.5.1.2)

If \( \Theta \) is division, it is assumed that zero does not belong to \( [a_2, b_2] \).

The equation (1.5.1.2) can be simplified as follows:

\[
[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]
\]

(1.5.1.3)

\[
[a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2]
\]

(1.5.1.4)

\[
[a_1, b_1] \cdot [a_2, b_2] = [\alpha, \beta]
\]

(1.5.1.5)

where

\[
\alpha = \min\{a_1, a_2, b_1, b_2\}
\]

(1.5.1.6)

\[
\beta = \max\{a_1, a_2, b_1, b_2\}
\]

(1.5.1.7)

\[
[a_1, b_1] / [a_2, b_2] = [a_1, b_1] \cdot \left[\frac{1}{b_2}, \frac{1}{a_2}\right]
\]

(1.5.1.8)

Further if \( a_1 \geq 0 \) and \( a_2 \geq 0 \), then

\[
[a_1, b_1] \cdot [a_2, b_2] = [a_1 a_2, b_1 b_2]
\]

(1.5.1.9)

If \( b_1 < 0 \) but \( a_2 \geq 0 \), then

\[
[a_1, b_1] \cdot [a_2, b_2] = [a_1 b_2, a_1 b_1]
\]

(1.5.1.10)

Also if \( b_1 < 0 \) but \( b_2 < 0 \), then

\[
[a_1, b_1] \cdot [a_2, b_2] = [b_1 a_2, b_2 a_2]
\]

(1.5.1.11)

And if \( a_1 \geq 0 \) and \( b_2 < 0 \), then

\[
[a_1, b_1] \cdot [a_2, b_2] = [a_2 b_1, b_2 a_1]
\]

(1.5.1.12)

Here, we will not discuss in detail the theories and properties of interval numbers due to the limited space in the thesis.

1.5.2. Fuzzy Arithmetic Based on \( \alpha \)-Cuts

Suppose we have two fuzzy numbers \( \tilde{A} \) and \( \tilde{B} \). Let, for \( \alpha \in [0,1] \),

\[
\tilde{A}_\alpha = [a_1(\alpha), a_2(\alpha)] \quad \text{and} \quad \tilde{B}_\alpha = [b_1(\alpha), b_2(\alpha)]
\]

be the \( \alpha \)-cuts of \( \tilde{A} \) and \( \tilde{B} \) respectively.
Then, if $\tilde{C} = \tilde{A} + \tilde{B}$ we have

$$\tilde{C}_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha$$

(1.5.2.1)

Setting $\tilde{C} = \tilde{A} - \tilde{B}$, we get

$$\tilde{C}_\alpha = \tilde{A}_\alpha - \tilde{B}_\alpha \text{ for all } \alpha \in [0,1]$$

(1.5.2.2)

For $\tilde{C} = \tilde{A} \tilde{B}$, we have $\tilde{C}_\alpha = \tilde{A}_\alpha \tilde{B}_\alpha$ and $\tilde{C}_\alpha = \tilde{A}_\alpha / \tilde{B}_\alpha$

(1.5.2.3)

where $\tilde{C} = \tilde{A} / \tilde{B}$ provided that $0 \notin \tilde{B}_\alpha$ for all $\alpha \in [0,1]$.

The procedure of $\alpha$-cut plus interval arithmetic is more user and computer friendly and is equivalent to the extension principle method of fuzzy arithmetic (Fang et al. [31], Klir and Yuan [63]).

1.6. Ordering of Fuzzy numbers

Let $F(\tilde{T})$ be a finite set of triangular (trapezoidal) fuzzy numbers $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n$. We want to order them from smallest to largest. For a finite set of real numbers, there is no problem in ordering them from the smallest to the largest. In the case of fuzzy numbers there is no universally accepted way to do this. However, there are probably more than 40 methods (Buckley [17]) proposed in the literature of defining $\tilde{A} \leq \tilde{B}$, for any two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ where the symbol $\leq$ means “less than or equal”. References on this topic can be given to the works of Buckley [17] where we can look up many of these methods and see their comparisons. Each method appears to have some advantages as well as disadvantages. The issue of choosing a proper ordering method in a given context is still a subject of active research.

Here we will present four simple methods for ordering fuzzy numbers. The first procedure is based upon defining Hamming distance on $F(\tilde{T})$. For any $\tilde{T}_i$ and $\tilde{T}_j$ belonging to $F(\tilde{T})$, the Hamming distance $d(\tilde{T}_i, \tilde{T}_j)$ is defined by the formula:

$$d(\tilde{T}_i, \tilde{T}_j) = \int_{\mathbb{R}} |\mu_{\tilde{T}_i}(x) - \mu_{\tilde{T}_j}(x)| \, dx$$

(1.6.1)
We first determine the least upper bound $\max(\tilde{T}_i, \tilde{T}_j)$ for $\tilde{T}_i$ and $\tilde{T}_j$. Then, we calculate the Hamming distance $d(\max(\tilde{T}_i, \tilde{T}_j), \tilde{T}_i)$ and $d(\max(\tilde{T}_i, \tilde{T}_j), \tilde{T}_j)$ and define

\[
\tilde{T}_i \leq \tilde{T}_j \text{ if } d(\max(\tilde{T}_i, \tilde{T}_j), \tilde{T}_i) \geq d(\max(\tilde{T}_i, \tilde{T}_j), \tilde{T}_j)
\]

(1.6.2)

If $\max(\tilde{T}_i, \tilde{T}_j) = \tilde{T}_i$, then, $\tilde{T}_i \leq \tilde{T}_j$.

It is also observed that we can define a similar ordering of $\tilde{T}_i$ and $\tilde{T}_j$ via the greatest lower bound $\min(\tilde{T}_i, \tilde{T}_j)$.

The second method is based on $\alpha$-cuts where a number of variations of this method have been suggested in the literature. A simple variation of this method proceeds as follows:

For given fuzzy numbers $\tilde{T}_i$ and $\tilde{T}_j$ to be compared, we select a particular value of $\alpha \in [0,1]$ and define the $\alpha$-cuts $[\tilde{T}_i]_{\alpha} = [a_i, a_2]$ and $[\tilde{T}_j]_{\alpha} = [b_1, b_2]$. Then, we define

\[
\tilde{T}_i \leq \tilde{T}_j \text{ if } a_2 \leq b_2
\]

(1.6.3)

It is usually required that $\alpha \geq 0.5$. There also exists a method based on $\alpha$-cuts where $\alpha$-cuts were aggregated to express the dominance of one fuzzy number over the other one for all $\alpha$-cuts.

The third method is based on the extension principle. In this method, we are to construct a fuzzy set on $\{\tilde{T}_i, \tilde{T}_2, \ldots, \tilde{T}_n\}$, called a “priority set”, $P(\tilde{T}_i)$, $i = 1, 2, \ldots, n$ where $P(\tilde{T}_i)$ denotes the degree to which $\tilde{T}_i$ is ranked as the greatest fuzzy number.

Using extension principle, $P$ is defined for each $i \in N_n$ (set of first $n$ natural numbers) by the formula:

\[
P(\tilde{T}_i) = \sup_{k \in N_n} \min_{\mu_{i_k}(r_k)}
\]

(1.6.4)

where the supremum is taken over all vectors $<r_1, r_2, \ldots, r_n> \in \mathbb{R}^n$ such that $r_i \geq r_j$ for all $j \in N_n$. Practically, this method proceeds as follows:
We define ‘<’ between two triangular (trapezoidal) fuzzy numbers $\tilde{T}_i$ and $\tilde{T}_j$ as follows:

Define
\[
P(\tilde{T}_i \leq \tilde{T}_j) = \max \{\min(\mu_{i_1}(x), \mu_{j_1}(y)), x \leq y\}
\]
which measures how much $\tilde{T}_i$ is less than or equal to $\tilde{T}_j$.

We write $\tilde{T}_j < \tilde{T}_i$ if
\[
P(\tilde{T}_j \leq \tilde{T}_i) = 1 \quad \text{and} \quad P(\tilde{T}_i \leq \tilde{T}_j) < \alpha_i
\]
where $\alpha_i$ is some fixed fraction in $[0,1]$; but usually used as $0.5 < \alpha_i < 0.8$.

$\therefore \tilde{T}_j < \tilde{T}_i$ if $P(\tilde{T}_j \leq \tilde{T}_i) = 1$ and $P(\tilde{T}_i \leq \tilde{T}_j) > 0.5$.

We define $\tilde{T}_j \approx \tilde{T}_i$ when both $\tilde{T}_j < \tilde{T}_i$ and $\tilde{T}_i < \tilde{T}_j$ are false. $\tilde{T}_i \leq \tilde{T}_j$ means $\tilde{T}_i < \tilde{T}_j$ or $\tilde{T}_j \approx \tilde{T}_i$. Note that this ‘≈’ may not be transitive in general.

This method of ordering is also useful in partitioning the set of fuzzy numbers $F(\tilde{T})$ up to disjoint sets $H_1, H_2, \ldots, H_K$ where

1. Given any $\tilde{T}_i, \tilde{T}_j$ in $H_k, 1 \leq k \leq K$, then $\tilde{T}_j \approx \tilde{T}_i$

2. Given $\tilde{T}_i \in H_i$ with $i < j$, there is also another fuzzy number $\tilde{T}_j \in H_j$ with $\tilde{T}_i < \tilde{T}_j$. We say that a fuzzy number $\tilde{T}_i$ is dominated if there is another fuzzy number $\tilde{T}_j$ so that $\tilde{T}_i < \tilde{T}_j$. Thus, $H_k$ will be all the undominated $\tilde{T}_i$. Now $H_k$ is nonempty and if it does not contain all the fuzzy numbers, we then define $H_{k-1}$ to be all the undominated fuzzy numbers after we delete all those in $H_k$. We continue this process to the last set $H_1$. Then, the highest ranked fuzzy numbers lie in $H_k$, the second highest ranked fuzzy numbers are in $H_{k-1}$ and so on. Then, the least ranked fuzzy numbers lie in $H_1$. 

There is an easy way to determine if \( \tilde{T}_i < \tilde{T}_j \) or \( \tilde{T}_i \approx \tilde{T}_j \) for many triangular (trapezoidal) fuzzy numbers. It is easy to see that if the core of \( \tilde{T}_j \) lies completely to the right of the core of \( \tilde{T}_i \), then \( P(\tilde{T}_j \leq \tilde{T}_i) = 1 \). If the core of \( \tilde{T}_i \) and the core of \( \tilde{T}_j \) overlap, then \( \tilde{T}_j \approx \tilde{T}_i \). Now, if we assume that the core of \( \tilde{T}_j \) lies to the right of the core of \( \tilde{T}_i \), the \( P(\tilde{T}_i \leq \tilde{T}_j) \) is the height of their intersection (see Fig. 1.6.1).

Fig. 1.6.1 Determining \( P(\tilde{T}_j \leq \tilde{T}_i) \)

### 1.7. Fuzzy Functions

In this thesis, a fuzzy function is defined by a mapping from fuzzy numbers into fuzzy members. Let \( \tilde{X} \) be a triangular (trapezoidal) fuzzy number. Then, we write \( H(\tilde{X}) = \tilde{Z} \) for a fuzzy function with one independent variable \( \tilde{X} \) where \( \tilde{Z} \) is usually a triangular (trapezoidal) shaped fuzzy number. Usually, the fuzzy functions are extensions of real valued functions.

Let \( h: [a, b] \rightarrow \mathbb{R} \) be a real valued function on \( [a, b] \). We use the notation \( z = h(x) \) which means that \( z \) is a real number for \( x \in [a, b] \) under \( h \). We extend \( h: [a, b] \rightarrow \mathbb{R} \) to \( H(\tilde{X}) = \tilde{Z} \) in two ways:

1. The extension principle
2. Using \( \alpha \)-cuts and interval arithmetic.
1.7.1. Extension Principle

Let $\tilde{X}$ be a triangular (trapezoidal) fuzzy number in $[a, b] \subseteq \mathbb{R}$. Then, any function $h : [a, b] \to \mathbb{R}$ may be extended to $H(\tilde{X}) = \tilde{Z}$ as follows:

$$
\mu_{\tilde{x}}(z) = \sup_x \{ \mu_x(x) \mid h(x) = z, a \leq x \leq b \}
$$

(1.7.1.1)

where $\mu_{\tilde{x}}(z)$ defines the membership function of $\tilde{Z}$ for $\tilde{X}$ in $[a, b]$.

Assuming $h$ is continuous, for any $\alpha \in [0, 1]$, the $\alpha$-cuts $\tilde{Z}_\alpha$ of $\tilde{Z}$ may be defined as $\tilde{Z}_\alpha = [z_1(\alpha), z_2(\alpha)]$

where $z_1(\alpha) = \min \{h(x) \mid x \in \tilde{X}_\alpha \}$  

(1.7.1.2)

and $z_2(\alpha) = \max \{h(x) \mid x \in \tilde{X}_\alpha \}$

(1.7.1.3)

For two independent variables $x \in [a_1, b_1]$ and $y \in [a_2, b_2]$ with $h(z) = h(x, y)$, we extend $h$ to $H(\tilde{X}, \tilde{Y}) = \tilde{Z}$ (Buckley [17]) as

$$
\mu_{\tilde{x}y}(z) = \sup_{x,y} \{ \min(\mu_x(x), \mu_y(y)) \mid h(x, y) = z \}
$$

(1.7.1.4)

where $\tilde{X}$ and $\tilde{Y}$ are triangular (trapezoidal) fuzzy numbers in $[a_1, b_1]$ and $[a_2, b_2]$ respectively.

If $h$ is continuous, the $\alpha$-cut $\tilde{Z}_\alpha$ of $\tilde{Z}$ is defined by $\tilde{Z}_\alpha = [z_1(\alpha), z_2(\alpha)]$. Then, for $\alpha \in [0, 1]$, we have

$$
z_1(\alpha) = \min \{h(x, y) \mid x \in \tilde{X}_\alpha, y \in \tilde{Y}_\alpha \}
$$

(1.7.1.5)

and

$$
z_2(\alpha) = \max \{h(x, y) \mid x \in \tilde{X}_\alpha, y \in \tilde{Y}_\alpha \}
$$

(1.7.1.6)

In application, we will use the following result.

**Result 1.7.1.** Given some triangular (trapezoidal) fuzzy numbers $\tilde{A}_i$ in $X_i = [a_i, b_i], i = 1, \ldots, m$ and $\theta_i \in (\tilde{A}_i)_\alpha, \alpha \in [0, 1]$ where $(\tilde{A}_i)_\alpha$ is the $\alpha$-cut of $\tilde{A}$. Let $f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_m)$ be a continuous function. Then
\[ \tilde{I}_\alpha = \{ f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_m) \mid \theta_i \in (\tilde{A}_i)_{\alpha} \} \]  

is the \( \alpha \)-cut of \( f(x_1, x_2, \ldots, x_n; \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m) \). The end points of \( \tilde{I}_\alpha \) may be found as in equation (1.7.1.2), (1.7.1.3), (1.7.1.5) and (1.7.1.6).

### 1.7.2. Alpha-cuts and Interval Arithmetic

All the functions we usually use in mathematical sciences have a computer usable algorithm which, by using a finite number of additions, subtractions, multiplications and divisions, can evaluate the function to required accuracy. Such function can be extended by using \( \alpha \)-cuts and interval arithmetic to fuzzy functions.

Let \( h: [a, b] \to \mathbb{R} \) be a continuous function. Then, its extension \( H(\tilde{X}) = \tilde{Z} \), \( \tilde{X} \) in \([a, b]\) is done via interval arithmetic in evaluating \( h(\tilde{X}_\alpha) = \tilde{Z}_\alpha \), \( \alpha \in [0,1] \). As a procedure, we input the interval \( \tilde{X}_\alpha \), perform the arithmetic operation needed to evaluate \( h \) on the interval and obtain the interval \( \tilde{Z}_\alpha \). Then, put these \( \alpha \)-cuts together to obtain the fuzzy number \( \tilde{Z} \). For example, consider the fuzzy function

\[ \tilde{Z} = H(\tilde{X}) = \frac{\tilde{A}\tilde{X} + \tilde{B}}{\tilde{C}X + \tilde{D}} \]  

which would be the extension of \( h(x_1, x_2, x_3, x_4, x) = \frac{x_1x + x_2}{x_3x + x_4} \)  

where \( \tilde{A}, \tilde{B}, \tilde{C} \) and \( \tilde{D} \) are triangular fuzzy numbers and \( \tilde{X} \) is a triangular fuzzy number in \([0,k]\), \( k > 0 \) such that \( \tilde{C} \geq 0, \tilde{D} \geq 0 \) and \( \tilde{C}\tilde{X} + \tilde{D} \geq 0 \). To obtain \( \tilde{Z}_\alpha \) for \( \tilde{Z} \), we would substitute the intervals \( \tilde{A}_\alpha \) for \( x_1 \), \( \tilde{B}_\alpha \) for \( x_2 \), \( \tilde{C}_\alpha \) for \( x_3 \), \( \tilde{D}_\alpha \) for \( x_4 \) and \( \tilde{X}_\alpha \) for \( x \) and then, perform interval arithmetic.

Let us denote \( \tilde{Z}^* = H(\tilde{X}) \) for the extension principle method of extending a function \( h: (a, b) \to \mathbb{R} \) to \( H \) for \( \tilde{X} \) in \([a, b]\) and \( H(\tilde{X}) = \tilde{Z} \) for \( \alpha \)-cut and interval arithmetic extension of \( h \). Then, it should be noted that \( \tilde{Z}^* \neq \tilde{Z} \) and in general \( \tilde{Z}^* \leq \tilde{Z} \).
1.8. Linguistic Variables and Linguistic Hedges

Our natural language can be broken down into fundamental terms and certain linguistic connectors of those terms. The fundamental terms in the natural language are characterized as atoms in the literature. The fundamental terms are known as atomic terms. Examples of some atomic terms are small, medium, young, beautiful etc.. A collection of atomic terms such as medium weight boy, young tree, fairly beautiful painting etc. is called a composite term. When the fuzzy numbers represent fundamental terms as interpreted in a particular context, the resulting constructs are usually called linguistic variables.

Each linguistic variable the states of which are expressed by linguistic terms interpreted as specific fuzzy numbers is defined in terms of a base variable (Klir and Yuan [63]), the value of which are real numbers within a specific range. A base variable is a variable in the classical sense exemplified by a physical variable (e.g. temperature, pressure, speed etc.) as well as any other numerical variable (e.g. age, salary, probability, reliability etc.). In a linguistic variable, linguistic terms representing approximate values of a base variable, generate to a particular context are captured by appropriate fuzzy numbers. Generally the notion of linguistic variable which was called a variable of higher order rather than a fuzzy variable is defined as follows (Zadeh [102]).

**Definition 1.8.1.** A linguistic variable is fully characterized by a quintuple \((v,T,X,g,\tilde{m})\) in which \(v\) is the name of the variable, \(T\) is the set of linguistic terms of \(v\) that refers to a base variable whose values range over a universal set \(X\), \(g\) is a syntactic rule (grammar) for generating linguistic terms, and \(\tilde{m}\) is a semantic rule that assigns to each linguistic terms \(t \in T\). Its meaning \(\tilde{m}(t)\) is a fuzzy set on \(X\) i.e., \(\tilde{m}: T \rightarrow \tilde{F}(X)\) (set of fuzzy subsets on \(X\)).

An example of a linguistic variable performance is shown in Fig 1.8.1. This variable expresses the performance (base variable) of goal-oriented entity in a
given context by five basic linguistic terms such as small, very small, medium, large, very large. Each of the basic linguistic terms is assigned one of the five fuzzy numbers by a semantic rule, as shown in Fig. 1.8.1. The fuzzy numbers whose membership functions have the usual triangular shapes are defined on the interval [0,16], the range of the base variable. Each of them expresses a fuzzy restriction on this range.

In the above example, each linguistic term in the universe of natural language $T$ is an element $\alpha$ and we define a fuzzy number (triangular) in the universe of base variables $X$ as a specific meaning for the term $\alpha$. Then, natural language can be expressed as a mapping $\tilde{m}$ from the set of atomic terms in $T$ to a corresponding set of interpretations defined on universe $X$. Each atomic term $\alpha$ in $T$ corresponds to a triangular fuzzy set $\tilde{A}$ in $X$ which is the interpretation of $\alpha$. This mapping which can be denoted as $\tilde{m}(\alpha, \tilde{A})$ (Ross [93]) schematically is shown in Fig.1.8.2.
The fuzzy set $\tilde{A}$ represents the fuzziness in the mapping between an atomic term and its interpretation and can be denoted by the membership function $\mu_m(\alpha, x)$ or simply by

$$\mu_m(\alpha, x) = \mu_{\tilde{A}}(x)$$

We can call $\alpha$ a natural language variable whose ‘value’ is defined by the fuzzy set $\mu_{\tilde{A}}(x)$. Define two atomic terms $\alpha$ and $\beta$ on the universe $T$. Then, the interpretation of the composite combined by various linguistic connectives such as ‘and’, ‘or’ and ‘not’ defined on the universe $X$ can be defined by the following set-theoretic operations (Zadeh [100]):

$$\begin{align*}
\alpha \lor \beta &: \mu_{\alpha \lor \beta} = \max(\mu_\alpha(x), \mu_\beta(x)) \\
\alpha \land \beta &: \mu_{\alpha \land \beta} = \min(\mu_\alpha(x), \mu_\beta(x)) \\
\text{Not } \alpha &: \mu_{\bar{\alpha}}(x) = 1 - \mu_\alpha(x)
\end{align*}$$

If the fundamental atomic terms are modified with adjectives or adverbs like very, low, sight, more or less, fairly, almost approximately etc, then, we will call these modifiers “linguistic hedges”. In such cases, the singular meaning of an atomic term is hedged or modified form its original interpretation. These linguistic hedges have the effect of modifying the membership function for a basic atomic term (Zadeh [103]). Some examples which are commonly used in the literature are given below:
Define $\alpha = \int_{x} \frac{\mu_{a}(x)}{x}$, then

"very $\alpha$" = $\alpha^{2} \int_{x} \frac{[\mu_{a}(x)]^{2}}{x}$

(1.8.3)

"very very $\alpha$" = $\alpha^{4}$

(1.8.4)

"plus $\alpha$" = $\alpha^{1.25}$

(1.8.5)

"slightly $\alpha$" = $\alpha^{1/2} \int_{x} \frac{[\mu_{a}(x)]^{1/2}}{x}$

(1.8.6)

"Minus $\alpha$" = $\alpha^{0.75}$

(1.8.7)

The expressions shown in Eqn. 1.8.3 to Eqn. 1.8.5 are linguistic hedges known as concentrations (Zadeh [103]). The expressions given in Eqn.1.8.6 and Eqn. 1.8.7 are linguistic hedges known as dilations. In concentrations, membership values of $\alpha^{2}, \alpha^{4}, \alpha^{1.25}$ are less than or equal to the membership values of $\alpha$ for all $x \in T$. Similarly, for dilations, the membership values are greater than or equal to the membership values of $\alpha$ for all $x \in T$. Another operation on linguistic fuzzy numbers is known as intensification which is combinations of concentration for $0 \leq \mu_{a} \leq 0.5$ and dilation for $0.5 \leq \mu_{a} \leq 1$ can be expressed by numerous membership functions, one of which proposed by Zadeh [103] is

"Intensify $\alpha$" = \begin{cases} 
2[\mu_{a}(x)]^{2} & \text{for } 0 \leq \mu_{a}(x) \leq 0.5 \\
1 - 2[1 - \mu_{a}(x)]^{2} & \text{for } 0.5 \leq \mu_{a}(x) \leq 1 
\end{cases}

(1.8.8)

1.9. Fuzzy Relations

A fuzzy relation generalizes the notion of a classical relation into one that allows partial membership. Fuzzy relations map elements of one universe say, $X$ to those of another universe, say, $Y$ through Cartesian product of the two universes. The “strength” of relation between ordered pairs of the two universes is measured with a membership function expressing various “degree” of strength of the relation on the unit interval $[0,1]$. Hence, a fuzzy relation $\bar{R}$ is a mapping from
the Cartesian space $X \times Y$ to the interval $[0,1]$ where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universe or $\mu_R(x,y)$. A few key references on this topic can be made to Chakraborty et al. [21], Fang et al. [31] Klir and Yuan [63] and Ross [93] where the interested reader can look up many of the properties of fuzzy relations.

**Operation on Fuzzy Relations, Fuzzy Cartesian Product and Composition:**

Let $R$ and $S$ be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations:

- **Union**: $\mu_{R\cup S}(x,y) = \max(\mu_R(x,y),\mu_S(x,y))$ (1.9.1)
- **Intersection**: $\mu_{R\cap S}(x,y) = \min(\mu_R(x,y),\mu_S(x,y))$ (1.9.2)
- **Complement**: $\mu_R^c(x,y) = 1 - \mu_R(x,y)$ (1.9.3)
- **Containment**: $R \subset S \Rightarrow \mu_R(x,y) \leq \mu_S(x,y)$ (1.9.4)

The Cartesian product of two or more fuzzy sets can be defined to be a fuzzy relation. Let $\tilde{A}$ be a fuzzy set in universe $X$ and $\tilde{B}$ a fuzzy set in universe $Y$. Then, the Cartesian product between $\tilde{A}$ and $\tilde{B}$ will result in a fuzzy relation $\tilde{R}$:

$\tilde{A} \times \tilde{B} = \tilde{R} \subset X \times Y$ (1.9.5)

where the fuzzy relation $\tilde{R}$ has a membership function

$\mu_{\tilde{R}}(x,y) = \mu_{\tilde{A} \times \tilde{B}}(x,y) = \min(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y))$ (1.9.6)

More generally, a fuzzy $n$-array relation $\tilde{R}$ in $x_1,x_2,...,x_n$ whose domains are $X_1, X_2,...,X_n$, respectively, is defined by a function that maps an $n$-tuple $<x_1,x_2,...,x_n>$ in $X_1 \times X_2 \times ... \times X_n$ to a number in the interval $[0,1]$ i.e.

$\tilde{R}: X_1 \times X_2 \times ... \times X_n \rightarrow [0,1]$

Fuzzy composition is just as it is for crisp (binary) relation (Ross [93]). Suppose $\tilde{R}$ is a fuzzy relation on the Cartesian space $X \times Y$, $\tilde{S}$ is a fuzzy relation on the Cartesian space $Y \times Z$ and $\tilde{T}$ is a fuzzy relation on the Cartesian space $X \times Z$. Then, fuzzy max-min composition (there are also other forms of the composition; see Klir and Yuan [63], Ross [93]) is defined in terms of membership function theoretic notation:
and the fuzzy max-product composition is defined in terms of membership function theoretic notation as

$$
\mu^*_{T}(x,z) = \bigvee_{y \in Y} \left( \mu^*_{R}(x,y) \land \mu^*_{S}(y,z) \right)
$$

(1.9.7)

where $\bigvee$, $\land$ and $\cdot$ are the maximum operator, minimum operator and product operator respectively. It should be pointed out that fuzzy composition like crisp composition has no inverse in general i.e.

$$
\tilde{R} \circ \tilde{S} \neq \tilde{S} \circ \tilde{R}
$$

(1.9.9)

Some important properties of fuzzy relations will be appeared in chapter 5.

1.10. Fuzzy Rules

The fuzzy if-then rule (or in short, the fuzzy rule) has wide range of successful applications to many disciplines such as control system, optimization problems, decision making pattern recognition and system modelling. Details of fuzzy rules can be had from (Klir and Yuan [63], Ross [93], Yen and Langari [99]). Mathematically, fuzzy rule based inference can be viewed as an interpolation scheme because it enables the fusion of multiple fuzzy rules when their conditions are all satisfied to a degree. The degree of which each rule is satisfied determines the weight of the rules conclusion. Using these weights, fuzzy rule based inference combines the conclusions of multiple fuzzy rules in a manner similar to linear interpolation.

From logical point of view, fuzzy rule based inference is a generalization of a classical logical reasoning scheme called modus ponens. In classical logic, if we know a rule is true and we know the antecedent of the rule is true, then, it can be inferred that the consequent of the rule is true. This is referred to as modus ponens. One limitation of modus ponens is that it cannot deal with partial matching. Viewing such a limitation, fuzzy rule based-inference generalizes modus ponens to
allow its inferred conclusion to be modified by the degree to which the antecedent is satisfied. This is the essence of fuzzy rule-based inference.

1.10.1. Structure of Fuzzy Rules- the Antecedent and the Consequent

A fuzzy rule can capture knowledge in many fuzzy systems. A fuzzy rule has two component: an if-part which is also referred to as the antecedent and a then-part also referred to as the consequent:

\[ \text{IF<antecedent>THEN<consequent>} \]  

(1.10.1.1)

The antecedent part (shortly antecedent) describes a condition and the consequent part (shortly consequent) describes a conclusion that can be drawn when the condition holds.

The antecedent of a fuzzy rule describes an elastic condition i.e. a condition that can be satisfied to a matter of degree. The antecedent of a fuzzy rule may combine multiple simple conditions into a complex one using three logical connectives: AND (conjunction), OR (disjunction) and NOT (negation).

The consequent of fuzzy rules can be classified into three categories (Yen and Langari [99]):

1. Crisp consequent: IF…THEN \( y = a \)
   where ‘\( a \)’ is a non-fuzzy numeric value or symbolic value.
2. Fuzzy consequent: IF…THEN \( y = \tilde{A} \)
   where \( \tilde{A} \) is a fuzzy set.
3. Functional consequent: IF \( x_1 \) is \( A_1 \) AND \( x_2 \) is \( A_2 \) AND…AND \( x_n \) is \( A_n \)
   THEN \( y = a_x + \sum_{i=1}^{n} a_i x_i \)
   where \( a_0, a_1, a_2, \ldots, a_n \) are constants.

1.10.2. Fuzzy Rule-Based Inference

The algorithm of fuzzy rule based inference consists of three basic steps and an additional optional steps as given below:

1. Fuzzy Matching: In this step, we calculate the degree to which the input data match the condition of fuzzy rules.
2. Inference: We calculate the rules’ conclusion based on the matching degree.

3. Combination (Aggregation): Aggregate the conclusion inferred by all fuzzy rules into a final conclusion.

4. Defuzzification: In this step, a fuzzy conclusion is converted into a crisp output.

In fuzzy matching, when a rule has multiple conditions combined using AND, we simply use a fuzzy conjunction operator to combine the matching degree of each condition. One of the most commonly used fuzzy conjunction operators is the min operator (or the product operator).

In general, the degree to which a rule of form:

\[
\text{IF } X_1 \text{ is } A_1^{i}, \text{AND } X_2 \text{ is } A_2^{i}, \text{AND…AND } X_n \text{ is } A_n^{i} \text{ THEN…}
\]

matches the input data

\[
X_1 = x_1^i, X_2 = x_2^i, \ldots, X_n = x_n^i
\]

is computed by the following formula:

\[
\text{Matching Degree} = \min \{ \mu_{A_1^i}(x_1^i), \mu_{A_2^i}(x_2^i), \ldots, \mu_{A_n^i}(x_n^i) \} \quad (1.10.2.1)
\]

If the antecedent of a rule includes conditions connected by OR, we use a fuzzy “max” operator to combine matching degrees accordingly. After the fuzzy matching step, a fuzzy inference step is followed for each of the relevant rules to produce the conclusion based on their matching degree. There are two methods to produce the conclusion. The methods are: 1. The clipping method and 2. The scaling method.

The clipping and scaling methods produce their inferred conclusion by suppressing the membership function of the consequent differently. The clipping method cuts off the top of the membership function whose value is higher than the matching degree. The scaling method scales down the membership function in proportion to the matching degree.
When the matching degree is 1, the inferred conclusion is identical to the rules consequent and when the matching degree is 0, no conclusion can be inferred from the rule. Combining fuzzy conclusions is accomplished typically by superimposing all fuzzy conclusions about a variable.

After combining fuzzy conclusions, the last step is defuzzification where the final combined fuzzy conclusion is to convert into a crisp one. There are at least seven defuzzification techniques in the literature among the many that have been proposed by investigators in recent years. We summarise these methods as follows.

Let \( \tilde{C} \) be the fuzzy set obtained after combining the fuzzy conclusions. Then, we have the following defuzzification methods:

1. Max- membership principle
2. Centroid method
3. Weighted average method
4. Mean- max membership
5. Center of sums
6. Center of largest area
7. First (or Last) of maxima

Here we will not discuss the above methods in detail due to the limitation of space in the thesis. The centroid method will be appeared in chapter 3.