Chapter 5

DIVERSIFICATION IN EOQ MODEL UNDER RANDOM YIELD
5.1 INTRODUCTION

5.2 DIVERSIFICATION IN EOQ MODEL FOR NON-DETERIORATING ITEMS UNDER RANDOM YIELD

5.3 DIVERSIFICATION IN EOQ MODEL FOR DETERIORATING ITEMS UNDER RANDOM YIELD

5.4 CONCLUSION
5.1 INTRODUCTION:

In previous three chapters, it is assumed that orders placed to two suppliers match with the ordered received. It has been assumed that order placed may not exactly match with the amount ordered due to variety of reasons viz., strike by workers, failure of electricity, unavailability of raw material, damage during transhipment etc. In all the above mentioned papers, it is assumed that there is only one supplier whose yield is random.

In this chapter, we extend the results found in Parlar and Wang (1993) by making the assumption that the variance of a random yield is partially constant and partially depends on the quantity actually ordered. It has been assumed that both the suppliers charge different unit prices. The cost function is developed as a function of the order quantities from each supplier for both the situations, defined in chapter 2. In section 5.2 the model is developed with non-deteriorating items and while section 5.3 deals with deteriorating items.

The optimal order quantities and the minimum value of the cost function are found explicitly in terms of the problem parameters. At the end of each section, effect of various parameters on decision variables and objective function is studied with the help of a numerical illustration.
5.2 DIVERSIFICATION IN EOQ MODEL FOR NON-DETERIORATION ITEMS UNDER RANDOM YIELD:

In this section, we order for \( Q_i > 0 \), \((i = 1,2)\) units while we requisition \( Y_i > 0 \). \((i = 1,2)\) units with expected value \( E(Y_i) = b_iQ_i \), \((i = 1,2)\), where \( b_i \) \((i = 1,2)\) is a non-negative constant and variance \( V(Y_i) = \sigma^2_i + \sigma^2_iQ^2_i \), \((i = 1,2)\), where \( \sigma^2_i \) and \( \sigma^2_i \) are constants. It is also assumed that shortages are not allowed and lead time is zero. The unit price charge by source \( i \) is \( C_i \) and the corresponding holding cost per unit per time is \( h_i \), \((i = 1,2)\).

SITUATION I: When units are ordered simultaneously to both the suppliers

When \( Q_i \) units are ordered from each supplier, the total amount received is \( Y = Y_1 + Y_2 \), which is initial inventory at the beginning of each cycle. The cycle length is \( \frac{Y}{R} \). Hence total random cost incurred per cycle is

\[
\text{Cost/cycle} = K + C_1Y_1 + C_2Y_2 + \frac{(h_1Y_1 + h_2Y_2)(Y_1 + Y_2)}{2R} \tag{5.1}
\]

\[
E(\text{cost/cycle}) = K + C_1b_1Q_{11} + C_2b_2Q_{21}
+ 0.5[h_1(\sigma^2_{01} + (b^2_1 + b^2_2)Q^2_{11})] + h_2(\sigma^2_{02} + (b^2_1 + b^2_2)Q^2_{21})
+ (h_1 + h_2)b_1b_2Q_{11}Q_{21} \tag{5.2}
\]
\[
E(\text{length of cycle}) = \frac{b_1 Q_{1I} + b_2 Q_{2I}}{R} \tag{5.3}
\]

Using, renewal theory of Ross (1983) the expected average cost \(TC_I(Q_{1I}, Q_{2I})\) is given by the ratio of (5.2) and (5.3), which is

\[
TC_I(Q_{1I}, Q_{2I}) = \frac{KR}{(b_1 Q_{1I} + b_2 Q_{2I})} + \frac{R(C_1 b_1 Q_{1I} + C_2 b_2 Q_{2I})}{(b_1 Q_{1I} + b_2 Q_{2I})}
\]

\[
+ \frac{h_1[\sigma_{01}^2 + (\sigma_1^2 + b_1^2)Q_{1I}^2]}{2(b_1 Q_{1I} + b_2 Q_{2I})}
\]

\[
+ \frac{h_2[\sigma_{02}^2 + (\sigma_2^2 + b_2^2)Q_{2I}^2]}{2(b_1 Q_{1I} + b_2 Q_{2I})}
\]

\[
+ \frac{(h_1 + h_2)b_1 b_2 Q_{1I} Q_{2I}}{2(b_1 Q_{1I} + b_2 Q_{2I})}
\]

Minimizing \(TC_I(Q_{1I}, Q_{2I})\) subject to \(Q_{1I} > 0\) and \(Q_{2I} > 0\) would give us the optimal quantity to order from each suppliers. To minimize \(TC_I\). We perform partial differentation and obtain the following

\[
\frac{\partial TC_I}{\partial Q_{1I}} = 0
\]

\[
h_1 b_1 (\sigma_1^2 + b_1^2) Q_{1I}^2 + 2b_2 h_1 (\sigma_1^2 + b_1^2) Q_{1I} Q_{2I} + (b_2^2 h_1 - h_2 \sigma_2^2) b_1 Q_{2I}^2
\]

\[
+ 2Rb_1 b_2 (C_1 - C_2) Q_{2I} - (2KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2) b_1 = 0
\]

\[
\tag{5.5}
\]
\[
\begin{align*}
\frac{\partial TC_1}{\partial Q_{2I}} &= 0 \\
h_2 b_2 (\sigma_2^2 + b_2^2) Q_{2I}^2 + 2b_1 h_2 (\sigma_2^2 + b_2^2) Q_{1I} Q_{2I} + (b_1^2 h_2 - h_1 \sigma_1^2) b_2 Q_{1I}^2 + \\
2 R b_1 b_2 (C_2 - C_1) Q_{1I} - (2 KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2) b_2 &= 0 \\
(5.6)
\end{align*}
\]

Assuming (5.5) to be quadratic in \( Q_{1I} \), we get

\[
Q_{1I} = -\frac{b_2}{b_1} Q_{2I}
\]

and

\[
+ \sqrt{\frac{b_2^2}{b_1^2} Q_{2I}^2 - \frac{(b_2^2 h_1 - h_2 \sigma_2^2) Q_{2I}^2 + 2 R b_2 (C_1 - C_2) Q_{2I} - (2 KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2)}{h_1 (\sigma_1^2 + b_1^2)}}
\]

substitute value of \( Q_{1I} \) in (5.6), we get quadratic equation which can be solved for \( Q_{2I} \).

**Proposition (P-1):** The objective function \( TC_I(Q_{1I}, Q_{2I}) \) is strictly convex in \((Q_{1I}, Q_{2I})\) when

\[
2 KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2 > \frac{R^2 b_1^2 b_2^2 (C_2 - C_1)^2}{h_1 \sigma_1^2 b_2^2 + h_2 \sigma_{2I}^2 b_1^2}
\]

Proof: The second partial and mixed partial derivatives of \( TC_I \) are given as follows:

\[
H_{11} = TC_I(Q_{1I} Q_{1I})
\]

\[
= \frac{[b_2^2 Q_{2I}^2/h_1 (\sigma_1^2 + b_1^2) - (b_2^2 h_1 - \sigma_2^2 h_2) b_1^2 Q_{2I}^2]}{(b_1 Q_{1I} + b_2 Q_{2I})^3}
\]

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\[
H_{12} = TC_1(\Omega_{11}\Omega_{21})
\]
\[
= \frac{[b_2^2\Omega_{21}\sigma_2^2 + \sigma_1^2 + b_2^2] + (b_2^2\sigma_1^2 - \sigma_2^2h_2)b_2^2\Omega_{21}\Omega_{11} + Rb_1^2b_2^2\Omega_{11}(C_1 - C_2)]}{(b_1\Omega_{11} + b_2\Omega_{21})^3}
\]
\[
- \frac{[2h_1b_2^2\sigma_1^2 + \sigma_1^2\Omega_{11}\Omega_{21} + Rb_1b_2^2\Omega_{11}(C_1 - C_2)]}{(b_1\Omega_{11} + b_2\Omega_{21})^3}
\]
\[
+ \frac{(2KR + h_1\sigma_0 + h_2\sigma_0^2)b_1b_2}{(b_1\Omega_{11} + b_2\Omega_{21})^3}
\]
\[
H_{21} = TC_1(\Omega_{21}\Omega_{11})
\]
\[
H_{22} = TC_1(\Omega_{21}\Omega_{21})
\]
\[
= \frac{[b_2^2\Omega_{11}\sigma_2^2 + \sigma_2^2 - (b_1^2\sigma_2^2 - \sigma_1^2h_1)b_2^2\Omega_{11}]}{(b_1\Omega_{11} + b_2\Omega_{21})^3}
\]
\[
- \frac{[2Rb_2^2b_1(C_2 - C_1)\Omega_{11} - (2KR + h_1\sigma_0 + h_2\sigma_0^2)b_2^2]}{(b_1\Omega_{11} + b_2\Omega_{21})^3}
\]

To establish \( H_{11} > 0 \), we only need to derive condition for numerator to be positive, as denominator of \( H_{11} \) is always positive write numerator as a quadratic in \( \Omega_{21} \) i.e.

\[
f(\Omega_{21}) = AQ_{21}^2 + B\Omega_{21} + C
\]
where

\[ A = h_1 \sigma_1^2 b_2^2 + h_2 \sigma_2^2 b_1^2 \]

\[ B = 2Rb_1 b_2 (C_2 - C_1) \]

\[ C = (2KR + h_1 \sigma_0^2 + h_2 \sigma_0^2)^2 b_1^2 \]

At \( Q_{2I} = 0 \), \( f(Q_{2I}) = C > 0 \). The function takes smallest value at \( Q_{2I} = \frac{b}{2A} \) and value is given by \( f(Q_{2I}) = C - \frac{B^2}{4A} \) This quantity is always positive when

\[ 2KR + h_1 \sigma_0^2 + h_2 \sigma_0^2 > \frac{R_2 b_1 b_2 (C_2 - C_1)^2}{h_1 \sigma_1^2 b_2^2 + h_2 \sigma_2^2 b_1^2}, (A.1) \]

Now the determinant of the Hessian matrix is

\[ \text{det.} H = H_{11} H_{22} - H_{12}^2 \]

\[ = \frac{R[(2KR + h_1 \sigma_0^2 + h_2 \sigma_0^2)(h_1 \sigma_1^2 b_2^2 + h_2 \sigma_2^2 b_1^2)]}{(b_1 Q_1 + b_2 Q_2)^4} \]

\[ - \frac{R^2 b_1^2 b_2^2 (C_2 - C_1)^2}{(b_1 Q_{1I} + b_2 Q_{2I})^4} \]

which is always positive when (A.1) holds.

This proves proposition (P-1).

To see whether diversification is worthwhile, one need to compare \( TC_i^* \) with \( TC_i^* \), the minimum cost incurred if source \( i \) is used exclusively. Defining \( K_i, i = 1, 2 \) to be set up cost if source \( i \) is used, the optimal order quantity \( Q_i^* \) and the corresponding optimum minimum cost, \( TC_i^* \), are as given by Silver (1976) :
\[ Q_i^* = \sqrt{\frac{2K_i R + C_i \sigma_{0i}^2}{C_i I (\sigma_i^2 + b_i^2)}}, \quad i = 1, 2 \]  

and

\[ T_{C_i}^* = \frac{K_i R}{b_i Q_i^*} - \frac{C_i I [\sigma_{0i}^2 + (\sigma_i^2 + b_i^2)Q_i^2]}{2b_i Q_i} + C_i R, \quad i = 1, 2 \]  

A comparison of (5.4) and (5.8) suggests that the optimal solution to the diversification problem will depend on which \( T_{C_1}, T_{C_1}^* \) and \( T_{C_2}^* \) is the smallest in terms of \( R, K, K_i, C_i, h_i \sigma_{0i}^2, \sigma_i^2, \) and \( b_i, i = 1, 2. \) We will discuss numerical examples where this comparison is made. Parlar and Wang (1993)'s article is special case of developed model, if \( \sigma_i^2 + b_i^2, i = 1, 2 \) is equated to zero.

**SITUATION II**: When order is placed one after the other to two suppliers.

Arguing, as above, in this situation, the expected average cost \( T_{C_{III}}(Q_{1III}, Q_{2III}) \) per time unit is given by

\[
T_{C_{III}}(Q_{1III}, Q_{2III}) = \frac{KR}{b_1 Q_{1III} + b_2 Q_{2III}} + \frac{R(C_1 b_1 Q_{1III} + C_2 b_2 Q_{2III})}{b_1 Q_{1III} + b_2 Q_{2III}} \]

\[
+ \frac{h_1[\sigma_{01}^2 + (\sigma_1^2 + b_1^2)Q_{1III}^2]}{2(b_1 Q_{1III} + b_2 Q_{2III})} \]

\[
+ \frac{h_2[\sigma_{02}^2 + (\sigma_2^2 + b_2^2)Q_{2III}^2]}{2(b_1 Q_{1III} + b_2 Q_{2III})} \]

\[ (5.9) \]
Minimizing $TC_{II}(Q_{1II}, Q_{2II})$ subject to $Q_{1II} > 0$ and $Q_{2II} > 0$ we get

$$\frac{\partial TC_{II}}{\partial Q_{1II}} = 0$$

$$h_1 b_1 (\sigma_1^2 + b_1^2) Q_{1II}^2 + 2b_2 h_1 (\sigma_1^2 + b_1^2) Q_{1II} Q_{2II} - h_2 b_1 (\sigma_2^2 + b_2^2) Q_{2II}^2 + 2Rb_1 b_2 (C_1 - C_2) Q_{2II} - (2KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2)b_1 = 0$$

(5.10)

$$\frac{\partial TC_{II}}{\partial Q_{2II}} = 0$$

$$h_2 b_2 (\sigma_2^2 + b_2^2) Q_{2II}^2 + 2b_1 h_2 (\sigma_2^2 + b_2^2) Q_{1II} Q_{2II} - h_1 b_2 (\sigma_1^2 + b_1^2) Q_{1II}^2 + 2Rb_1 b_2 (C_2 - C_1) Q_{1II} - (2KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2)b_2 = 0$$

(5.11)

Assuming (5.10) to be quadratic in $Q_{1II}$, we get

$$Q_1 = -\frac{b_2}{b_1} Q_{2II} + \sqrt{\frac{b_2^2}{b_1^2} Q_{2II}^2 + \frac{(\sigma_2^2 + b_2^2) Q_{2II}^2 h_2 + 2Rb_2 (C_2 - C_1) Q_{2II} + (2KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2)b_1}{h_1 (\sigma_1^2 + b_1^2)}}$$

Substituting value of $Q_{1II}$ in (5.10) results solution of quadratic equation in $Q_{2II}$.

**Proposition (P-2)**: The objective function $TC_{II}(Q_{1II}, Q_{2II})$ is strictly convex in $(Q_{1II}, Q_{2II})$ when

$$2KR + h_1 \sigma_{01}^2 + h_2 \sigma_{02}^2 > \frac{b^2 [(h_2 - h_1)(\sigma_1^2 + b_1^2) Q_{1II} + 2b_1 b_2 R(C_2 - C_1)]^2}{h_1 b_2^2 (\sigma_1^2 + b_1^2) + h_2 b_1^2 (\sigma_2^2 + b_2^2)}$$

**Proof**: The second partial and mixed partial derivatives of $TC_{II}$ are given as follows

$$H_{II} = TC_{II}(Q_{1II} Q_{1II})$$
\[ H_{12} = T C_{11}(Q_{111}Q_{211}) \]

\[ = \frac{[2Rb_1 b_2(C_1 - C_2)Q_{211} - h_2 b_1^2(\sigma_2^2 + b_2^2)Q_{211}^2 - (2KR + h_1 \sigma_{01} + h_2 \sigma_{02})b_1 b_2]}{(b_1 Q_{111} + b_2 Q_{211})^3} \]

\[ H_{21} = T C_{11}(Q_{211}Q_{111}) \]

\[ = \frac{[h_1 b_2^2(\sigma_1^2 + b_1^2)Q_{111}Q_{211} - (2KR + h_1 \sigma_{01} + h_2 \sigma_{02})b_1 b_2]}{(b_1 Q_{111} + b_2 Q_{211})^3} \]

\[ H_{22} = T C_{11}(Q_{211}Q_{211}) \]

\[ = \frac{[ - h_2 b_1 b_2(\sigma_2^2 + b_2^2)Q_{111}Q_{211} + h_1 b_1 b_2(\sigma_2^2 + b_2^2)Q_{111}Q_{211} + h_2 b_1^2(\sigma_2^2 + b_2^2)Q_{111}^2 - 2Rb_2^2 b_1(C_2 - C_1)Q_{111} + (2KR + h_1 \sigma_{01} + h_2 \sigma_{02})b_1 b_2^2]}{(b_1 Q_{111} + b_2 Q_{211})^3} \]

\[ + \frac{[h_1 b_2^2(\sigma_1^2 + b_1^2)Q_{111}^2 - 2Rb_2^2 b_1(C_2 - C_1)Q_{111} + (2KR + h_1 \sigma_{01} + h_2 \sigma_{02})b_2^2]}{(b_1 Q_{111} + b_2 Q_{211})^3} \]

\[ f(Q_2) = AQ_{211}^2 + BQ_{211} + C \]

where

\[ A = h_1 b_2^2(\sigma_1^2 + b_1^2) + h_2 b_1^2(\sigma_2^2 + b_2^2) \]

\[ B = 2Rb_1 b_2^2(C_2 - C_1) + (h_2 - h_1) b_1 b_2(\sigma_1^2 + b_1^2)Q_{111} \]

\[ C = (2KR + h_1 \sigma_{01} + h_2 \sigma_{02})b_1^2 \]

\[ 2KR + h_1 \sigma_{01} + h_2 \sigma_{02} > \frac{b^2[(h_2 - h_1)(\sigma_1^2 + b_1^2)Q_{111} + 2b_1 b_2 R(C_2 - C_1)^2]}{h_1 b_2^2(\sigma_1^2 + b_1^2) + h_2 b_1^2(\sigma_2^2 + b_2^2)} \]

\[ (5.12) \]

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\[ det.H = H_{11}H_{22} - H_{21}H_{12} \]
\[ = \frac{R[(2KR + h_1\sigma_{01}^2 + h_2\sigma_{02}^2)(h_1b_2^2(\sigma_1^2 + b_1^2) + h_2b_1^2(\sigma_2^2 + b_2^2))]}{(b_1Q_{1II} + b_2Q_{2II})^4} - \frac{R[b_2^2((h_2 - h_1)(\sigma_1^2 + b_1^2)Q_1 + 2b_1b_2R(C_2 - C_1))^2]}{(b_1Q_{1II} + b_2Q_{2II})^4} \]

which is always positive when (B.1) holds.

This proves proposition (P-2).

To see whether diversification is worthwhile, to compare \( TC_{1II}^* \) with \( TC_i^* \), the minimum cost incurred in (5.8), if source \( i \) is used.

A comparison of (5.8) and (5.9) suggests that the optimal solution to the diversification problem will depend on the values of \( TC_{1II}^* \), \( TC_1^* \) or \( TC_2^* \), whichever is the smallest.

**NUMERICAL EXAMPLE**

In this section, we support developed model by numerical example.

We use following parametric values:

Ordering cost, \( K = $ 25.00 \) per order.

Demand rate, \( R = 1000 \) units per year.

Unit price charged by supplier 1, \( C_1 = $ 35.00 \) per unit.

Unit price charged by supplier 2, \( C_2 = $ 30.00 \) per unit.

Inventory carrying charge fraction, \( I = 30 \% \) per annum.

Inventory holding cost by supplier 1, \( h_1 = C_1I \).

Inventory holding cost by supplier 2, \( h_2 = C_2I \).
\[
\begin{align*}
\sigma_0 &= \sigma_0 = 5. \\
\sigma_1 &= 2. \\
\sigma_2 &= 30. \\
b_1 &= 1.2 \\
b_2 &= 0.9
\end{align*}
\]

We assumed that the first supplier charges more \((C_1 > C_2)\), is more realistic if \(\sigma_1 < \sigma_2\) and on an average, it supplies more than what is requisitioned.

With above values, the optimal order quantities and the annual costs for both the situations are obtained as:
TABLE : 1

<table>
<thead>
<tr>
<th></th>
<th>situation - I</th>
<th>situation - II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1^*$</td>
<td>29.00 /order</td>
<td>55.97 /order</td>
</tr>
<tr>
<td>$Q_2^*$</td>
<td>01.71 /order</td>
<td>01.90 /order</td>
</tr>
<tr>
<td>$TC^*$</td>
<td>$36357.57</td>
<td>$35750.63</td>
</tr>
<tr>
<td>cycle time</td>
<td>0.03071</td>
<td>0.05779</td>
</tr>
</tbody>
</table>

TABLE : 1 suggest that situation II is preferable, because of lower average expected cost.

Now suppose that the order can be placed only at one source. If the order costs at individual suppliers are, respectively, $K_1 = $13 and $K_2 = $12 with the other data as above, we obtain.

\[
\begin{align*}
Q_1^* &= 21.44 \text{ per order } TC_1^* = $36020.66 \\
Q_2^* &= 01.73 \text{ per order } TC_2^* = $45571.39
\end{align*}
\]

Comparing these $TC_i^*$ values with $TC_i^*$ and $TC_{II}^*$ obtained when two suppliers are used (K = $25.00) we see that it would be preferable to use both the suppliers and situation II, since

\[
TC_{II}^* < TC_1^* < TC_i^* < TC_2^*
\]

When values of $K_1$ and $K_2$ are interchanged, i.e. $K_1 = $12 and $K_2 = $13, we get

\[
\begin{align*}
Q_1^* &= 20.61 \text{ per order } TC_1^* = $35981.03 \\
Q_2^* &= 01.80 \text{ per order } TC_2^* = $46201.42
\end{align*}
\]
In this section, we extend, Parlar and Wang (1993)'s mathematical model for deteriorating items, when input is random. In an inventory, units are subject to deterioration at a constant rate (say) $\theta$ of on hand inventory. The deteriorated units can neither be repaired or replaced during the cycle time. We develop the total cost expression as a function of cycle times, cycle times as a function of order quantities for each supplier when both the orders are placed simultaneously and when orders are placed one after the other from the two suppliers. The lead time is assumed to be zero and shortages are not allowed. The benefits of diversification are achieved through possible reduction in total annual costs which include purchasing deteriorating, ordering, and holding costs. The model concludes with a discussion of a numerical example for each situations.

**SITUATION I**: When orders are simultaneously placed to two suppliers.

Let $T_1$ be the time point at which satisfy the order and $T$ be the time point which supplier 2 satisfy the order.

Let $Q_I(t/Y)$ denotes the number of units on hand at any instant $t$, $0 \leq t \leq T+Y$. Then the differential equation that governs the instantaneous states of units on hand
is given by

\[ \frac{\partial Q_1(t/Y)}{\partial t} + \theta Q_1(t/Y) = -R, \quad 0 \leq t \leq T_1(Y) \]

\[ = -R, \quad T_1(Y) \leq t \leq T(Y) \]

with boundary conditions

\[ Q_1(0/Y) = Y_1, \quad Q_1(T_1(Y)/Y) = Y_2 \quad \text{and} \quad Q_1(T(Y)) = 0 \]

Then, the solution of above differential equation is given by

\[ Q_1(t/Y) = e^{-\theta t}[Y_1 - \frac{R}{\theta}(e^{\theta t} - 1)], \quad 0 \leq t \leq T_1(Y_1) \]

\[ = Y_2 e^{-\theta t}, \quad 0 \leq t \leq T_1(Y_1) \]

\[ = e^{-\theta(t - T_1(Y_1))}[Y_2 e^{-\theta T_1(Y_1)} - \frac{R}{\theta}[e^{\theta(t - T_1(Y_1))} - 1]], \quad T_1(Y_1) \leq t \leq T(Y) \]

so

\[ T_1(Y_1) = \frac{1}{\theta} \log[1 + \frac{\theta Y_1}{R}] \]

and

\[ T(Y) = T_1(Y_1) + \frac{1}{\theta} \log[1 + \frac{\theta Y_2 e^{-\theta T_1(Y_1)}}{R}] \]

The different components of total random cost incurred per cycle are as follows:

a) Purchase cost :

\[ PC = \frac{C_1 Y_1 + C_2 Y_2}{Y_1 + Y_2} \]

b) Cost due to deterioration of units :

\[ DC = C_1(Y_1 - RT_1(Y_1)) + C_2(Y_2 - R(T - T_1(Y_1))) \]
c) Ordering cost:

\[ OC = K \]

d) Inventory holding cost:

\[
IHC = C_1 I \int_0^{T_1(Y_1)} Q_1(t/Y) dt + C_2 I \int_0^{T_1(Y_1)} Q_2(t/Y) dt + C_2 I \int_{T_1(Y)}^T Q_2(t/Y) dt
\]

\[
= C_1 I Y_1 + C_2 I Y_2 [1 - e^{-\theta T_1(Y_1)}]
\]

\[
+ \frac{h_1 R}{\theta^2} [1 - \theta T_1(Y_1) - e^{-\theta T_1(Y_1)}]
\]

\[
+ \frac{h_2 Y_2}{\theta} [e^{-\theta T_1(Y_1)} - e^{-\theta Y}]
\]

\[
+ \frac{h_2 R}{\theta^2} [1 - \theta(T_1(Y_1) - T(Y)) - e^{-\theta(Y) - T_1(Y_1)}]
\]

Using (a) - (d), total random cost incurred per cycle is

\[
Cost/cycle = PC + DC + OC + IHC
\]

(5.13)

since it is not possible to take expectation of terms involved in (5.13), we take series approximation up to power \( \theta \), neglecting \( \theta^2 \) higher power, the random cost incurred per cycle is given by

\[
Cost/cycle = \frac{C_1 Y_1 + C_2 Y_2}{Y_1 + Y_2} + K + \frac{C_1 \theta Y_1^2 + C_2 \theta Y_2^2}{2R}
\]

\[
+ \frac{C_2 \theta Y_1 Y_2 + C_2 I Y_1 Y_2}{R}
\]

\[
+ \frac{C_2 I Y_2^2 + C_2 I Y_2^2}{2R}
\]
\[
- \frac{\theta C_2 I Y_1^2 Y_2}{R^2} + \frac{\theta C_2 I Y_1 Y_2^2}{R^2}
\]
\[
- \frac{\theta C_2 I Y_2^3 + \theta C_1 I Y_1^3}{3R^2}
\]

and

\[
T(Y_1, Y_2) = \frac{Y_1 + Y_2}{R} - \frac{\theta(Y_1 + Y_2)^2}{2R^2}
\]  \hspace{1cm} (5.14)

Assuming, \( Y \) to be normally distributed we get

\[
E(Y_1^2) = \sigma_{y1}^2 + (\sigma_1^2 + b_1^2)Q_{11}^2
\]
\[
E(Y_2^2) = \sigma_{y2}^2 + (\sigma_2^2 + b_2^2)Q_{21}^2
\]
\[
E(Y_1^3) = b_1^3 Q_{11}^3 + 3b_1 Q_{11}(\sigma_{y1}^2 + \sigma_1^2 Q_{11}^2)
\]
\[
E(Y_2^3) = b_2^3 Q_{21}^3 + 3b_2 Q_{21}(\sigma_{y2}^2 + \sigma_2^2 Q_{21}^2)
\]

hence

\[
E(\text{Cost/cycle}) = \frac{C_1 b_1 Q_{11} + C_2 b_2 Q_{21}}{Q_{11} + Q_{21}} + K
\]
\[
+ \frac{C_1 \theta \sigma_{y1}^2 + (\sigma_1^2 + b_1^2)Q_{11}^2}{2R}
\]
\[
+ \frac{C_2 \theta \sigma_{y2}^2 + (\sigma_2^2 + b_2^2)Q_{21}^2}{2R}
\]
\[
+ \frac{C_2 \theta b_1 Q_{11} b_2 Q_{21}}{R} + \frac{C_2 I b_1 Q_{11} b_2 Q_{21}}{R}
\]
\[
+ \frac{C_2 I \sigma_{y1}^2 + (\sigma_1^2 + b_1^2)Q_{11}^2}{2R}
\]
\[
+ \frac{C_2 I \sigma_{y2}^2 + (\sigma_2^2 + b_2^2)Q_{21}^2}{2R}
\]

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\[
\frac{\theta C_2 I[\sigma_{01}^2 + (\sigma_1^2 + b_2^2)Q_{11}^2]b_2Q_{21}}{R^2} - \frac{\theta C_2 I[\sigma_{02}^2 + (\sigma_2^2 + b_2^2)Q_{21}^2]b_1Q_{11}}{R^2} - \frac{\theta C_2 I[b_2^3Q_{21}^3 + 3b_2Q_{21}(\sigma_{02}^2 + \sigma_2^2Q_{21}^2)]}{3R^2} - \frac{\theta C_1 I[b_1^3Q_{11}^3 + 3b_1Q_{11}(\sigma_{01}^2 + \sigma_1^2Q_{11}^2)]}{3R^2}
\]

(5.15)

and

\[
E(T(Y_1, Y_2))^{-1} = \frac{R}{b_1Q_{11} + b_2Q_{21}} + \frac{\theta(\sigma_{01}^2 + \sigma_{02}^2 + (\sigma_1^2 + b_2^2)Q_{11}^2 + (\sigma_2^2 + b_2^2)Q_{21}^2 + 2b_1b_2Q_{11}Q_{21})}{2(b_1Q_{11} + b_2Q_{21})^2}
\]

(5.16)

Using, renewal theory of Ross (1983), the average expected cost \(TC_1(Q_{11}, Q_{21})\) is given by multiplying equations (5.14) and (5.15) i.e.

\[
TC_1(Q_{11}, Q_{21}) = (\alpha_1Q_{21}^2 + \alpha_2Q_{21}^3 + \alpha_3Q_{21} + \alpha_4Q_{11}^3 + \alpha_5Q_{11}^2 + \alpha_6Q_{11}) + \alpha_7Q_{11}Q_{21} + \alpha_8Q_{11}Q_{21}^2 + \alpha_9Q_{11}Q_{21}^2 + \alpha_{10})
\]

\((T_1(Q_{11}, Q_{21}))^{-1}\)

(5.17)

where

\[
\alpha_1 = \frac{-\theta C_2 Ib_2}{R^2} \left[ \frac{b_2^2}{3} + \sigma_2^2 \right]
\]

\[
\alpha_2 = \frac{C_2(I + \theta)(\sigma_2^2 + b_2^2)}{2R}
\]

\[
\alpha_3 = \frac{-C_2 I\theta b_2}{R^2} (\sigma_{01}^2 + \sigma_{02}^2)
\]

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\[
\alpha_4 = \frac{-C_1 \theta b_1}{R^2} \left( \frac{b_2}{3} + \sigma_1^2 \right)
\]
\[
\alpha_5 = \frac{\sigma_1^2 + b_1^2}{2R} (C_2 I + C_1 \theta)
\]
\[
\alpha_6 = \frac{-\theta b_1 I}{R^2} \left[ C_2 \sigma_0^2 + C_1 \sigma_1^2 \right]
\]
\[
\alpha_7 = \frac{b_1 b_2}{R} C_2 (I + \theta)
\]
\[
\alpha_8 = \frac{-\theta C_2 I b_1}{R^2} (\sigma_2^2 + b_2^2)
\]
\[
\alpha_9 = \frac{-\theta C_2 I b_2}{R^2} (\sigma_1^2 + b_1^2)
\]
\[
\alpha_{10} = K + \sigma_0^2 + \sigma_0^2 \frac{C_2 I}{2R} + \frac{C_2 \sigma_0^2}{2R} \left( \frac{C_2 \sigma_0^2 + C_1 \sigma_0^2 \theta}{2R} \right)
\]

We want to minimize $TC_I$ with respect to $Q_{1I}$ and $Q_{2I}$. The optimal value of $Q_{1I} = Q_{1I}^*$ and $Q_{2I} = Q_{2I}^*$ can be obtained by solving $\frac{\partial TC_I}{\partial Q_{1I}} = 0$ and $\frac{\partial TC_I}{\partial Q_{2I}} = 0$ simultaneously by Gass-seidal iterative method. Since $Q_{1I}$ and $Q_{2I}$ take only finite values, the iteration will coverage after finite number of steps. It can be seen that it is not possible to obtain the explicit form of the minimum value of the objective function $TC_I^*$ in terms of all parameter values.

**SITUATION II:** When orders are placed one after the other, to two supplier.

We assume that $C_1 > C_2$. WLOG, we assume that initial demand in the cycle is satisfied by $Y_2$ units (ordered $Q_{2II}$ units) from supplier 2 and than by $Y_1$ units (ordered $Q_{1II}$ units) from supplier 1.

Let $Q_{1II}(t/Y_1)$ denotes the on hand inventory of the system at any instant $t$ for the
$Y_1$ units received from supplier 1 and $Q_{1II}(t/Y_2)$ denotes the on hand inventory of the system at any instant $t$ for the $Y_2$ units received from supplier 2. The instantaneous states of units on hand at any instant $t$ is given by following differential equations:

$$\frac{\partial Q_{1II}(t/Y_1)}{\partial t} + \theta Q_{1II}(t/Y_1) = 0, 0 \leq t \leq T_1(Y_1)$$
$$\frac{\partial Q_{2II}(t/Y_2)}{\partial t} + \theta Q_{2II}(t/Y_2) = -R, 0 \leq t \leq T_1(Y_1)$$
$$\frac{\partial Q_{1II}(t/Y_1)}{\partial t} + \theta Q_{1II}(t/Y_1) = -R, T_1(Y_1) \leq t \leq T(Y_1, Y_2)$$
$$\frac{\partial Q_{2II}(t/Y_2)}{\partial t} = 0, T_1(Y_1) \leq t \leq T(Y_1, Y_2)$$

with boundary conditions

$Q_{2II}(0/Y_2) = Y_2$, $Q_{2II}(T_1(Y_1)) = 0$, $Q_{1II}(0/Y_1) = Y_1$, and $Q_{2II}(T_1(Y_1, Y_2)) = 0$

Arguing as in situation I, Cycle time is given by

$$T(Y_1, Y_2) = \frac{1}{\theta} \log[1 + \frac{\theta Y_1}{R}] + \frac{1}{\theta} \log[1 + \frac{\theta Y_2}{R}]$$

The solution of above differential equation is given by

$$Q_{1II}(t/Y_1) = \frac{R}{\theta} e^{\theta T_1(Y_1)} [e^{\theta T(Y_1, Y_2)} - e^{\theta T_1(Y_1)}] e^{-\theta t}, 0 \leq t \leq T_1(Y_1)$$
$$Q_{1II}(t/Y_1) = \frac{R}{\theta} e^{\theta T_1(Y_1)} [e^{\theta (T(Y_1, Y_2) - t)} - 1]$$

$T_1(Y_1) \leq t \leq T(Y_1, Y_2)$

$$Q_{2II}(t/Y_2) = \frac{R}{\theta} [e^{\theta (T(Y_1, Y_2) - t)} - 1]$$

$0 \leq t \leq T_1(Y_1)$

$$Q_{2II}(t/Y_2) = 0, T_1(Y_1) \leq t \leq T(Y_1, Y_2)$$

Then, the different costs of the total cost per cycle are as follows:

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e) Cost due to deterioration of units:

\[ DC = C_1(Y_1 - RT(Y_1)) + C_2(Y_2 - RT(Y_1, Y_2)) \]

where

\[ T(Y_1) = \frac{1}{\theta} \log[1 + \frac{\theta Y_1}{R}] \]

\[ T(Y_1, Y_2) = T_1 + \frac{1}{\theta} \log[1 + \frac{\theta Y_2}{R}] \]

f) Inventory holding cost for cycle:

\[ IHC = \frac{C_1 IR}{\theta^2} [e^{\theta T(Y_1)} - \theta T(Y_1) - 1] \]
\[ + \frac{C_2 IR}{\theta^2} [e^{\theta (T(Y_1, Y_2) - T(Y_1))} - \theta (T(Y_1, Y_2) - T_1(Y_1)) - 1] \]

Using (a), (c), (e) and (f), total random cost incurred per cycle is

\[ \text{Cost/cycle} = PC + DC + OC + IHC \] (5.18)

Under first order approximation in terms of \( \theta \) (neglecting \( \theta^2 \) and higher powers) of the series involved in eq.(5.17), the random cost incurred per cycle is given by

\[ \text{Cost/cycle} = C_1 Y_1 + C_2 Y_2 + K + \frac{C_1 \theta Y_1^2 + C_2 \theta Y_2^2}{2R} \]
\[ + \frac{C_1 I Y_1^3 + C_2 I Y_2^3}{2R} \]
\[ - \frac{C_1 I Y_1^3 + C_2 I Y_2^3}{3R^2} \]

and cycle time as in eq. (5.14).

Using expectations defined earlier in the text, we have
\[ E(\text{Cost/cycle}) = C_1 b_1 Q_{111} + C_2 b_2 Q_{211} \]
\[ + \frac{C_1 \theta \sigma_{01}^2 + (\sigma_1^2 + b_1^2) Q_{111}^2}{2R} \]
\[ + \frac{C_2 \theta \sigma_{02}^2 + (\sigma_2^2 + b_2^2) Q_{211}^2}{2R} \]
\[ + \frac{C_1 I \sigma_{01}^2 + (\sigma_1^2 + b_1^2) Q_{111}^2}{2R} \]
\[ + \frac{C_2 I \sigma_{02}^2 + (\sigma_2^2 + b_2^2) Q_{211}^2}{2R} \]
\[ - \frac{\theta C_2 I [b_2^3 Q_{211}^3 + 3b_2 Q_{211} (\sigma_{02}^2 + \sigma_{211}^2)]]}{3R^2} \]
\[ - \frac{\theta C_1 I [b_1^3 Q_{111}^3 + 3b_1 Q_{111} (\sigma_{01}^2 + \sigma_{111}^2)]]}{3R^2} \]

Using, Renewal theory of Ross (1983), the average expected cost \( TC_{111}(Q_{111}, Q_{211}) \) is given by

\[ TC_{111}(Q_{111}, Q_{211}) = (\gamma_1 Q_{111}^3 + \gamma_2 Q_{111}^2 + \gamma_3 Q_{111} + \gamma_4 Q_{211}^3 + \gamma_5 Q_{211}^2 + \gamma_6 Q_{211}) \]
\[ \frac{R}{b_1 Q_{111} + b_2 Q_{211}} + \frac{\theta (b_1 + b_2 Q_{111}^2 + b_2 Q_{211})}{2 (b_1 Q_{111} + b_2 Q_{211})^2} \]

\[ \gamma_1 = \frac{C_1 \theta b_1^3}{3R^3} - \frac{C_1 \theta b_1 \sigma_{01}^2}{R^2} \]
\[ \gamma_2 = \frac{C_1 (\theta + I) \sigma_1^2 + b_1^2}{2R} \]
\[ \gamma_3 = C_1 [b_1 - \frac{\theta I b_1 \sigma_{01}^2}{R^2}] \]
\[ \gamma_4 = -\frac{C_2 \theta b_2^3}{3R^2} - \frac{C_2 \theta b_2 \sigma_{02}^2}{R^2} \]
\[ \gamma_5 = \frac{C_2 (\theta + I) (\sigma^2 + b_2^2)}{2R} \]
\[ \gamma_6 = C_2[b_2 - \frac{\theta I b_2 \sigma_{02}^2}{R^2}] \]

\[ \gamma_7 = (\theta + I)\left[\frac{C_1 \sigma_{01}^2 + C_2 \sigma_{02}^2}{2R}\right] + K \]

\[ \beta_1 = \sigma_{01}^2 + \sigma_{02}^2 \]

\[ \beta_2 = \sigma_1^2 + b_1^2 \]

\[ \beta_3 = \sigma_2^2 + b_2^2 \]

The optimal value of \( Q_{1II} \) and \( Q_{2II} \) can be obtained by solving \( \frac{\partial TC_{II}}{\partial Q_{1II}} = 0 \) and \( \frac{\partial TC_{II}}{\partial Q_{2II}} = 0 \) simultaneously by Gass-Seidal iterative method. Since \( Q_{1II} \) and \( Q_{2II} \) take only finite values, the iteration will coverage after finite number of steps. It can be seen that it is not possible to obtain the explicit form of the minimum value of the objective function \( TC_{II}^* \) in terms of all parameter values.

**NUMERICAL EXAMPLE:**

We use following parametric values:

Ordering cost, \( K = \$25.00 \) per order.

Demand rate, \( R = 1000 \) units per year.

Unit price charged by supplier 1, \( C_1 = \$35.00 \) per unit.

Unit price charged by supplier 2, \( C_2 = \$30.00 \) per unit.

Inventory holding charge fraction, \( I = 10 \% \) per annum.

(Note: we assume that second source charge mores \( C_1 > C_2 \))
We construct tables:

**TABLE : 1.** Effect of deterioration on order quantities and total cost (situation 1)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$TC_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>30.08</td>
<td>0.103</td>
<td>36321.78</td>
</tr>
<tr>
<td>0.015</td>
<td>30.34</td>
<td>0.104</td>
<td>36334.47</td>
</tr>
<tr>
<td>0.02</td>
<td>30.61</td>
<td>0.105</td>
<td>36347.13</td>
</tr>
<tr>
<td>0.025</td>
<td>30.88</td>
<td>0.105</td>
<td>36359.60</td>
</tr>
<tr>
<td>0.03</td>
<td>31.21</td>
<td>0.106</td>
<td>36372.10</td>
</tr>
<tr>
<td>0.035</td>
<td>31.45</td>
<td>0.106</td>
<td>36384.50</td>
</tr>
</tbody>
</table>
TABLE 2. Effect of deterioration on order quantities and total cost (situation II)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$TC_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>25.88</td>
<td>0.58</td>
<td>36431.10</td>
</tr>
<tr>
<td>0.015</td>
<td>26.21</td>
<td>0.59</td>
<td>36466.78</td>
</tr>
<tr>
<td>0.02</td>
<td>26.56</td>
<td>0.61</td>
<td>36501.70</td>
</tr>
<tr>
<td>0.025</td>
<td>26.91</td>
<td>0.63</td>
<td>36535.95</td>
</tr>
<tr>
<td>0.03</td>
<td>27.29</td>
<td>0.65</td>
<td>36569.52</td>
</tr>
<tr>
<td>0.035</td>
<td>27.68</td>
<td>0.67</td>
<td>36602.65</td>
</tr>
</tbody>
</table>

From tables, it can be observed that as rate of deterioration of units in inventory increases, the number of units to be procured at lower units cost (i.e. from supplier 1) is higher than that from supplier 2. Because of deterioration of units in an inventory total cost of an inventory system increases with increase in deterioration of units.

5.4 CONCLUSIONS:

In this chapter, we have provided an analysis of the diversification situations in the EOQ model when the yield is random. We have obtained analytic solutions for the optimal order quantities from two sources and the minimum average cost in terms of all the parametric values for the EOQ model. The model is developed for two different situations viz, when order is placed simultaneously to two suppliers and order is placed one after the other to two suppliers. Practically, simultaneous ordering to two suppliers increases inventory holding cost and as a result the total cost of an
inventory system increases. The stated model also agree with this fact. Model suggest that order only, once first order depletes, is favourable situation.